Design of Nonlinear Lead and/or Lag Compensators

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Abstract: A known nonlinear compensator design approach is adapted to allow design of nonlinear lead and/or lag compensators, and a number of MATLAB functions are developed that automate the compensator design procedure. With this design tool, control engineers would be able to rapidly design nonlinear lead and/or lag compensators. An example of a tutorial nature is presented.

Keywords: Describing functions, nonlinear systems, mechatronics, robotics.

1. INTRODUCTION

Methods for design of linear lead and/or lag compensators for linear plants have received considerable attention (see [1] and references therein). In many cases, a linear lead and/or lag compensator designed on the basis of a linear model does not produce adequate level of robustness over a wide variety of operating regimes, especially if the plant is highly nonlinear. Note that operating regimes, unlike operating points, are characterized by expected range of amplitudes and frequencies of excitation.

There is a limited literature on design of nonlinear lead/lag compensators [2-4]. A general describing function based approach for design of nonlinear compensators (PID, lead/lag, ...) is outlined in [2]. Then, in [3] the approach for design of nonlinear PID compensators is developed; the corresponding MATLAB routines are developed in [5]. Later in [4] three approaches (one of which is basically the approach outlined in [3]) are presented for design of nonlinear lead/lag compensators. In this work, the nonlinear controller design approach of [3] and [6] is adapted to allow design of nonlinear lead/lag compensators, and a computer-aided design approach is developed; no use of M-circles is made. One of the advantages of this approach (unlike previous works) is that allows inversion of nonlinearities with memories.

In this research, a multi-range and computer-aided nonlinear lead and/or lag compensator design technique is presented. The approach is based on several describing function (DF) models of the nonlinear plant at various operating regimes, and the amplitude dependent gains of the compensator are obtained by application of a new describing function inversion technique [7]; for alternative approaches for inverting memory-less nonlinearities see [3,4,6-11]. The method of design has no restriction on nonlinearity type, configuration, arrangement, and system order. There is no assumption that only single-valued nonlinear terms are considered. The design results in a nonlinear closed-loop system whose dynamic behavior is insensitive to various operating regimes of interest. The primary contribution of this work is the demonstration that approaches of [3] and [6] are also applicable to design of nonlinear lead and/or lag controllers.

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2. DESCRIBING FUNCTIONS

Describing functions have been used to analyze and diagnose nonlinear systems [12-14]. However, there is a body of literature in the past two decades that successfully uses DF in order to design robust nonlinear feedback systems [2-11,14-18]. Generally speaking, sinusoidal-input describing function (SIDF) models are used for the following reasons: (1) to capture the amplitude dependency of the original nonlinear plant, (2) to characterize the dependency of the nonlinear plant on the expected range of frequencies of interest, (3) to have strong basis for a robust design as the dependency of the nonlinear plant on the amplitudes of excitation is an important issue in design of robust nonlinear closed-loop systems, (4) to achieve robust stable closed-loop systems without sacrificing performance, (5) to characterize the behavior of the nonlinear plant only by one parameter which is the amplitude of excitation; hence, design is much simpler and restrictive than if the design is based on several parameters that are obtained by replacing each nonlinear term by a linear gain, and (6) to allow controller design for nonlinear plants with discontinuous or multi-valued nonlinear terms; in such cases, small signal linear models do not exit.

SIDF models may be obtained by a procedure similar to that used in limit cycle analysis; in this approach each nonlinearity term is replaced by a quasi-linear term, and a set of nonlinear algebraic equations, that correspond to harmonic balance, are solved to determine the parameters of the quasi-linear term [2]. This method assumes that input to each nonlinear term is nearly sinusoidal. Such assumption may be removed if the SIDF models are obtained by direct simulation and evaluation of Fourier integrals as outlined in [3].

Consider the following class of nonlinear systems.

$$\dot{x}(t) = f(x,u,t), \tag{1}$$

$$y(t) = g(x,u,t), \tag{2}$$

where, $x \in R^n$ is the vector of state variables, $u \in R^1$ is the input, $y \in R^1$ is the output, and t is the time variable. The stable plant is excited by a sinusoid of the following form.

$$u(t) = u_0 + a\cos(\omega t), \tag{3}$$

where u is the input to the plant, u_0 is the DC value of the input, a is the amplitude of the excitation, and ω is the frequency of excitation. The equations of motion are numerically integrated to obtain the output, $y(t)$. Once the output reaches steady state, the Fourier integrals for period k are evaluated; these integrals are of the following form.

$$I_{m,k} = \int_{(k-1)T}^{kT} y(t)e^{-jm\omega t}dt, \tag{3}$$

where $k = 1,2,3, \cdots$ is the period index, $m = 0,1, 2, \cdots$ is the harmonic index, and $T = 2\pi/\omega$ is the period. $I_{0,k}$ is the constant or DC component of the response, and the harmonic dependent transfer functions, $G_{m,k}$, are given by the following relation.

$$G_{m,k}(jm\omega;\, u_0, a) = \omega I_{m,k} / a\pi. \tag{4}$$

The first harmonic SIDF models are given by $G_{1,k}$. This procedure for generation of SIDF models of a nonlinear plant is automated by a MATLAB function. The MATLAB function [19] that is used to generate the SIDF models of multivariable nonlinear plants may be adapted to automate the generation of the SISO SIDF models. See [19] for printout of the SIDF generation software.

3. MAIN DEVELOPMENTS

The problem statement is to design a nonlinear lead and/or lag compensator for a nonlinear plant in a unity feedback configuration such that the resulting closed-loop system would be as insensitive to the amplitude level of the excitation command as possible. The structure of the considered nonlinear feedback system is depicted in Fig. 1.

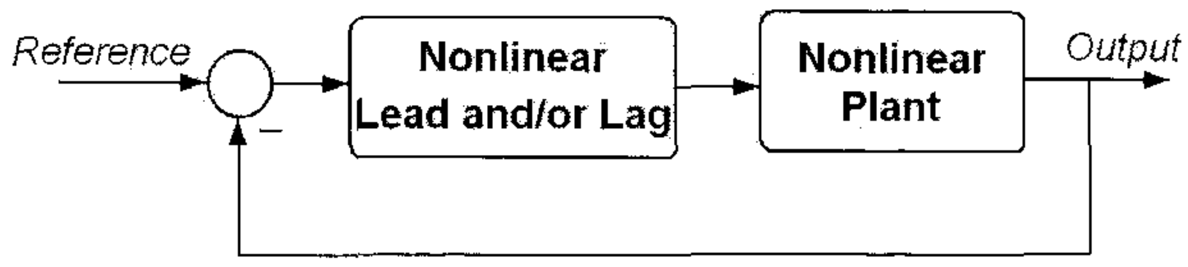

Fig. 1. The structure of the considered nonlinear feedback system.

The class of considered nonlinear plants is SISO with no other restrictions on the nonlinear plant; however, if higher than first harmonic effects are pronounced, then, adequate low pass filtering should be introduced. The considered nonlinear compensator is of a lead and/or lag type whose parameters are functions of the amplitude levels of the error signal.

The design procedure is composed of six steps, and they are basically the same as that developed in [3] with noted minor differences.

1. Generate the SIDF models for the user defined amplitude set $\{a_i\}$ and the frequency set $\{\omega_k\}$ as was outlined in the previous section.
2. Select one of the SIDF models of the previous step as the nominal model, $G^*(j\omega, a^*)$, and design a lead and/or lag compensator denoted $C^*(j\omega)$. In this research, the invfreqs function of MATLAB is used to identify a linear model for the nominal SIDF model followed by application of the MATLAB function described in [1] to obtain $C^*(j\omega)$. There is no selection rule. If a different model is selected, then identical results would be obtained. The only difference would be a small difference in the desired open-loop model of Step 3, and therefore, a small difference in final results would be expected. But as long as the performance measures are satisfied in this Step, then the final results would also be satisfactory.
3. Generate the open-loop SIDF model of the system comprised of $C^*(j\omega)$ in series with the nonlinear plant; note that excitation signal would be of the form $e(t) = e^*\cos(\omega t)$, and e^* is consistent with a^*:

$$e^* = a^* / \left|C^*_{co}\right|, \tag{5}$$

where $\left|C^*_{co}\right|$ is the gain of the nominally designed compensator at the phase cross-over frequency.

4. Design a set of linear lead and/or lag

compensators at various operating regimes of interest that mimic the desired open-loop behavior obtained in the previous section. This is accomplished by minimizing the following objective function.

$$E(j\omega)=\left|\begin{array}{l}C^{*}G^{*}(j\omega,e^{*})\\-C_i(j\omega)G(j\omega,e_i\left|C_i(j\omega)\right|)\end{array}\right|. \quad (6)$$

The major problem with this step is that software for generation of SIDF models must be placed inside an optimization routine to evaluate the open-loop frequency response at each iteration. This will result in an unreasonable execution time for this step. In order to lift this restriction, $G(j\omega,e_i\left|C_i(j\omega)\right|)$ is evaluated by using the $G(j\omega,a_i)$ data of Step 1 coupled with interpolation. This will allow one to execute this step with standard available personal computers. The output of this step is a set of values for the parameters of the lead and/or lag compensator at various operating regimes of interest. These amplitude dependent parameters are treated as describing function models of the desired nonlinear gains of the nonlinear compensator.

5. Determine a set of nonlinear functions whose DF models mimic the DF models of the compensator parameters that were determined in the previous step. A typical nonlinear function, that user may select, is shown in Fig. 2.
The parameters of this function are determined in such a manner that the function DF model approximates the amplitude dependent gains of the compensator parameters at various operating regimes of interest. Such a function will allow for high gains at low amplitudes, constant gains at specific amplitude ranges, and gain reductions or jumps at specific amplitudes. Note that this step is the inverse of activities of Step 1. In Step 1, the model is available and the SIDF models are to be obtained; in this step, the SIDF models are known, and the nonlinearity model must be obtained. In order to determine the parameters of the nonlinearity function shown in Fig. 2, optimization is used, and the fminsearch function of the MATLAB software is utilized. The details of this inversion process are: (a) use the describing function generator MATLAB function [19] and generate the describing function model of the nonlinearity, (b) if satisfactory results are obtained, then stop; otherwise, use fminsearch function of the MATLAB in order to modify the independent variables and go to (a).
6. Finally, the closed-loop system at various operating regimes of interest is simulated to verify design. Additionally, in order to quantify the amount of achieved reduction in sensitivity the followings are numerically evaluated.

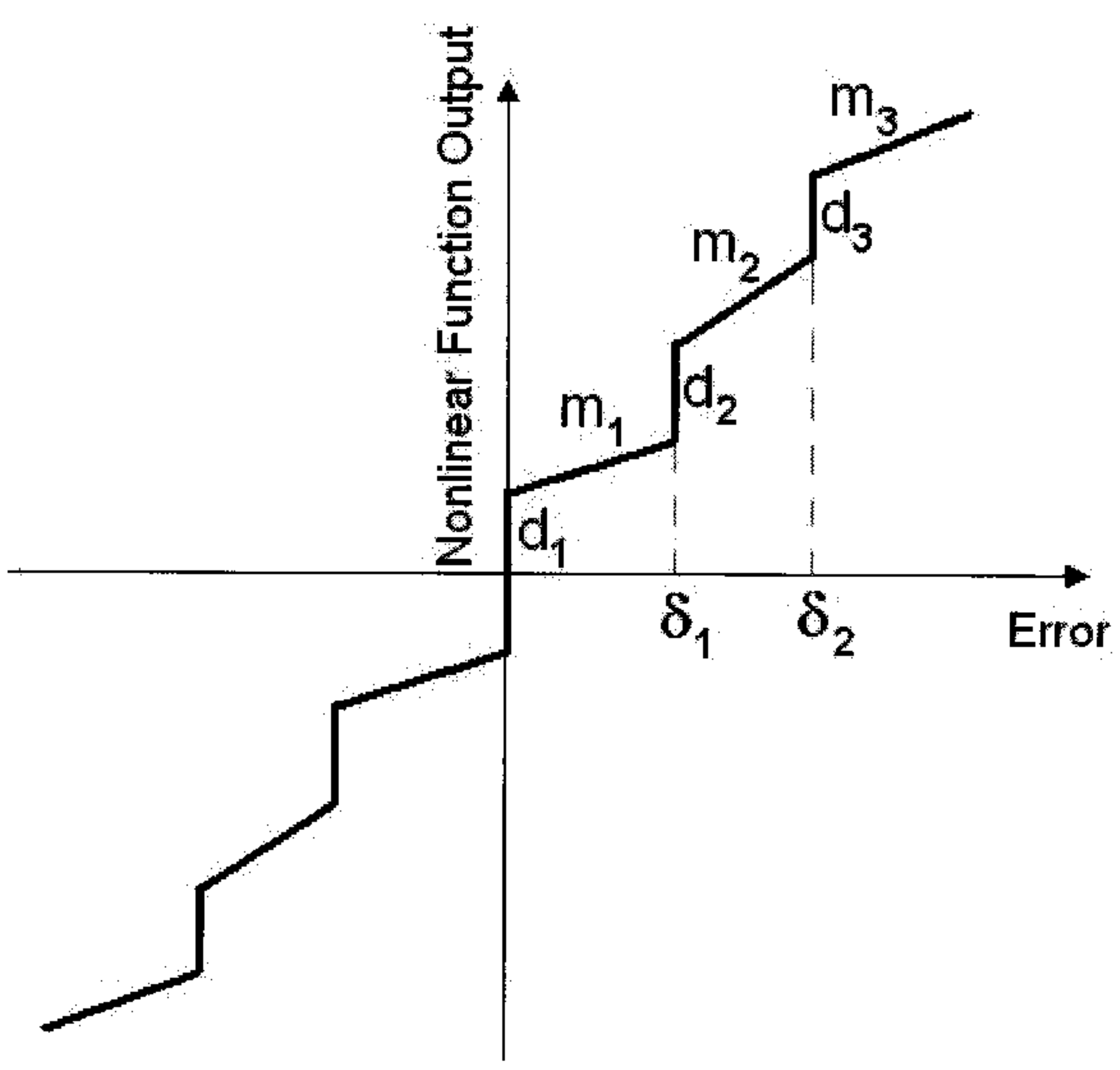

Fig. 2. The candidate nonlinear function that is used in the describing function inversion process.

$$\sigma_0=\sum_i\left[\int_{\omega_1}^{\omega_2}\left|1-\frac{G_i(j\omega,a_i)}{G^{*}(j\omega,a^{*})}\right|d\omega\right], \quad (7)$$

$$\sigma_1=\sum_i\left[\int_{\omega_1}^{\omega_2}\left|1-\frac{\Gamma(C_{NL},NL_i)}{\Gamma(C_{NL},NL^{*})}\right|d\omega\right], \quad (8)$$

where $G_i(j\omega,a_i)$ is the SIDF model of the open-loop nonlinear plant at operating regime i, $\Gamma_i(C_{NL},NL_i)$ is the SIDF model of the open-loop system comprised of the nonlinear controller, C_{NL}, and the nonlinear plant at operating regime i, NL_i, and $\Gamma^{*}(C_{NL},NL^{*})$ is the SIDF model of the open-loop system comprised of the nonlinear controller, C_{NL}, and the nonlinear plant at nominal operating regime, NL^{*}.

The closer σ_i $(i=0,1)$ is to zero, the smaller is the level of sensitivity. The percent amount of reduction in sensitivity would be given by the following relation.

$$R=(\sigma_0-\sigma_1)/\sigma_0\times 100 \quad (9)$$

The interested reader may obtain a copy of the developed MATLAB functions that automate the above procedure by sending an email to the first author at control727@yahoo.com.

4. TUTORIAL EXAMPLE PROBLEM

Consider the problem of designing a nonlinear lead compensator for a nonlinear process of the sort encountered in robotics [6]. The schematic block diagram of the nonlinear plant considered is depicted

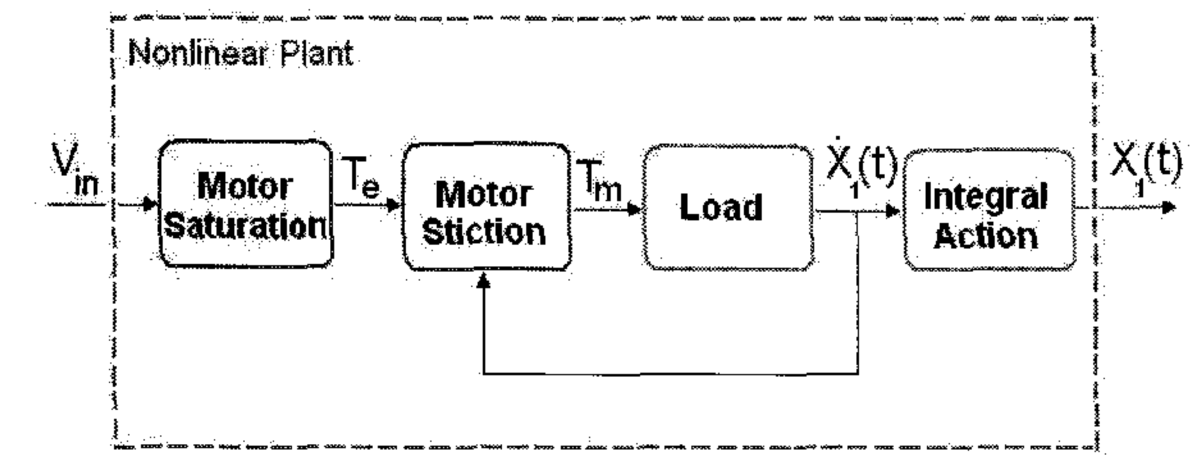

Fig. 3. The schematic block diagram for the tutorial example problem.

in Fig. 3.

This nonlinear plant is of the sort encountered in robotics/mechatronics, and it is typical of position control problems that exist in industry. The mathematical model of the nonlinear plant is given by (9)-(12).

$$\dot{x}_1 = x_2, \tag{9}$$

$$\dot{x}_2 = T_m / J, \tag{10}$$

where $J = 0.01$ kg-m^2 and the servomotor friction characteristic includes Coulomb and viscous effects.

$$T_m = \begin{cases} T_e - f_v\dot{x}_1 - f_c Sign(\dot{x}_1) & \text{if } |T_e| > f_c, \\ T_e - f_v\dot{x}_1 - f_c Sign(\dot{x}_1) & \text{if } \dot{x}_1 \neq 0, \\ 0.0 \text{ if } |T_e| < f_c \text{ and } \dot{x}_1 = 0, \end{cases} \tag{11}$$

where $f_v = 0.1\text{Nm} - \text{s/rad}$, $f_c = 1.0\text{Nm}$, and servomotor saturation effects are modeled by

$$T_e = \begin{cases} m_1 V_{in} & \text{if } |V_{in}| \le \delta, \\ Sign(V_{in}) \cdot \left(m_1\delta + m_2\left(|V_{in}| - \delta\right)\right) & \text{if } |V_{in}| > \delta, \end{cases} \tag{12}$$

where

$\delta = 0.5$ volts, $m_1 = 5$ Nm/V, and $m_2 = 1.0$ Nm/V.

The computer model [6] of this process in terms of a FORTRAN subroutine is given in reference [20], and it will not be repeated here. The problem statement is that a feedback system with a nonlinear lead controller must be designed such that resulting closed-loop system would be as insensitive to the amplitude level of the excitation command as possible. The expected amplitudes of excitation in units of Volts are 0.25, 0.325, 0.4, 0.8, 1.6, 3.2, 6.4, 12.8, 30, and 60. The nonlinear effects are most pronounced at an excitation of 0.25 or less, and they become less important at an excitation of 12.8 and higher. The synthesis procedure is executed as follows.

1. The SIDF models are generated at various operating regimes of interest as outlined before; these models are shown in Figs. 4, and 5.
   It is apparent that system is highly nonlinear because the surfaces are not flat. There is a 20 dB spread across the magnitude plots, and there is a 60 degrees spread across the phase plots. At small amplitudes, where stiction effects become pronounced, phase behavior is not smooth.

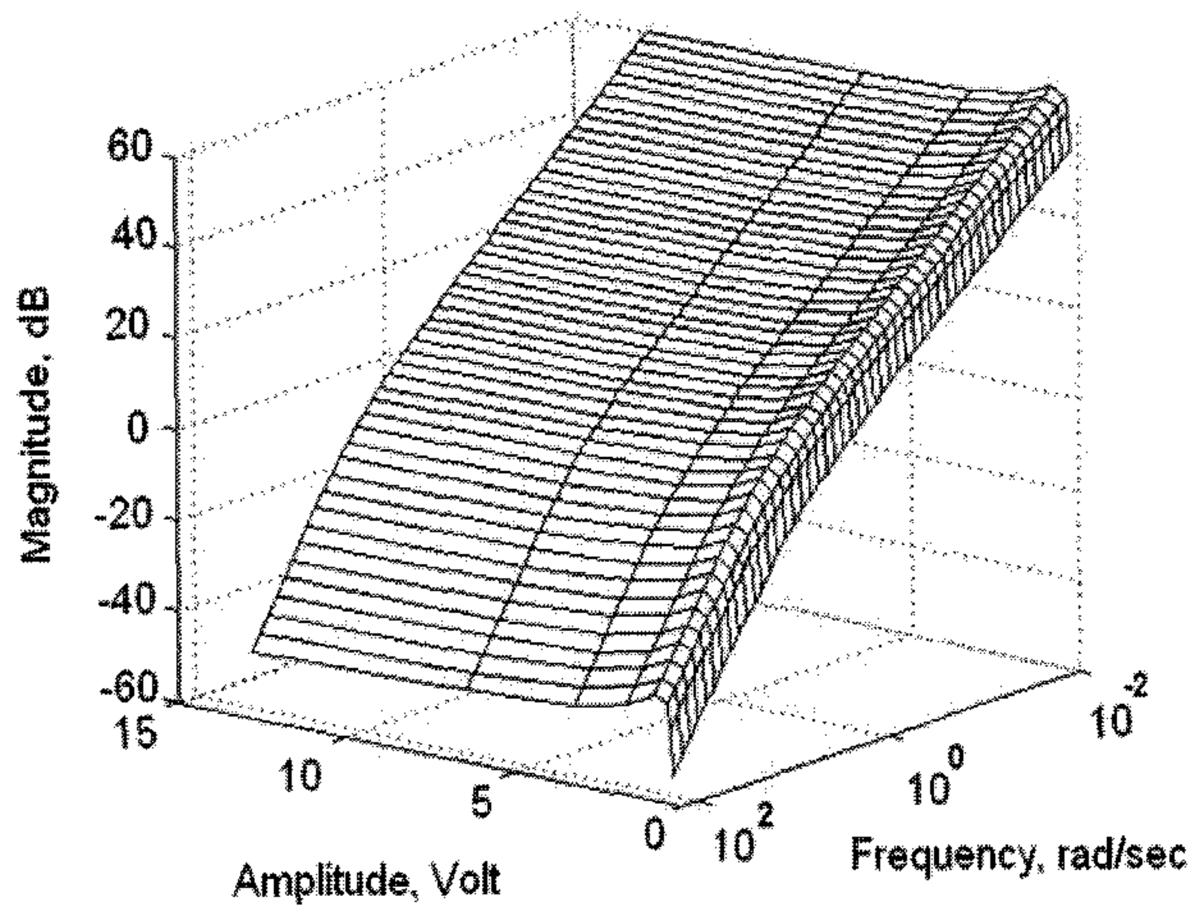

Fig. 4. The magnitude surface of the nonlinear plant.

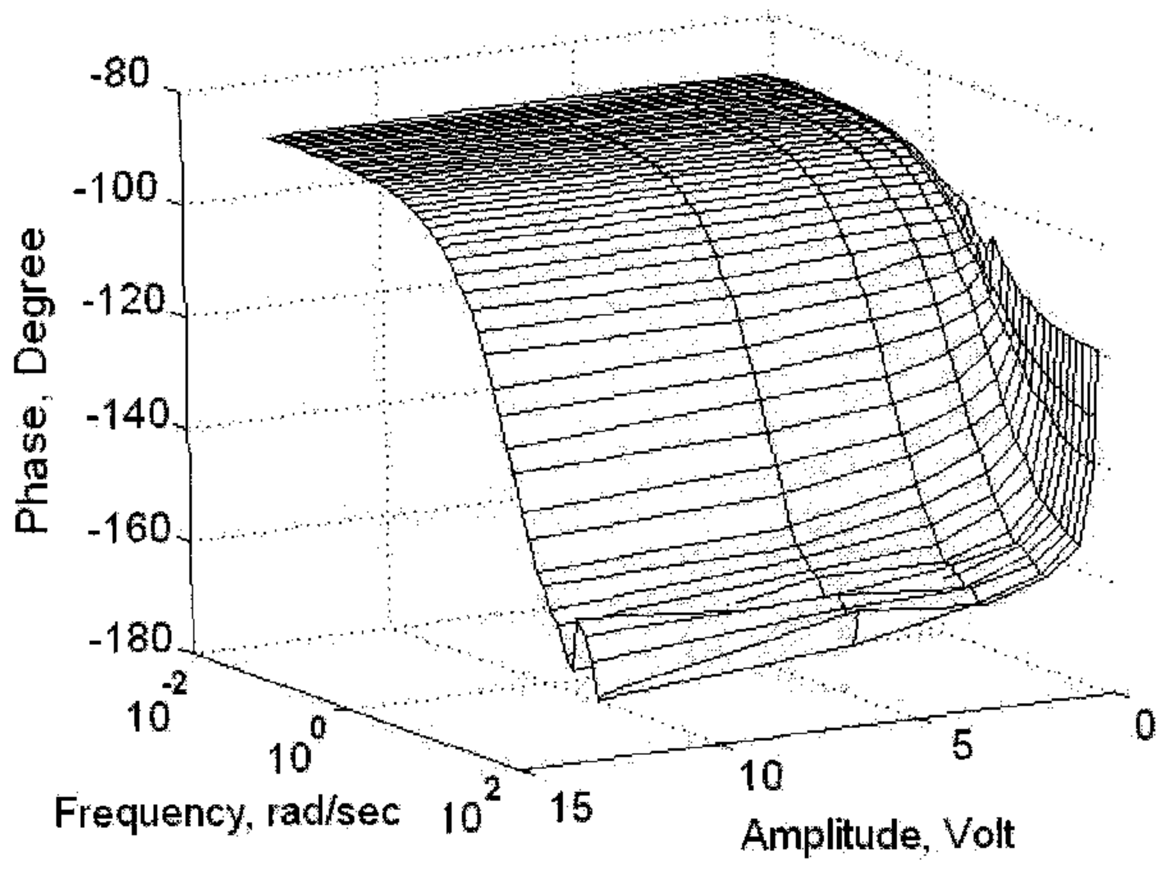

Fig. 5. Phase surface of the nonlinear plant.

2. The middle model (at excitation of 0.325) is selected as the nominal model; a linear model is identified using invfreqs function of MATLAB, and the MATLAB function developed in [1] is used to design a lead compensator. The compensator gain is adjusted to obtain a satisfactory step response when the resulting nonlinear system is simulated. The designed lead compensator is of the following form.

$$C^* = 1.5(0.3514s + 3.36)/(0.03303s + 1) \tag{13}$$

3. The desired open-loop behavior is determined by obtaining the pseudo frequency response of the open-loop system comprised of the nominal compensator, C^*, and the nonlinear plant. The desired open-loop system behavior is shown in Fig. 6.
   This quasi-linear model serves as the objective to obtain the parameters of the lead compensator at

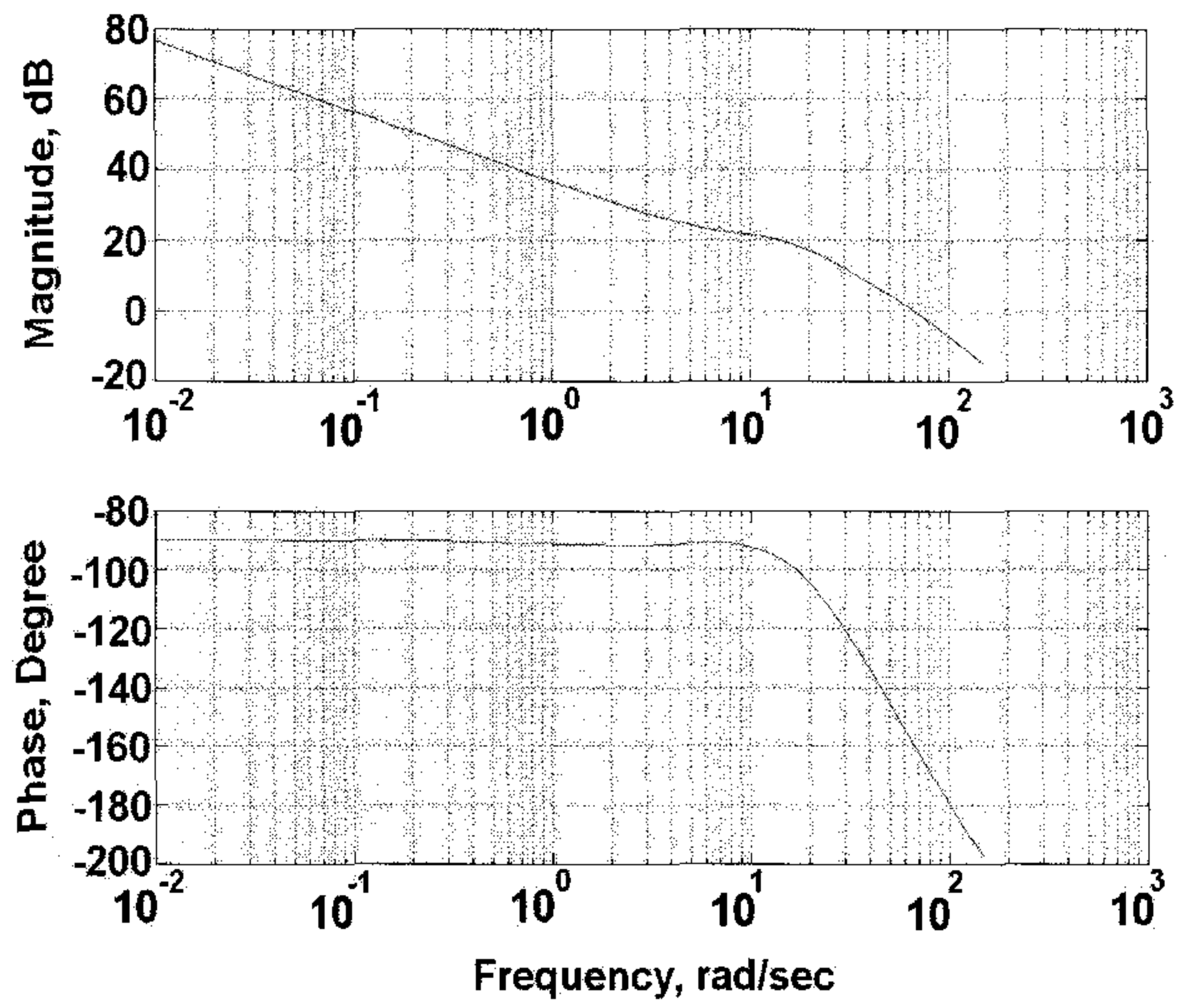

Fig. 6. The SIDF model of the open-loop system comprised of the nominal compensator, C^*, and the nonlinear plant.

various operating regimes that cause insensitivity of the nonlinear plant to various amplitude levels of excitation. The target model has approximately a 30 degrees phase margin and an approximate gain margin of 3.

4. The set of linear lead compensators at various excitation amplitudes are obtained as outlined previously; those are: A_0={6.1141, 5.0390, 4.3959, 3.3293, 3.3834, 5.0765, 5.9311, 6.3986, 6.7062, 6.8369}, A_1={0.6507, 0.6001, 0.5737, 0.5447, 0.7374, 1.0177, 1.1620, 1.2460, 1.3322, 1.1587}, and B_1={0.0423, 0.0380, 0.0344, 0.0241, 0.0235, 0.0259, 0.0271, 0.0272, 0.0264, 0.0302}.
The parameters of the lead compensator are defined as below

$$C = (A_1 s + A_0)/(B_1 s + 1). \tag{13}$$

The constants are nonlinear function of the error signal: (a) for a given value of the error signal, the nonlinearities (obtained in Step 5) are evaluated and this gives a set of constants, (b) the values obtained in (a) are divided by error value, and (c) the values obtained in (b) sit in place of the lead and/or lag coefficients and the resulting controller is excited by the error signal.

5. With reference to Fig. 2, the parameters of the synthesized nonlinear functions for compensator parameters are defined in Table 1.
The quality of fit is shown in Figs. 7-9. These figures show the results of DF inversion process for A_0, A_1, and B_1. The nonlinearity parameters are optimized in a fashion that DF model of that nonlinearity mimics the desired amplitude dependent gains of the compensator parameters.

6. Finally, the design is verified using numerical

Table 1. Parameters of the synthesized nonlinear functions.

	δ_1	δ_2	d_1	d_2
A_0	0.1486	0.2811	0.1822	0.1452
A_1	0.1762	0.5761	0.0079	0.1111
B_1	0.0583	0.2373	0.0006	-0.0001
	d_3	m_1	m_2	m_3
A_0	0.3085	1.4951	1.1223	6.9811
A_1	0.3338	0.4475	0.4105	1.2591
B_1	0.0005	0.0282	0.0168	0.0286

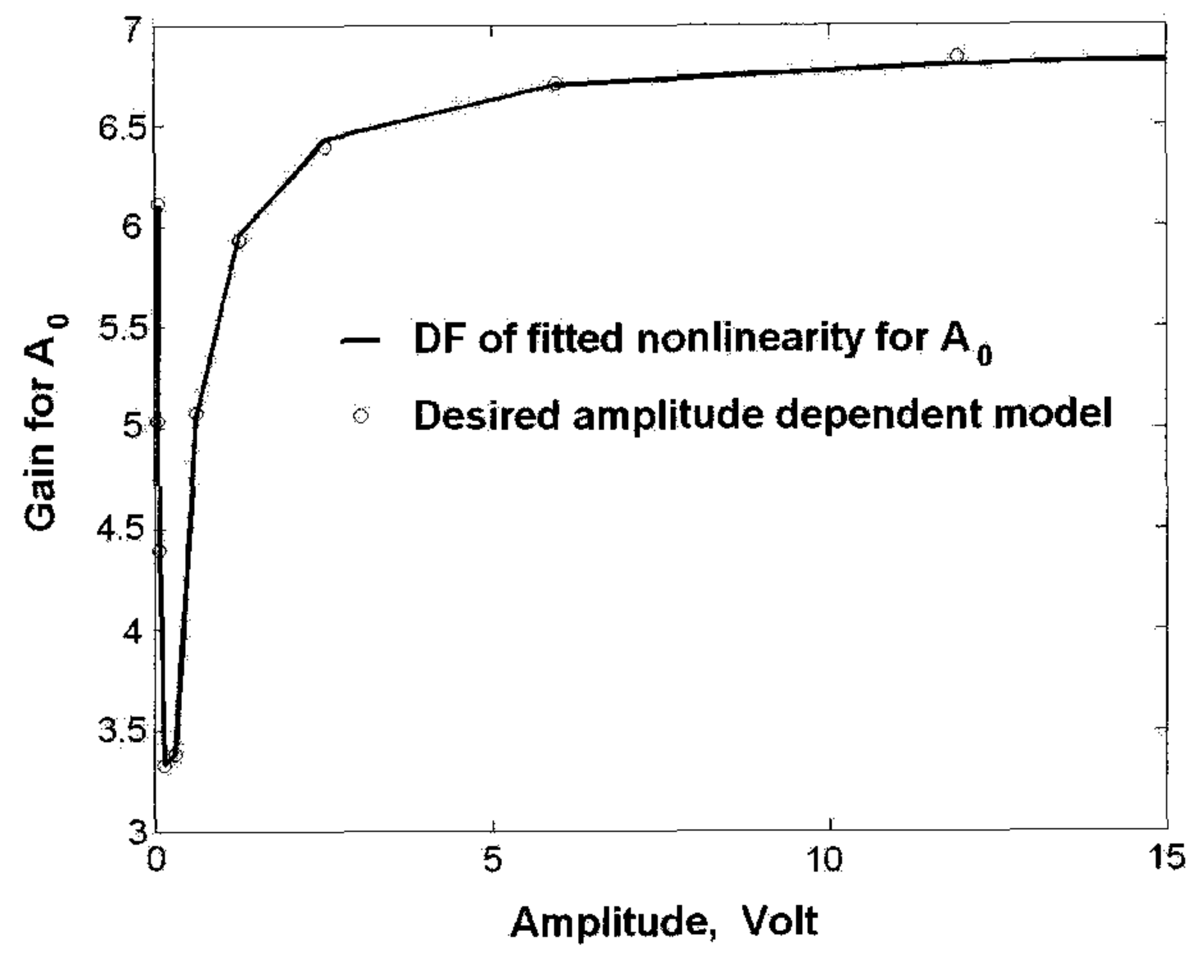

Fig. 7. The results of describing function inversion process for A_0.

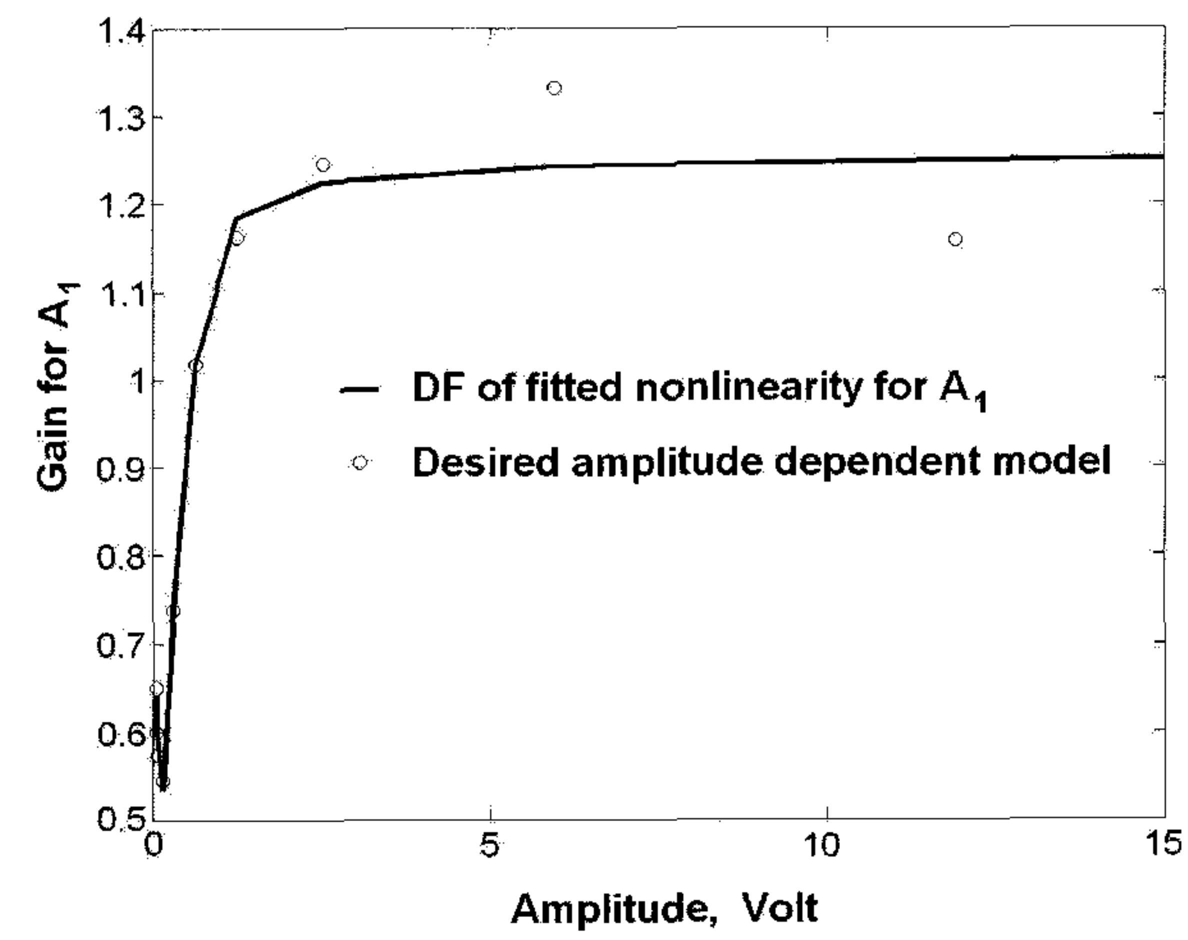

Fig. 8. The results of describing function inversion process for A_1.

simulation. The normalized step response plots of the closed loop system at various operating regimes of interest are depicted in Fig. 10. This figure demonstrates the results of design verification step. The normalized step responses at

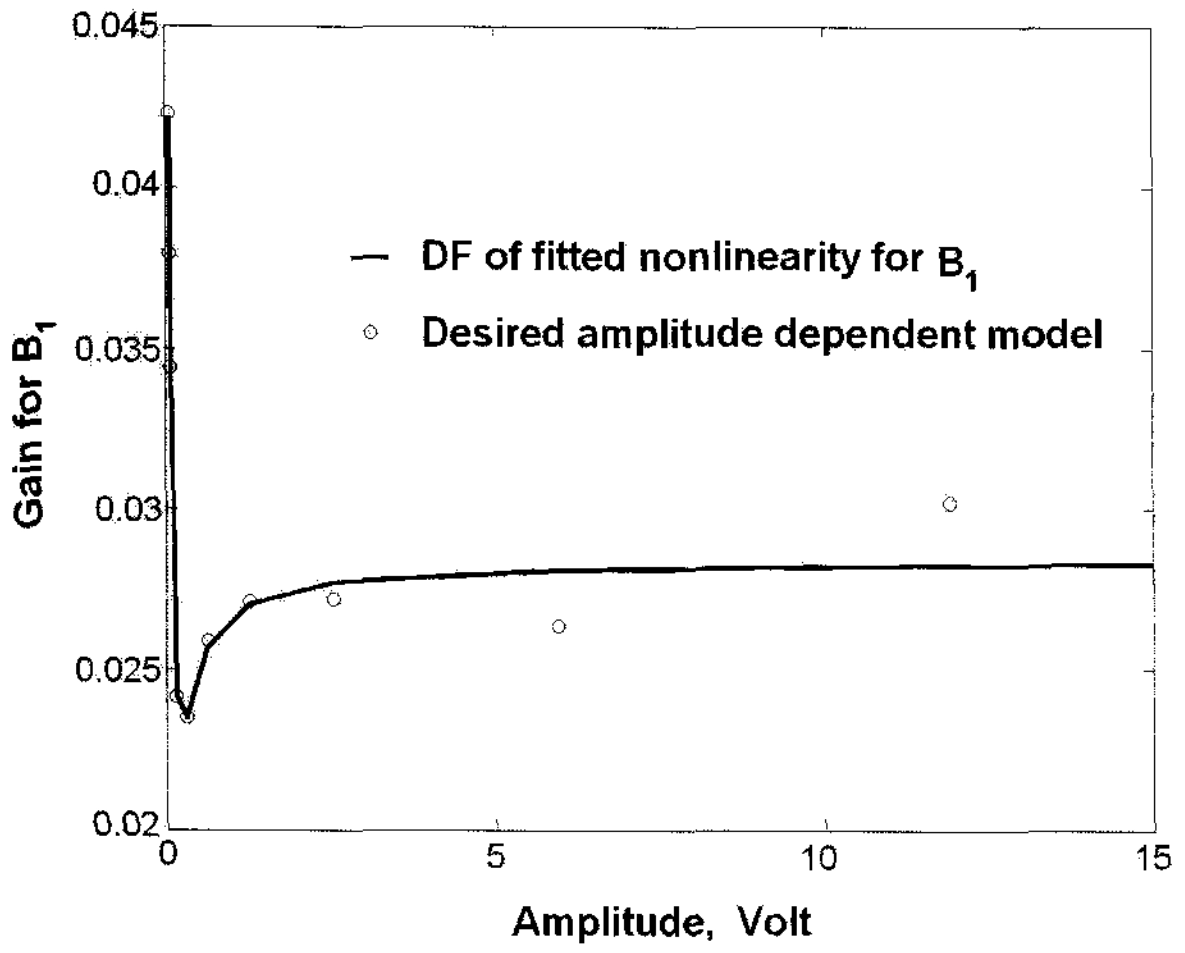

Fig. 9. The results of describing function inversion process for B_1.

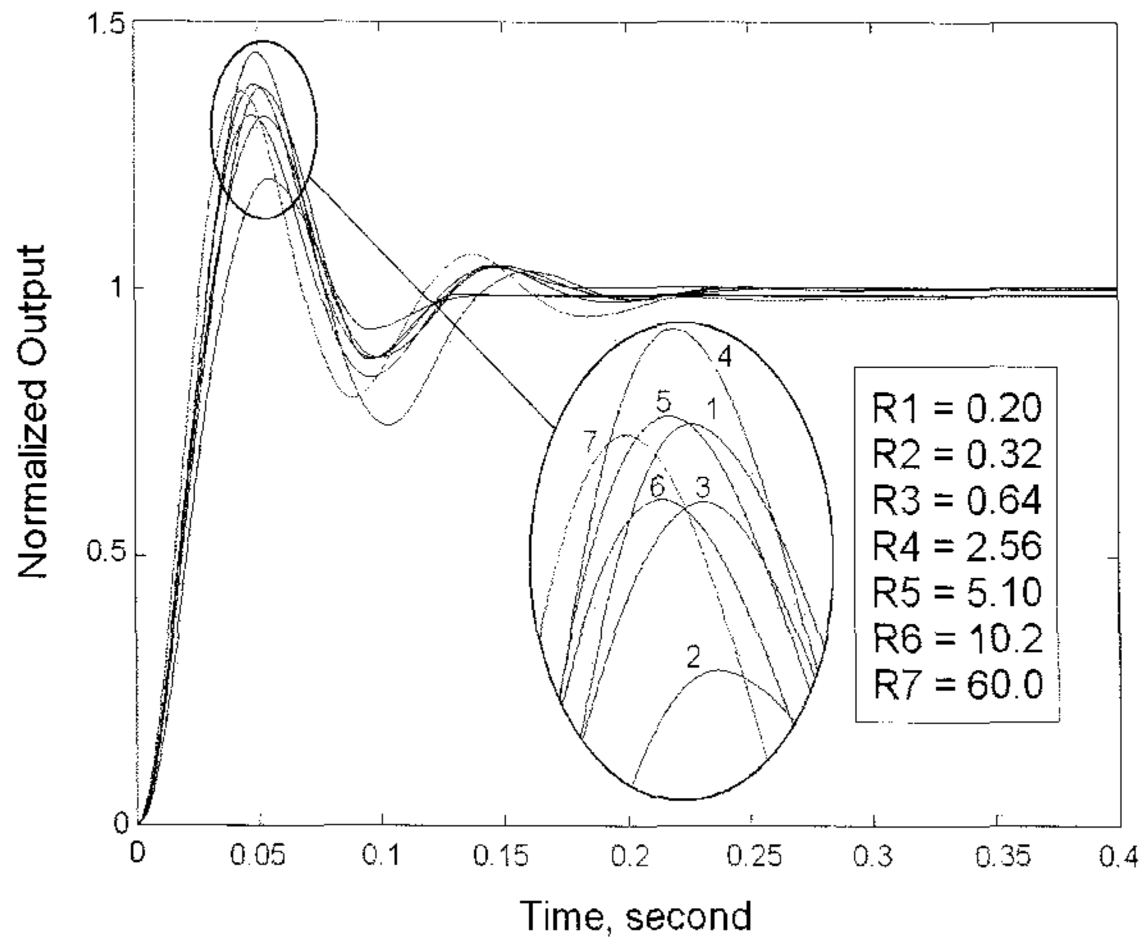

Fig. 10. Normalized step-response plots of the synthesized nonlinear control system.

various amplitudes of reference signals (R1, R2, R3, ...) indicate that closed loop system behavior is fairly insensitive to the amplitude level of the command signal.

The stability test is shown in Fig. 11. In this figure, the pseudo frequency response plots of the open-loop system comprised of the designed nonlinear lead compensator and the nonlinear process are depicted. Phase margin is about 30 degrees and the open-loop pseudo frequency response does not encircle the -1 point at all operating regimes; hence, it may be concluded that the designed closed loop nonlinear system would be stable. Also, it is apparent that open-loop system is fairly insensitive to the amplitudes of excitations. The legend on the magnitude plot identifies the amplitudes of the error signals (e1, e2, e3, ...) consistent with the amplitudes of excitations that define the operating regimes. As was mentioned in the previous section σ_0 and σ_2 may be evaluated to measure the percent reduction in

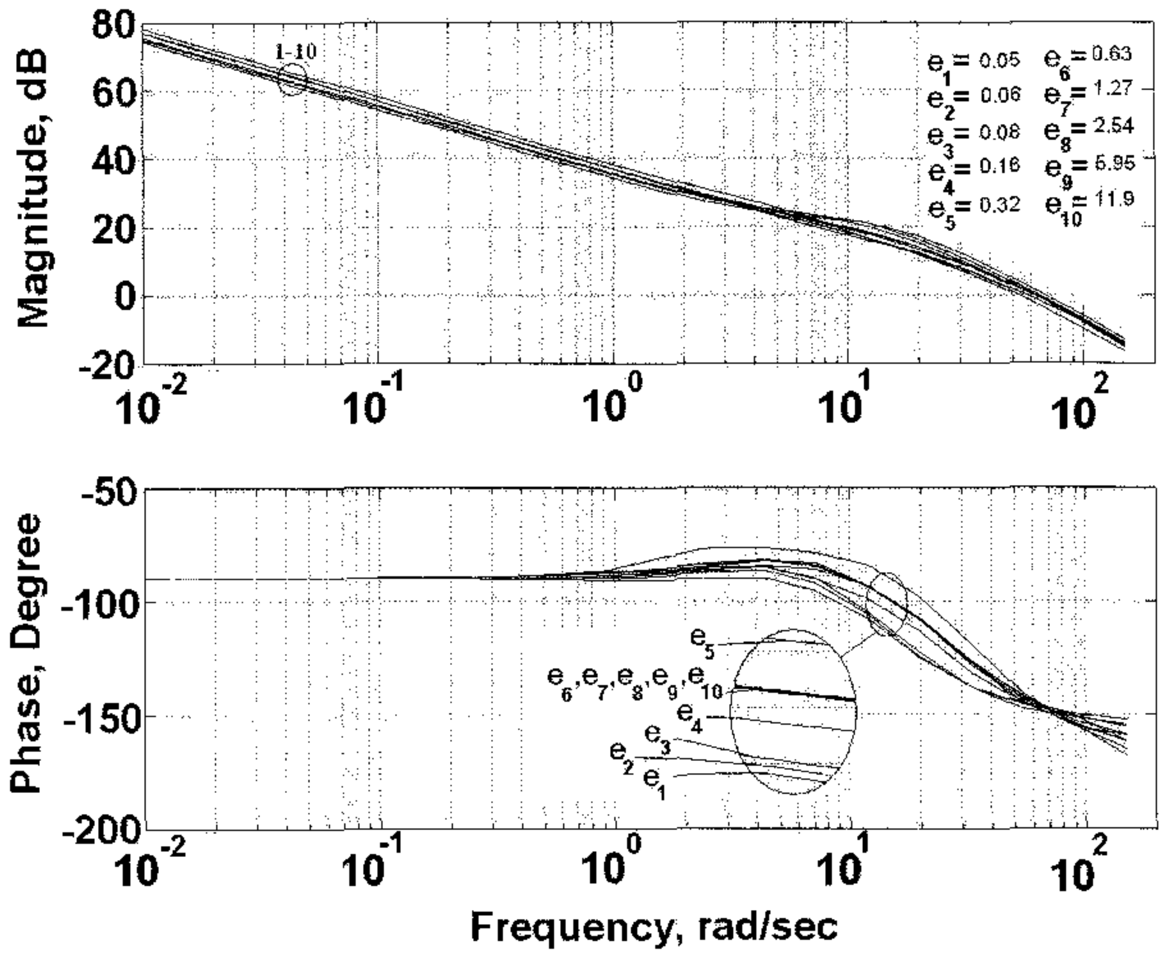

Fig. 11. Results of stability test.

sensitivity; in this case, $\sigma_0 = 2308$, $\sigma_1 = 322$, and $R = 86$. In other words, 86percent reduction in sensitivity is achieved.

5. SUMMARY AND CONCLUSIONS

A computer-aided design procedure for design of nonlinear lead and/or lag compensators is presented. The procedure is composed of six steps, and each step is a functional unit whose output drives the other steps of the procedure. The procedure is not trial-and-error free; however, it is considered to be systematic [2]. The areas of trial-and-error are: (1) the original closed-loop performance measures may have to be modified since the reference linear lead and/or lag compensator design (C^*) is based on linear or quasi-linear model of the nonlinear system, (2) the gain of the designed reference linear compensator may have to be adjusted to account for nonlinear effects not accounted for in applying a linear control theory for design of C^*, (3) the supplied starting solution in the inversion process of Step 5 may require subjective judgment, and this usually results in a few iterations. For immediate future work it is recommended to extend the presented design procedure and the associated software to allow the design of nonlinear multivariable lead and/or lag compensators.

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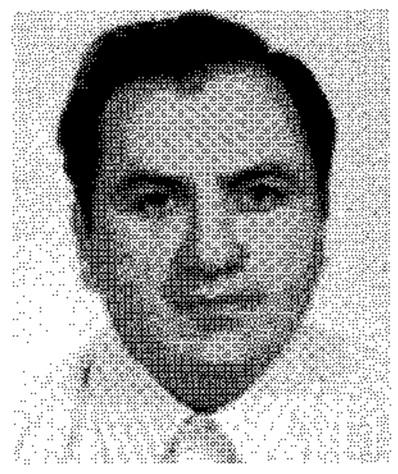

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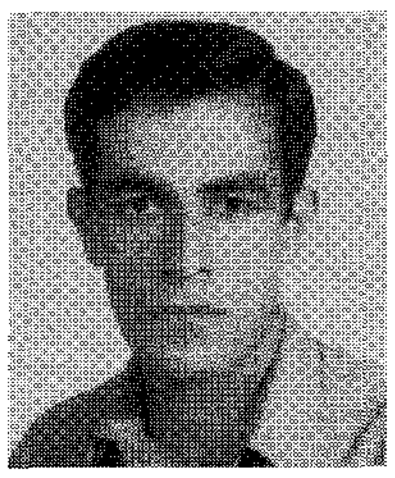

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