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JORDAN DERIVATIONS OF SEMIPRIME RINGS AND NONCOMMUTATIVE BANACH ALGEBRAS, I

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ABSTRACT. Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \to A$ such that $D(x)[D(x), x]^2 \in \mathrm{rad}(A)$ or $[D(x), x]^2 D(x) \in \mathrm{rad}(A)$ for all $x \in A$. In this case, we have $D(A) \subseteq \mathrm{rad}(A)$.

1. INTRODUCTION

Throughout, R represents an associative ring and A will be a complex Banach algebra. We write $[x, y]$ for the commutator $xy - yx$ for x, y in a ring. Let $\mathrm{rad}(R)$ denote the (*Jacobson*) *radical* of a ring R. And a ring R is said to be (*Jacobson*) *semisimple* if its Jacobson radical $\mathrm{rad}(R)$ is zero.

A ring R is called *n-torsion free* if $nx = 0$ implies $x = 0$. Recall that R is *prime* if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is *semiprime* if $aRa = (0)$ implies $a = 0$. On the other hand, let X be an element of a normed algebra. Then for every $a \in X$ the *spectral radius* of a, denoted by $r(a)$, is defined by $r(a) = \inf\{||a^n||^{\frac{1}{n}} : n \in \mathbb{N}\}$. It is well-known that the following theorem holds: If a is an element of a normed algebra, then $r(a) = \lim_{n\to\infty} ||a^n||^{\frac{1}{n}}$ (see F.F. Bonsall and J. Duncan [1]).

An additive mapping D from R to R is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. And an additive mapping D from R to R is called a *Jordan derivation* if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$.

B.E. Johnson and A.M. Sinclair [5] have proved that any linear derivation on a semisimple Banach algebra is continuous. A result of I.M. Singer and J. Wermer [9] states that every continuous linear derivation on a commutative Banach algebra

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maps the algebra into its radical. From these two results, we can conclude that there are no nonzero linear derivations on a commutative semisimple Banach algebra. M.P. Thomas [10] has proved that any linear derivation on a commutative Banach algebra maps the algebra into its radical.

J. Vukman [11] has proved the following: Let R be a 2-torsion free prime ring. If $D : R \longrightarrow R$ is a derivation such that $[D(x), x]D(x) = 0$ for all $x \in R$, then $D = 0$.

Moreover, using the above result, he has proved that the following holds: Let A be a noncommutative semisimple Banach algebra. Suppose that $[D(x), x]D(x) = 0$ holds for all $x \in A$. In this case, $D = 0$.

B.D. Kim [6] has showed that the following results hold: Let R be a 3!-torsion free semiprime ring. Suppose there exists a Jordan derivation $D : R \to R$ such that

$$[D(x), x]D(x)[D(x), x] = 0$$

for all $x \in R$. In this case, we have $[D(x), x]^5 = 0$ for all $x \in R$.

And, B.D. Kim [7] has showed that the following results hold: Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \to A$ such that $D(x)[D(x), x]D(x) \in \text{rad}(A)$ for all $x \in A$. In this case, we have $D(A) \subseteq \text{rad}(A)$.

In this paper, our first aim is to prove the following results in the ring theory in order to apply it to the Banach algebra theory:

(i) Let R be a 3!-torsion free semiprime ring.

Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that

$$D(x)[D(x), x]^2 = 0$$

for all $x \in R$. In this case, we obtain $[D(x), x]^4 = 0$ for all $x \in R$.

(ii) Let R be a 3!-torsion free semiprime ring.

Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that

$$[D(x), x]^2 D(x) = 0$$

for all $x \in R$. In this case, we obtain $[D(x), x]^4 = 0$ for all $x \in R$.

Using the above results, theses generalize J. Vukman's result [11] as follows: Let A be a noncommutative Banach algebra and let $D : A \longrightarrow A$ be a continuous linear Jordan derivation.

(iii) Suppose that $D(x)[D(x), x]^2 \in \text{rad}(A)$ holds for all $x \in A$. In this case, $D(A) \subseteq \text{rad}(A)$.

(iv) Suppose that $[D(x),x]^2D(x) \in \text{rad}(A)$ holds for all $x \in A$. In this case, $D(A) \subseteq \text{rad}(A)$.

2. Preliminaries and Results

The following lemma is due to L.O. Chung and J. Luh [4].

Lemma 2.1. *Let R be a $n!$-torsion free ring. Suppose there exist elements $y_1, y_2, \cdots, y_{n-1}, y_n$ in R such that $\sum_{k=1}^{n} t^k y_k = 0$ for all $t = 1, 2, \cdots, n$. Then we have $y_k = 0$ for every positive integer k with $1 \leq k \leq n$.*

The following theorem is due to M. Brešar [3].

Theorem 2.2. *Let R be a 2-torsion free semiprime ring and let $D : R \longrightarrow R$ be a Jordan derivation. In this case, D is a derivation.*

We write $Q(A)$ for the set of all quasinilpotent elements in A. M. Brešar [2] has proved the following theorem.

Theorem 2.3. *Let D be a bounded derivation of a Banach algebra A. Suppose that $[D(x), x] \in Q(A)$ for every $x \in A$. Then D maps A into rad(A).*

3. Main Results in Semiprime Rings

We need the lemma to prove the main theorem. After this, by S_m we denote the set $\{k \in \mathbb{N} \mid 1 \leq k \leq m\}$ where m is a positive integer.

We need the following results to obtain the main theorems for Banach algebra theory.

Theorem 3.1. *Let R be a $3!$-torsion free noncommutative semiprime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$D(x)[D(x), x]^2 = 0$$

for all $x \in R$. In this case we have $[D(x), x]^4 = 0$ for all $x \in R$.

Proof. By Theorem 2.2, we can see that D is a derivation on R. For simplicity, we shall denote the maps $B : R \times R \longrightarrow R$, $f, g : R \longrightarrow R$ by $B(x,y) \equiv [D(x), y] + [D(y), x], f(x) \equiv [D(x), x]$, $g(x) \equiv [f(x), x]$, $h(x) \equiv [g(x), x]$ for all $x, y \in R$ respectively. Then we have the basic properties:

$$B(x,y) = B(y,x), \ B(x,yz) = B(x,y)z + yB(x,z) + D(y)[z,x] + [y,x]D(z),$$

$$B(x,x)=2f(x),\ B(xy,z)=B(y,z)x+zB(y,x)+D(z)[x,y]+[z,y]D(x),$$
$$B(x,x^2)=2(f(x)x+xf(x)),\ x,y,z\in R.$$

After this, we use the above relations without specific reference. By assumption,

(1) $$D(x)f(x)^2=0,\ x\in R.$$

Replacing $x+ty$ for x in (1), we have

(2) $$\begin{aligned}&D(x+ty)f(x+ty)^2\\ &\equiv D(x)f(x)^2+t\{D(y)f(x)^2+D(x)B(x,y)f(x)\\ &+D(x)f(x)B(x,y)\}+t^2H_1(x,y)+t^3H_2(x,y)+t^4H_3(x,y)\\ &+t^5D(y)f(y)^2=0,\ x,y\in R,\ t\in S_3\end{aligned}$$

where $H_i, 1\le i\le 3$, denotes the term satisfying the identity (2).
From (1) and (2), we obtain

(3) $$\begin{aligned}&t\{D(y)f(x)^2+D(x)B(x,y)f(x)+D(x)f(x)B(x,y)\}\\ &\quad+t^2H_1(x,y)+t^3H_2(x,y)+t^4H_3(x,y)=0,\ x,y\in R,\ t\in S_4.\end{aligned}$$

Since R is 3!-torsion free by assumption, by Lemma 2.1 the relation (3) yields

(4) $$D(y)f(x)^2+D(x)B(x,y)f(x)+D(x)f(x)B(x,y)=0,\ x,y\in R.$$

Let $y=x^2$ in (4). Then using (1), we have

(5) $$\begin{aligned}&(D(x)x+xD(x))f(x)^2+D(x)(2(f(x)x+xf(x)))f(x)\\ &\quad+D(x)f(x)(2(f(x)x+xf(x)))\\ &=f(x)^3-2D(x)f(x)g(x)+2f(x)^3-2D(x)f(x)g(x)\\ &=3f(x)^3-4D(x)f(x)g(x)=0,\ x\in R.\end{aligned}$$

From (1), we arrive at

(6) $$\begin{aligned}0&=[D(x)f(x)^2,x]\\ &=f(x)^3+D(x)g(x)f(x)+D(x)f(x)g(x),\ x\in R.\end{aligned}$$

From (5) and (6), we get

(7) $$7f(x)^3+4D(x)g(x)f(x)=0,\ x\in R.$$

Combining (5) with (7),

$$4(3D(x)g(x)f(x)+7D(x)f(x)g(x))=0,\ x\in R.$$

Since R is 3!-torsion-free, the above relation gives

$$3D(x)g(x)f(x) + 7D(x)f(x)g(x) = 0, \ x \in R. \tag{8}$$

Writing xy for y in (4), we get

$$\begin{aligned} &xD(y)f(x)^2 + D(x)yf(x)^2 + D(x)(xB(x,y) + 2f(x)y + D(x)[y,x])f(x) \\ &+D(x)f(x)(xB(x,y) + 2f(x)y + D(x)[y,x]) = 0, \ x, y \in R. \end{aligned} \tag{9}$$

Left multiplication of (4) by x leads to

$$xD(y)f(x)^2 + xD(x)B(x,y)f(x) + xD(x)f(x)B(x,y) = 0, \ x, y \in R. \tag{10}$$

From (9) and (10), we arrive at

$$\begin{aligned} &D(x)yf(x)^2 + f(x)B(x,y)f(x) + 2D(x)f(x)yf(x) + D(x)^2[y,x]f(x) \\ &+(f(x)^2 + D(x)g(x))B(x,y) + 2D(x)f(x)^2y + D(x)f(x)D(x)[y,x] \\ &= 0, \ x, y \in R. \end{aligned} \tag{11}$$

By (1) and (11), it is obvious that

$$\begin{aligned} &D(x)yf(x)^2 + f(x)B(x,y)f(x) + 2D(x)f(x)yf(x) + D(x)^2[y,x]f(x) \\ &+(f(x)^2 + D(x)g(x))B(x,y) + D(x)f(x)D(x)[y,x] = 0, \ x, y \in R. \end{aligned} \tag{12}$$

Replacing yx for y in (12), it follows from that

$$\begin{aligned} &D(x)yxf(x)^2 + f(x)(B(x,y)x + 2yf(x) + [y,x]D(x))f(x) \\ &+2D(x)f(x)yxf(x) + D(x)^2[y,x]xf(x) \\ &+(f(x)^2 + D(x)g(x))(B(x,y)x + 2yf(x) + [y,x]D(x)) \\ &+D(x)f(x)D(x)[y,x]x = 0, \ x, y \in R. \end{aligned} \tag{13}$$

Right multiplication of (12) by x leads to

$$\begin{aligned} &D(x)yf(x)^2x + f(x)B(x,y)f(x)x + 2D(x)f(x)yf(x)x \\ &+D(x)^2[y,x]f(x)x + (f(x)^2 + D(x)g(x))B(x,y)x + D(x)f(x)D(x)[y,x]x \\ &= 0, \ x, y \in R. \end{aligned} \tag{14}$$

Combining (13) with (14), we see that

$$\begin{aligned} &-D(x)y(f(x)g(x) + g(x)f(x)) - f(x)B(x,y)g(x) + 2f(x)yf(x)^2 \\ &+f(x)[y,x]D(x)f(x) - 2D(x)f(x)yg(x) - D(x)^2[y,x]g(x) \\ &+2(f(x)^2 + D(x)g(x))yf(x) + (f(x)^2 + D(x)g(x))[y,x]D(x) = 0, \ x, y \in R. \end{aligned} \tag{15}$$

Left multiplication of (15) by $D(x)$ leads to

$$\begin{aligned} (16)\quad & -D(x)^2y(f(x)g(x)+g(x)f(x)) - D(x)f(x)B(x,y)g(x) + 2D(x)f(x)yf(x)^2 \\ & +D(x)f(x)[y,x]D(x)f(x) - 2D(x)^2f(x)yg(x) - D(x)^3[y,x]g(x) \\ & +2(D(x)f(x)^2 + D(x)^2g(x))yf(x) + (D(x)f(x)^2 + D(x)^2g(x))[y,x]D(x) \\ & = 0, \ x,y \in R. \end{aligned}$$

Let $y = x$ in (16). Then we get

$$\begin{aligned} (17)\quad & -D(x)^2x(f(x)g(x)+g(x)f(x)) - 2D(x)f(x)^2g(x) + 2D(x)f(x)xf(x)^2 \\ & -2D(x)^2f(x)xg(x) + 2(D(x)f(x)^2 + D(x)^2g(x))xf(x) = 0, \ x,y \in R. \end{aligned}$$

Left multiplication of (6) by $D(x)$ leads to

$$(18)\quad D(x)f(x)^3 + D(x)^2g(x)f(x) + D(x)^2f(x)g(x) = 0, \ x \in R.$$

Comparing (1) and (18), we get

$$(19)\quad D(x)^2g(x)f(x) + D(x)^2f(x)g(x) = 0, \ x \in R.$$

From (1), (7) and (19), it follows from (17) that

$$\begin{aligned} (20)\quad & -(f(x)D(x) + D(x)f(x))f(x)g(x) - (f(x)D(x) + D(x)f(x))g(x)f(x) \\ & -2D(x)f(x)^2g(x) + 2(f(x)^2 + D(x)g(x))f(x)^2 + 2D(x)^2f(x)h(x) \\ & -2D(x)^2g(x)^2 \\ & = -f(x)D(x)f(x)g(x) - D(x)f(x)^2g(x) - f(x)D(x)g(x)f(x) \\ & -D(x)f(x)g(x)f(x) - 2D(x)f(x)^2g(x) + 2f(x)^4 + 2D(x)g(x)f(x)^2 \\ & +2D(x)^2f(x)h(x) - 2D(x)^2g(x)^2 \\ & = -f(x)D(x)f(x)g(x) - f(x)D(x)g(x)f(x) \\ & -D(x)f(x)g(x)f(x) + 2f(x)^4 + 2D(x)g(x)f(x)^2 \\ & +2D(x)^2f(x)h(x) - 2D(x)^2g(x)^2 = 0, \quad x \in R. \end{aligned}$$

Combining (1), (6) with (20), we obtain

$$(21)\quad 4f(x)^4 + 3D(x)g(x)f(x)^2 + 2D(x)^2f(x)h(x) - 2D(x)^2g(x)^2 = 0, \ x \in R.$$

Right multiplication of (7) by $f(x)$ leads to

$$(22)\quad 7f(x)^4 + 4D(x)g(x)f(x)^2 = 0, \quad x \in R.$$

Comparing (21) and (22), we have

$$(23)\quad 5f(x)^4 - 8D(x)^2f(x)h(x) + 8D(x)^2g(x)^2 = 0, \quad x \in R.$$

From (5), it follows that

$$(24)\qquad \begin{aligned} 0 &= [3f(x)^3 - 4D(x)f(x)g(x), x] \\ &= 3g(x)f(x)^2 + 3f(x)g(x)f(x) - f(x)^2g(x) \\ &\quad -4D(x)g(x)^2 - 4D(x)f(x)h(x),\ x \in R. \end{aligned}$$

Left multiplication of (24) by $D(x)$ leads to

$$(25)\qquad \begin{aligned} &3D(x)g(x)f(x)^2 + 3D(x)f(x)g(x)f(x) - D(x)f(x)^2g(x) \\ &-4D(x)^2g(x)^2 - 4D(x)^2f(x)h(x) = 0,\ x \in R. \end{aligned}$$

Combining (1) with (25), we get

$$(26)\qquad \begin{aligned} &3D(x)g(x)f(x)^2 + 3D(x)f(x)g(x)f(x) - 4D(x)^2g(x)^2 \\ &-4D(x)^2f(x)h(x) = 0,\ x \in R. \end{aligned}$$

Right multiplication of (5) by $f(x)$ leads to

$$(27)\qquad 3f(x)^4 - 4D(x)f(x)g(x)f(x) = 0,\ x \in R.$$

Comparing (22), (26) and (27), we have

$$(28)\qquad 12f(x)^4 + 16D(x)^2g(x)^2 + 16D(x)^2f(x)h(x) = 0,\ x \in R.$$

Thus since R is 3!-torsion-free, (28) gives

$$(29)\qquad 3f(x)^4 + 4D(x)^2g(x)^2 + 4D(x)^2f(x)h(x) = 0,\ x \in R.$$

Comparing (23) and (29),we obtain

$$(30)\qquad 11f(x)^4 + 16D(x)^2g(x)^2 = 0,\ x \in R.$$

Left multiplication of (8) by $D(x)$ leads to

$$(31)\qquad 3D(x)^2g(x)f(x) + 7D(x)^2f(x)g(x) = 0,\ x \in R.$$

Comparing (19) and (31), we have

$$(32)\qquad 4D(x)^2f(x)g(x) = 0,\ x \in R.$$

Thus since R is 3!-torsion-free, (32) gives

$$(33)\qquad D(x)^2g(x)f(x) = 0,\ x \in R.$$

Combining (31) with (33), we get

$$(34)\qquad D(x)^2f(x)g(x) = 0,\ x \in R.$$

On the other hand, we have the basic relations

$$(35)\quad D(x^3) = D(x)x^2 + xD(x)x + x^2D(x),$$

$$(36)\quad [D(x), x^2] = f(x)x + xf(x),$$

$$[D(x), x^3] = f(x)x^2 + xf(x)x + x^2f(x),$$

$$(37)\quad B(x, x^3) = [D(x^3), x] + [D(x), x^3] = [D(x)x^2 + xD(x)x + x^2D(x), x]$$
$$+[D(x), x^3] = 2f(x)x^2 + 2xf(x)x + 2x^2f(x),$$

$$(38)\quad [D(x)f(x), x] = f(x)^2 + D(x)g(x),$$

$$(39)\quad [D(x)f(x), x^2] = D(x)[f(x), x^2] + [D(x), x^2]f(x)$$
$$= D(x)g(x)x + D(x)xg(x) + f(x)xf(x) + xf(x)^2,$$

$$(40)\quad [D(x)^2, x] = f(x)D(x) + D(x)f(x),\ x \in R.$$

Let $y = x^3$ in (4). Then using (35),(36), (37) and (38), we get

$$(41)\ D(x^3)f(x)^2 + D(x)B(x, x^3)f(x) + D(x)f(x)B(x, x^3)$$
$$= (D(x)x^2 + xD(x)x + x^2D(x))f(x)^2 + D(x)B(x, x^3)f(x) + D(x)f(x)B(x, x^3)$$
$$= (D(x)x^2 + xD(x)x + x^2D(x))f(x)^2 + 2D(x)(f(x)x^2 + xf(x)x + x^2f(x))f(x)$$
$$+2D(x)f(x)(f(x)x^2 + xf(x)x + x^2f(x))$$
$$= 5f(x)xf(x)^2 + 6xf(x)^3 + 2D(x)g(x)xf(x) + 2D(x)xg(x)f(x)$$
$$+2D(x)xf(x)xf(x) = 0,\ x \in R.$$

Left multiplication of (41) by $D(x)$ leads to

$$(42)\quad 5D(x)f(x)xf(x)^2 + 6D(x)xf(x)^3 + 2D(x)^2g(x)xf(x)$$
$$+2D(x)^2xg(x)f(x) + 2D(x)^2xf(x)xf(x) = 0,\ x \in R.$$

Comparing (38), (40) and (42), we have

$$(43)\quad 5(f(x)^2 + D(x)g(x))f(x)^2 + 6f(x)^4 - 2D(x)^2g(x)^2$$
$$+2(f(x)D(x) + D(x)f(x))g(x)f(x) + 2D(x)^2xf(x)xf(x)$$
$$= 11f(x)^4 + 5D(x)g(x)f(x)^2 - 2D(x)^2g(x)^2$$
$$+2f(x)D(x)g(x)f(x)+2D(x)f(x)g(x)f(x)+2D(x)^2xf(x)xf(x) = 0,\ x \in R.$$

Combining (5), (7) with (43), we get

$$(44)\quad f(x)^4 - 8D(x)^2g(x)^2 + 8D(x)^2xf(x)xf(x) = 0,\ x \in R.$$

Comparing (1) and (34), we obtain

$$D(x)^2f(x)xf(x) = 0,\ x \in R. \tag{45}$$

From (40), (44) and (45), we get

$$f(x)^4 - 8D(x)^2g(x)^2 + 8(f(x)D(x) + D(x)f(x))f(x)xf(x) = 0,\ x \in R. \tag{46}$$

Combining (1) with (46),

$$f(x)^4 - 8D(x)^2g(x)^2 + 8f(x)D(x)f(x)xf(x) = 0,\ x \in R. \tag{47}$$

From (1) and (47),

$$f(x)^4 - 8D(x)^2g(x)^2 - 8f(x)D(x)f(x)g(x) = 0,\ x \in R. \tag{48}$$

Comparing (5) and (48), we obtain

$$5f(x)^4 + 8D(x)^2g(x)^2 = 0,\ x \in R. \tag{49}$$

From (30) and (49), we can conclude that $f(x)^4 = 0,\ x \in R$. □

Theorem 3.2. *Let R be a 3!-torsion free noncommutative semiprime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$[D(x), x]^2D(x) = 0$$

for all $x \in R$. In this case we have $[D(x), x]^4 = 0$ for all $x \in R$.

Proof. By Theorem 2.2, we can see that D is a derivation on R. For simplicity, we shall denote the maps $B : R \times R \longrightarrow R$, $f, g : R \longrightarrow R$ by $B(x, y) \equiv [D(x), y] + [D(y), x]$, $f(x) \equiv [D(x), x]$, $g(x) \equiv [f(x), x]$, $h(x) \equiv [g(x), x]$ for all $x, y \in R$ respectively. Then we have the basic properties:

$$B(x, y) = B(y, x),\ B(x, yz) = B(x, y)z + yB(x, z) + D(y)[z, x] + [y, x]D(z),$$
$$B(x, x) = 2f(x),\ B(xy, z) = B(y, z)x + zB(y, x) + D(z)[x, y] + [z, y]D(x),$$
$$B(x, x^2) = 2(f(x)x + xf(x)),\ x, y, z \in R.$$

After this, we use the above relations without specific reference. By assumption,

$$f(x)^2D(x) = 0,\ x \in R \tag{50}$$

Replacing $x + ty$ for x in (50), we have

$$\begin{aligned} &f(x + ty)^2D(x + ty) \\ &\equiv f(x)^2D(x) + t\{f(x)^2D(y) + f(x)B(x, y)D(x) \\ &+B(x, y)f(x)D(x)\} + t^2H_1(x, y) + t^3H_2(x, y) + t^4H_3(x, y) \end{aligned} \tag{51}$$

$$+t^5 f(y)^2 D(y) = 0, \; x, y \in R, \; t \in S_4$$

where $H_i, 1 \le i \le 3$, denotes the term satisfying the identity (51).
From (50) and (51), we obtain

$$\text{(52)} \quad \begin{aligned} &t\{f(x)^2 D(y) + f(x)B(x,y)D(x) + B(x,y)f(x)D(x)\} \\ &+t^2 H_1(x,y) + t^3 H_2(x,y) + t^4 H_3(x,y) = 0, \; x, y \in R, \; t \in S_4 \end{aligned}$$

Since R is 3!-torsion free by assumption, by Lemma 2.1 the relation (52) yields

$$\text{(53)} \quad f(x)^2 D(y) + f(x)B(x,y)D(x) + B(x,y)f(x)D(x) = 0, \; x, y \in R.$$

Let $y = x^2$ in (53). Then using (50), we obtain

$$\text{(54)} \quad \begin{aligned} &f(x)^2(D(x)x + xD(x)) + 2f(x)(f(x)x + xf(x))D(x) \\ &\quad +2(f(x)x + xf(x))f(x)D(x) \\ &= f(x)^2 D(x)x + 3f(x)^2 xD(x) + 4f(x)xf(x)D(x) \\ &\quad +2xf(x)^2 D(x) \\ &= -3f(x)^3 + 4g(x)f(x)D(x) = 0, \; x \in R. \end{aligned}$$

From (50), it follows that

$$\text{(55)} \quad 0 = [f(x)^2 D(x), x] = g(x)f(x)D(x) + f(x)g(x)D(x) + f(x)^3, \; x \in R.$$

From (54) and (55), we get

$$\text{(56)} \quad 7f(x)^3 + 4f(x)g(x)D(x) = 0, \; x \in R.$$

Combining (54) with (56),

$$4(3f(x)g(x)D(x) + 7g(x)f(x)D(x)) = 0, \; x \in R.$$

Since R is 3!-torsion-free, the above relation gives

$$\text{(57)} \quad 3f(x)g(x)D(x) + 7g(x)f(x)D(x) = 0, \; x \in R.$$

Writing yx for y in (53), we have

$$\text{(58)} \quad \begin{aligned} &f(x)^2 D(y)x + f(x)^2 yD(x) + f(x)(B(x,y)x + 2yf(x) + [y,x]D(x))D(x) \\ &+(B(x,y)x + 2yf(x) + [y,x]D(x))f(x)D(x) = 0, \; x, y \in R. \end{aligned}$$

Right multiplication of (53) by x leads to

$$\text{(59)} \quad f(x)^2 D(y)x + f(x)B(x,y)D(x)x + B(x,y)f(x)D(x)x = 0, \; x, y \in R.$$

From (58) and (59), we obtain

$$(60)\qquad f(x)^2yD(x) - f(x)B(x,y)f(x) + 2f(x)yf(x)D(x) + f(x)[y,x]D(x)^2 \\ -B(x,y)(f(x)^2 + g(x)D(x)) + 2yf(x)^2D(x) \\ +[y,x]D(x)f(x)D(x) = 0, \ x,y \in R.$$

By (50) and (60), it is obvious that

$$(61)\qquad f(x)^2yD(x) - f(x)B(x,y)f(x) + 2f(x)yf(x)D(x) + f(x)[y,x]D(x)^2 \\ -B(x,y)(f(x)^2 + g(x)D(x)) + [y,x]D(x)f(x)D(x) = 0, \ x,y \in R.$$

Replacing xy for y in (61), it follows that

$$(62)\ f(x)^2xyD(x) - f(x)(xB(x,y) + 2f(x)y + D(x)[y,x])f(x) + 2f(x)xyf(x)D(x) \\ +f(x)x[y,x]D(x)^2 - (xB(x,y) + 2f(x)y + D(x)[y,x])(f(x)^2 + g(x)D(x)) \\ +x[y,x]D(x)f(x)D(x) = 0, \ x,y \in R.$$

Left multiplication of (61) by x leads to

$$(63)\qquad xf(x)^2yD(x) - xf(x)B(x,y)f(x) + 2xf(x)yf(x)D(x) + xf(x)[y,x]D(x)^2 \\ -xB(x,y)(f(x)^2 + g(x)D(x)) + x[y,x]D(x)f(x)D(x) = 0, \ x,y \in R.$$

Combining (62) with (63), we see that

$$(64)\qquad (f(x)g(x) + g(x)f(x))yD(x) - g(x)B(x,y)f(x) - 2f(x)^2yf(x) \\ -f(x)D(x)[y,x]f(x) + 2g(x)yf(x)D(x) + g(x)[y,x]D(x)^2 \\ -2f(x)y(f(x)^2 + g(x)D(x)) - D(x)[y,x](f(x)^2 + g(x)D(x)) = 0, \ x,y \in R.$$

Right multiplication of (64) by $D(x)$ leads to

$$(65)\qquad (f(x)g(x) + g(x)f(x))yD(x)^2 - g(x)B(x,y)f(x)D(x) - 2f(x)^2yf(x)D(x) \\ -f(x)D(x)[y,x]f(x)D(x) + 2g(x)yf(x)D(x)^2 + g(x)[y,x]D(x)^3 \\ -2f(x)y(f(x)^2D(x) + g(x)D(x)^2) - D(x)[y,x](f(x)^2D(x) + g(x)D(x)^2) \\ = 0, \ x,y \in R.$$

Let $y = x$ in (65). Then we get

$$(66)\qquad (f(x)g(x) + g(x)f(x))xD(x)^2 - 2g(x)f(x)^2D(x) - 2f(x)^2xf(x)D(x) \\ +2g(x)xf(x)D(x)^2 - 2f(x)x(f(x)^2D(x) + g(x)D(x)^2) = 0, \ x,y \in R.$$

Right multiplication of (55) by $D(x)$ leads to

$$(67)\qquad f(x)^3D(x) + f(x)g(x)D(x)^2 + g(x)f(x)D(x)^2 = 0, \ x \in R.$$

Comparing (50) and (67), we obtain

$$f(x)g(x)D(x)^2+g(x)f(x)D(x)^2=0,\ x\in R. \tag{68}$$

Right multiplication of (57) by $D(x)$ leads to

$$3f(x)g(x)D(x)^2+7g(x)f(x)D(x)^2=0,\ x\in R. \tag{69}$$

Combining (68) with (69), we have

$$4g(x)f(x)D(x)^2=0,\ x\in R. \tag{70}$$

And since R is 3!-torsion-free, (70) gives

$$g(x)f(x)D(x)^2=0,\ x\in R. \tag{71}$$

From (68) and (71),

$$f(x)g(x)D(x)^2=0,\ x\in R. \tag{72}$$

From (50),(71), (72), it follows from (66) that

$$\begin{aligned}
&-(f(x)g(x)+g(x)f(x))(f(x)D(x)+D(x)f(x))\\
&+2f(x)^2(g(x)D(x)+f(x)^2)+2h(x)f(x)D(x)^2\\
&+2f(x)(2g(x)f(x)D(x)+f(x)g(x)D(x)+f(x)^3\\
&+h(x)D(x)^2+g(x)D(x)f(x))\\
&=-f(x)g(x)f(x)D(x)-f(x)g(x)D(x)f(x)-g(x)f(x)^2D(x)\\
&-g(x)f(x)D(x)f(x)+2f(x)^2g(x)D(x)+2f(x)^4\\
&+2h(x)f(x)D(x)^2+4f(x)g(x)f(x)D(x)\\
&+2f(x)^2g(x)D(x)+2f(x)^4+2f(x)h(x)D(x)^2+2f(x)g(x)D(x)f(x)\\
&=3f(x)g(x)f(x)D(x)+f(x)g(x)D(x)f(x)\\
&-g(x)f(x)D(x)f(x)+4f(x)^2g(x)D(x)\\
&+4f(x)^4+2h(x)f(x)D(x)^2+2f(x)h(x)D(x)^2=0,\ x\in R.
\end{aligned} \tag{73}$$

Combining (54), (56) with (73), we have

$$-13f(x)^4+8h(x)f(x)D(x)^2+8f(x)h(x)D(x)^2=0,\ x\in R. \tag{74}$$

From (57), it follows that

$$\begin{aligned}
0 &= [3f(x)g(x)D(x)+7g(x)f(x)D(x),x]\\
&= 10g(x)^2D(x)+3f(x)h(x)D(x)+3f(x)g(x)f(x)\\
&\quad +7h(x)f(x)D(x)+7g(x)f(x)^2,\ x\in R.
\end{aligned} \tag{75}$$

Right multiplication of (75) by $D(x)$ leads to

$$(76)\qquad 10g(x)^2D(x)^2 + 3f(x)h(x)D(x)^2 + 3f(x)g(x)f(x)D(x) \\ +7h(x)f(x)D(x)^2 + 7g(x)f(x)^2D(x) = 0,\ x \in R.$$

Comparing (50) and (76), we get

$$(77)\qquad 10g(x)^2D(x)^2 + 3f(x)h(x)D(x)^2 + 3f(x)g(x)f(x)D(x) \\ +7h(x)f(x)D(x)^2 = 0,\ x \in R.$$

Combining (74) with (77), we have

$$(78)\qquad 57f(x)^4 + 80g(x)^2D(x)^2 + 32h(x)f(x)D(x)^2 = 0,\ x \in R.$$

From (71), it follows that

$$(79)\qquad \begin{aligned} 0 &= [g(x)f(x)D(x)^2, x] \\ &= h(x)f(x)D(x)^2 + g(x)^2D(x)^2 + g(x)f(x)^2D(x), \\ &\quad +g(x)f(x)D(x)f(x)x \in R. \end{aligned}$$

And from (50) and (79),

$$(80)\qquad h(x)f(x)D(x)^2 + g(x)^2D(x)^2 + g(x)f(x)D(x)f(x) = 0,\ x \in R.$$

Thus, combining (54) and (80), we get

$$(81)\qquad 3f(x)^4 + 4g(x)^2D(x)^2 + 4h(x)f(x)D(x)^2 = 0,\ x \in R.$$

Combining (78) with (81), we have

$$(82)\qquad 33f(x)^4 + 48g(x)^2D(x)^2 = 0,\ x \in R.$$

Thus since R is 3!-torsion-free, (82) gives

$$(83)\qquad 11f(x)^4 + 16g(x)^2D(x)^2 = 0,\ x \in R.$$

On the other hand, we have the basic relations

$$(84)\qquad [D(x)^2, x^2] = [D(x)^2, x]x + x[D(x)^2, x] = D(x)f(x)x + f(x)D(x)x \\ +xD(x)f(x) + xf(x)D(x),$$

$$(85)\qquad [f(x)D(x), x] = g(x)D(x) + f(x)^2,$$

$$(86)\qquad [f(x)D(x), x^2] = [f(x), x^2]D(x) + f(x)[D(x), x^2] \\ = (g(x)xD(x) + xg(x)D(x) + f(x)^2x + f(x)xf(x), \\ x \in R.$$

Let $y = x^3$ in (53). Then using (35), (50),(36), (37) and (85), we get

$$
\begin{aligned}
(87)\, & f(x)^2D(x^3) + f(x)B(x,x^3)D(x) + B(x,x^3)f(x)D(x) \\
&= f(x)^2(D(x)x^2 + xD(x)x + x^2D(x)) + f(x)B(x,x^3)D(x) + B(x,x^3)f(x)D(x) \\
&= f(x)^2(D(x)x^2 + xD(x)x + x^2D(x)) + 2f(x)(f(x)x^2 + xf(x)x + x^2f(x))D(x) \\
&\quad +2(f(x)x^2 + xf(x)x + x^2f(x))f(x)D(x) \\
&= f(x)^2xD(x)x + 3f(x)^2x^2D(x) + 2f(x)xf(x)xD(x) + 4f(x)x^2f(x)D(x) \\
&\quad +2xf(x)xf(x)D(x) = 0, \ \ x \in R.
\end{aligned}
$$

Right multiplication of (87) by $D(x)$ leads to

$$
\begin{aligned}
(88) \qquad & f(x)^2xD(x)xD(x) + 3f(x)^2x^2D(x)^2 + 2f(x)xf(x)xD(x)^2 \\
& +4f(x)x^2f(x)D(x)^2 + 2xf(x)xf(x)D(x)^2 = 0, \ \ x \in R.
\end{aligned}
$$

Comparing (40), (50), (84),(85) and (88), we have

$$
\begin{aligned}
(89)\ \ & f(x)^2xD(x)xD(x) + 3f(x)^2x^2D(x)^2 + 2f(x)xf(x)xD(x)^2 \\
& +4f(x)x^2f(x)D(x)^2 + 2xf(x)xf(x)D(x)^2 \\
&= -f(x)^3xD(x) - 3f(x)^2[D(x)^2, x^2] + 2f(x)xf(x)xD(x)^2 \\
&\quad +4[f(x), x^2]f(x)D(x)^2 + 2x[f(x), x]f(x)D(x)^2 \\
&= -4f(x)^3xD(x) - 3f(x)^2(D(x)f(x)x + f(x)D(x)x + xD(x)f(x) \\
&\quad +xf(x)D(x)) + 2f(x)xf(x)xD(x)^2 + 4g(x)xf(x)D(x)^2 + 6xg(x)f(x)D(x)^2 \\
&\quad +6xg(x)f(x)D(x)^2 \\
&= -4f(x)^3xD(x) - 3f(x)^2xD(x)f(x) - 3f(x)^2xf(x)D(x) \\
&\quad +2f(x)xf(x)xD(x)^2 + 4g(x)xf(x)D(x)^2 \\
&\quad +6xg(x)f(x)D(x)^2 = 0, \ \ x \in R.
\end{aligned}
$$

Combining (50), (71) with (89), we get

$$(90)\quad \begin{aligned}&-4f(x)^3xD(x) - 3f(x)^2xD(x)f(x) - 3f(x)^2xf(x)D(x)\\ &+2f(x)xf(x)xD(x)^2 + 4g(x)xf(x)D(x)^2\\ &\quad +6xg(x)f(x)D(x)^2\\ &= 4f(x)^4 + 3f(x)^2[D(x)f(x),x] + 3f(x)^2[f(x)D(x),x] + 2f(x)xf(x)xD(x)^2\\ &\quad -4g(x)[f(x)D(x)^2,x]\\ &= 10f(x)^4 + 3f(x)^2g(x)D(x) + 2f(x)xf(x)xD(x)^2 - 4g(x)^2D(x)^2\\ &\quad -4g(x)f(x)D(x)f(x) = 0,\ x \in R.\end{aligned}$$

Comparing (50) and (71), we obtain

$$(91)\qquad f(x)xf(x)D(x)^2 = 0,\ x \in R.$$

Comparing (50), (54), (56), (90), and (91), we obtain

$$(92)\quad \begin{aligned}&40f(x)^4 + 3f(x)(4f(x)g(x)D(x)) - 8f(x)xf(x)[D(x)^2,x]\\ &-16g(x)^2D(x)^2 - 4(4g(x)f(x)D(x))f(x)\\ &= 40f(x)^4 + 3f(x)(-7f(x)^3) - 8f(x)xf(x)(f(x)D(x) + D(x)f(x))\\ &\quad -16g(x)^2D(x)^2 - 4(3f(x)^3)f(x)\\ &= 7f(x)^4 - 8f(x)xf(x)D(x)f(x) - 16g(x)^2D(x)^2 = 0,\ x \in R.\end{aligned}$$

From (50) and (92), we get

$$(93)\qquad 7f(x)^4 - 8g(x)f(x)D(x)f(x) - 16g(x)^2D(x)^2 = 0,\ x \in R.$$

Combining (54) with (93),

$$(94)\qquad \begin{aligned}&7f(x)^4 - 2(3f(x)^3)f(x) - 16g(x)^2D(x)^2\\ &= f(x)^4 - 16g(x)^2D(x)^2 = 0,\ x \in R.\end{aligned}$$

From (83) and (94),

$$(95)\qquad 12f(x)^4 = 0,\ x \in R.$$

Since R is 3!-torsion-free, (95) gives $f(x)^4 = 0,\ x \in R$. □

Theorem 3.3. *Let R be a 2!-torsion free noncommutative semiprime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$[D(x),x]D(x) = 0$$

for all $x \in R$. In this case we have $[D(x),x]^2 = 0$ for all $x \in R$.

Proof. By Theorem 2.2, we can see that D is a derivation on R. For simplicity, we shall denote the maps $B : R \times R \longrightarrow R$, $f, g : R \longrightarrow R$ by $B(x,y) \equiv [D(x),y] + [D(y),x], f(x) \equiv [D(x),x]$, $g(x) \equiv [f(x),x]$, $h(x) \equiv [g(x),x]$ for all $x, y \in R$ respectively. Then we have the basic properties:

$$B(x,y) = B(y,x),\ B(x,yz) = B(x,y)z + yB(x,z) + D(y)[z,x] + [y,x]D(z),$$
$$B(x,x) = 2f(x),\ B(xy,z) = B(y,z)x + zB(y,x) + D(z)[x,y] + [z,y]D(x),$$
$$B(x,x^2) = 2(f(x)x + xf(x)),\ x,y,z \in R.$$

After this, we use the above relations without specific reference. By assumption,

(96) $$f(x)D(x) = 0,\ x \in R$$

Replacing $x + ty$ for x in (96), we have

(97) $$\begin{aligned} &f(x+ty)D(x+ty) \\ &\equiv f(x)D(x) + t\{f(x)D(y) + B(x,y)D(x)\} \\ &\quad +t^2 J(x,y) + t^3 f(y)D(y) = 0,\ x,y \in R,\ t \in S_2 \end{aligned}$$

where J denotes the term satisfying the identity (97).
From (96) and (97), we obtain

(98) $$t\{f(x)D(y) + B(x,y)D(x)\} + t^2 J(x,y) = 0,\ x,y \in R,\ t \in S_2$$

Since R is 2!-torsion free by assumption, by Lemma 2.1 the relation (98) yields

(99) $$f(x)D(y) + B(x,y)D(x) = 0,\ x,y \in R.$$

Let $y = x^2$ in (99). Then using (96), we obtain

(100) $$\begin{aligned} &f(x)(D(x)x + xD(x)) + 2(f(x)x + xf(x))D(x) \\ &= f(x)D(x)x + 3f(x)xD(x) + 2xf(x)D(x) \\ &= -3f(x)^2 = 0,\ x \in R. \end{aligned}$$

From (96), it follows that

(101) $$\begin{aligned} 0 &= [f(x)D(x),x] \\ &= g(x)D(x) + f(x)^2,\ x \in R. \end{aligned}$$

Writing yx for y in (99), we have

(102) $$\begin{aligned} &f(x)D(y)x + f(x)yD(x) + (B(x,y)x + 2yf(x) + [y,x]D(x))D(x) \\ &= 0,\ x,y \in R. \end{aligned}$$

Right multiplication of (99) by x leads to

(103) $$f(x)D(y)x + B(x,y)D(x)x = 0,\ x, y \in R.$$

From (102) and (103), we obtain

(104) $$f(x)yD(x) - B(x,y)f(x) + 2yf(x)D(x) + [y,x]D(x)^2 = 0,\ x, y \in R.$$

By (96) and (104), it is obvious that

(105) $$f(x)yD(x) - B(x,y)f(x) + [y,x]D(x)^2 = 0,\ x, y \in R.$$

Replacing xy for y in (105), it follows that

(106) $$f(x)xyD(x) - (xB(x,y) + 2f(x)y + D(x)[y,x])f(x) + x[y,x]D(x)^2 = 0,\ x, y \in R.$$

Left multiplication of (105) by x leads to

(107) $$xf(x)yD(x) - xB(x,y)f(x) + x[y,x]D(x)^2 = 0,\ x, y \in R.$$

Combining (106) with (107), we see that

(108) $$g(x)yD(x) - 2f(x)yf(x) - D(x)[y,x]f(x) = 0,\ x, y \in R.$$

Let $y = x$ in (108). Then we get

(109) $$g(x)xD(x) - 2f(x)xf(x) = 0,\ x \in R.$$

Comparing (96) and (109), we obtain

(110) $$-g(x)f(x) + 2f(x)g(x) = 0,\ x \in R.$$

Right multiplication of (110) by $D(x)$ leads to

(111) $$-g(x)f(x)D(x) + 2f(x)g(x)D(x) = 0,\ x \in R.$$

Combining (96) with (110), we have

(112) $$2f(x)g(x)D(x) = 0,\ x \in R.$$

And since R is 2!-torsion-free, (112) gives

(113) $$f(x)g(x)D(x) = 0, x \in R.$$

Left multiplication of (101) by $f(x)$ leads to

(114) $$f(x)g(x)D(x) + f(x)^3 = 0,\ x, y \in R.$$

Combining (113) with (114), we see that $f(x)^3 = 0,\ x \in R$. □

Theorem 3.4. *Let R be a 3!-torsion free noncommutative semiprime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$[D(x), x]D(x) = 0$$

for all $x \in R$. In this case we have $[D(x), x]^2 = 0$ for all $x \in R$.

Proof. By Theorem 2.2, we can see that D is a derivation on R. For simplicity, we shall denote the maps $B : R \times R \longrightarrow R$, $f, g : R \longrightarrow R$ by $B(x, y) \equiv [D(x), y] + [D(y), x], f(x) \equiv [D(x), x],\ g(x) \equiv [f(x), x],\ h(x) \equiv [g(x), x]$ for all $x, y \in R$ respectively. Then we have the basic properties:

$$B(x,y) = B(y,x),\ B(x,yz) = B(x,y)z + yB(x,z) + D(y)[z,x] + [y,x]D(z),$$
$$B(x,x) = 2f(x),\ B(xy,z) = B(y,z)x + zB(y,x) + D(z)[x,y] + [z,y]D(x),$$
$$B(x,x^2) = 2(f(x)x + xf(x)),\ x, y, z \in R.$$

After this, we use the above relations without specific reference. By assumption,

$$f(x)D(x) = 0,\ x \in R \tag{115}$$

Replacing $x + ty$ for x in (115), we have

$$\begin{aligned} &f(x+ty)D(x+ty) \\ &\equiv f(x)D(x) + t\{f(x)D(y) + B(x,y)D(x)\} + t^2L(x,y) + t^3f(y)D(y) \\ &= 0,\ x, y \in R,\ t \in S_3 \end{aligned} \tag{116}$$

where L denotes the term satisfying the identity (116).
From (115) and (116), we obtain

$$t\{f(x)D(y) + B(x,y)D(x)\} + t^2J(x,y) = 0,\ x, y \in R,\ t \in S_3 \tag{117}$$

Since R is 3!-torsion free by assumption, by Lemma 2.1 the relation (117) yields

$$f(x)D(y) + B(x,y)D(x) = 0,\ x, y \in R. \tag{118}$$

Let $y = x^2$ in (118). Then using (115), we obtain

$$\begin{aligned} &f(x)(D(x)x + xD(x)) + 2(f(x)x + xf(x))D(x) \\ &= f(x)D(x)x + 3f(x)xD(x) + 2xf(x)D(x) = -3f(x)^2 = 0,\ x \in R. \end{aligned} \tag{119}$$

and so, since R is 3!-torsion-free, (119) gives $f(x)^2 = 0,\ x \in R.$ □

4. Applications to Banach Algebra Theory

The proof of the following theorem as our main theorem is the same argument as in the proof of J. Vukman's theorem [12].

Theorem 4.1. *Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that*

$$D(x)[D(x), x]^2 \in rad(A)$$

for all $x \in A$. Then we have $D(A) \subseteq rad(A)$.

Proof. According to the result of B.E. Johnson and A.M. Sinclair [5] every linear derivation on a semisimple Banach algebra is continuous. A.M. Sinclair [8] has proved that any continuous linear derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a continuous linear Jordan derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P$, $\hat{x} = x + P$. Then A/P is a prime Banach algebra, by Theorem 2.2 it is obvious that D_P is a continuous derivation on A/P. And also, from the given assumptions $D(x)f(x)^2 = 0$, $x \in A$, it follows that $(D_P(\hat{x}))[D_P(\hat{x}), \hat{x}]^2 = 0$, $\hat{x} \in A/P$, since all the assumptions of Theorem 2.4 are fulfilled. The factor algebra A/P is noncommutative, by Theorem 2.5 we have $[D_P(\hat{x}), \hat{x}]^4 = 0$, $\hat{x} \in A/P$. Then $r_P([D_P(\hat{x}), \hat{x}])^4 = r([D_P(\hat{x}), \hat{x}]^4) = 0$ for all $\hat{x} \in A/P$. Hence we obtain $r_P([D_P(\hat{x}), \hat{x}]) = 0$ for all $\hat{x} \in A/P$. Thus $[D_P(\hat{x}), \hat{x}] \in Q(A/P)$ for all $\hat{x} \in A/P$. On the other hand, since D is continuous, we see that D_P is also continuous. Thus by Theorem 2.3, one obtains $D_P(A/P) \subseteq \text{rad}(A/P)$. But since A/P is semisimple, $D_P(A/P) = \{0\}$. Hence we get $D(A) \subseteq P$ for all primitive ideals P of A. Thus $D(A) \subseteq \text{rad}(A)$. On the other hand, in case A/P is a commutative Banach algebra, one can conclude that $D_P = 0$ as well since A/P is semisimple and prime Banach algebra and since we know that there are no nonzero linear derivations on a commutative semisimple Banach algebras. Hence $D(x) \in P$ for all primitive ideals and all $x \in A$ i.e., we $D(A) \subseteq \text{rad}(A)$. Therefore in any case, we $D(A) \subseteq \text{rad}(A)$. □

Theorem 4.2. *Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that*

$$[D(x), x]^2 D(x) \in rad(A)$$

for all $x \in A$. Then we have $D(A) \subseteq rad(A)$.

Proof. The proof is similar as in the proof of Theorem 2.7. □

We will give another proof of J. Vukman's result in [11]. We will give a new proof to make use of the results of Theorem 3.3 and 3.4. Of course, it is also a special case of the above Theorem.

Corollary 4.3. *Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that*

$$[D(x), x]D(x) \in rad(A)$$

for all $x \in A$. Then we have $D(A) \subseteq rad(A)$.

Proof. According to the result of B.E. Johnson and A.M. Sinclair [5] every linear derivation on a semisimple Banach algebra is continuous. A.M. Sinclair [8] has proved that any continuous linear derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a continuous linear Jordan derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P$, $\hat{x} = x + P$. Then A/P is a prime Banach algebra, by Theorem 2.2 it is obvious that D_P is a continuous derivation on A/P. And also, from the given assumptions $f(x)D(x) = 0$, $x \in A$, it follows that $[D_P(\hat{x}), \hat{x}]D_P(\hat{x}) = 0$, $\hat{x} \in A/P$, since all the assumptions of Theorem 3.4 are fulfilled. The factor algebra A/P is noncommutative, by Theorem 3.4 we have $[D_P(\hat{x}), \hat{x}]^2 = 0$, $\hat{x} \in A/P$. Then $r_P([D_P(\hat{x}), \hat{x}])^2 = r([D_P(\hat{x}), \hat{x}]^2) = 0$ for all $\hat{x} \in A/P$. Hence we obtain $r_P([D_P(\hat{x}), \hat{x}]) = 0$ for all $\hat{x} \in A/P$. Thus $[D_P(\hat{x}), \hat{x}] \in Q(A/P)$ for all $\hat{x} \in A/P$. On the other hand, since D is continuous, we see that D_P is also continuous. Thus by Theorem 2.3, one obtains $D_P(A/P) \subseteq \text{rad}(A/P)$. But since A/P is semisimple, $D_P(A/P) = \{0\}$. Hence we get $D(A) \subseteq P$ for all primitive ideals P of A. Thus $D(A) \subseteq \text{rad}(A)$. On the other hand, in case A/P is a commutative Banach algebra, one can conclude that $D_P = 0$ as well since A/P is semisimple and prime Banach algebra and since we know that there are no nonzero linear derivations on a commutative semisimple Banach algebras. Hence $D(x) \in P$ for all primitive ideals and all $x \in A$ i.e., we $D(A) \subseteq \text{rad}(A)$. Therefore in any case, we $D(A) \subseteq \text{rad}(A)$. □

Remark 4.4. By the same arguments of Theorem 3.3 and 3.4, it is easily checked that $[D(x), x]^3 = 0$ or $[D(x), x]^2 = 0$ for all $x \in R$ if $D(x)[D(x), x] = 0$ for all $x \in R$. And using the results, we can obtain the analytic version by the same arguments as in Corollary 4.11.

Theorem 4.5. *Let A be a noncommutative semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that*

$$D(x)[D(x), x]^2 = 0$$

for all $x \in A$. Then we have $D = 0$.

Proof. According to the result of B.E. Johnson and A.M. Sinclair [5] every linear derivation on a semisimple Banach algebra is continuous. A.M. Sinclair [8] has proved that any continuous linear derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a linear Jordan derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P$, $\hat{x} = x + P$. Then A/P is a prime Banach algebra, by Theorem 2.2 it is obvious that D_P is a continuous derivation on A/P. And also, from the given assumptions $D(x)f(x)^2 = 0$, $x \in A$, it follows that $(D_P(\hat{x}))[D_P(\hat{x}), \hat{x}]^2 = 0$, $\hat{x} \in A/P$, since all the assumptions of Theorem 2.4 are fulfilled. In the case that the prime Banach algebra A/P is noncommutative, by Theorem 2.5 we have $[D_P(\hat{x}), \hat{x}]^4 = 0$, $\hat{x} \in A/P$. Then $r_P([D_P(\hat{x}), \hat{x}])^4 = r([D_P(\hat{x}), \hat{x}]^4) = 0$ for all $\hat{x} \in A/P$. Hence we obtain $r_P([D_P(\hat{x}), \hat{x}]) = 0$ for all $\hat{x} \in A/P$. Thus $[D_P(\hat{x}), \hat{x}] \in Q(A/P)$ for all $\hat{x} \in A/P$. On the other hand, since D is continuous, we see that D_P is also continuous. Thus by Theorem 2.3, one obtains $D_P(A/P) \subseteq \mathrm{rad}(A/P)$. But since A/P is semisimple, $D_P(A/P) = \{0\}$. Hence we get $D(A) \subseteq P$ for all primitive ideals P of A. Thus $D(A) \subseteq \mathrm{rad}(A) = \{0\}$ since A is semisimple. That is $D \equiv 0$. On the other hand, in case A/P is a commutative semisimple Banach algebra, one can conclude that $D_P = 0$ as well since A/P is a semisimple and prime Banach algebra and since we know that there are no nonzero linear derivations on a commutative semisimple Banach algebras. Hence $D(A) \subseteq P$ for all primitive ideals of A i.e., we $D(A) \subseteq \mathrm{rad}(A)$. By the assumption that A is semisimple, we get $D \equiv 0$. □

The following theorem generalizes Vukman's result [11].

Theorem 4.6. *Let A be a noncommutative semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that*

$$[D(x), x]^2 D(x) = 0$$

for all $x \in A$. Then we have $D = 0$.

Proof. The proof is similar as in the proof of Theorem 2.9. □

The following Theorem 2.8 is essentially due to J. Vukman [11].

Corollary 4.7. *Let A be a noncommutative semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that*

$$D(x)[D(x), x] = 0$$

for all $x \in A$. Then we have $D = 0$.

Corollary 4.8. *Let A be a noncommutative semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that*

$$[D(x), x]D(x) = 0$$

for all $x \in A$. Then we have $D = 0$.

As a special case of Theorem 2.9 we get the following result which characterizes commutative semisimple Banach algebras.

Corollary 4.9. *Let A be a semisimple Banach algebra. Suppose*

$$[x, y][[x, y], x]^2 = 0$$

for all $x, y \in A$. In this case, A is commutative.

As a special case of Theorem 2.10 we get the following result which characterizes commutative semisimple Banach algebras.

Corollary 4.10. *Let A be a semisimple Banach algebra. Suppose*

$$[[x, y], x]^2[x, y] = 0$$

for all $x, y \in A$. In this case, A is commutative.

As a special case of Theorem 2.7 we have the following result.

Corollary 4.11. *Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that*

$$D(x)[D(x), x] \in rad(A)$$

and for all $x \in A$. Then we have $D(A) \subseteq rad(A)$.

As a special case of Theorem 2.8 we have the following result.

Corollary 4.12. *Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that*

$$[D(x), x]D(x) \in rad(A)$$

and for all $x \in A$. Then we have $D(A) \subseteq rad(A)$.

REFERENCES

1. F.F. Bonsall & J. Duncan: *Complete Normed Algebras*, Berlin-Heidelberg-New York, 1973.
2. M. Brešar: Derivations of noncommutative Banach algebras II. *Arch. Math.* **63** (1994), 56-59.
3. M. Brešar: Jordan derivations on semiprime rings. *Proc. Amer. Math. Soc.* **104** (1988), no. 4, 1003-1006.
4. L.O. Chung & J. Luh: Semiprime rings with nilpotent derivatives. *Canad. Math. Bull.* **24** (1981), no. 4, 415-421.
5. B.E. Johnson & A.M. Sinclair: Continuity of derivations and a problem of Kaplansky. *Amer. J. Math.* **90** (1968), 1067-1073.
6. B.D. Kim: On the derivations of semiprime rings and noncommutative Banach algebras. *Acta Mathematica Sinica* **16** (2000), no. 1, 21-28.
7. B.D. Kim: Derivations of semiprime rings and noncommutative Banach algebras. *Commun. Korean Math. Soc.* **17** (2002), no. 4, 607-618.
8. A.M. Sinclair: Jordan homohorphisms and derivations on semisimple Banach algebras. *Proc. Amer. Math. Soc.* **24** (1970), 209-214.
9. I.M. Singer & J. Wermer: Derivations on commutative normed algebras. *Math. Ann.* **129** (1955), 260-264.
10. M.P. Thomas: The image of a derivation is contained in the radical. *Annals of Math.* **128** (1988), 435-460.
11. J. Vukman: A result concerning derivations in noncommutative Banach algebras. *Glasnik Mathematicki* **26** (1991), no. 46, 83-88.
12. J. Vukman: On derivations in prime rings and Banach algebras. *Proc. Amer. Math. Soc.* **116** (1992), no. 4, 877-884.

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