East Asian Math. J. **24** (2008), No. 1, pp. 125–137

APPROXIMATIONS OF THE ITERATIVE SEQUENCES FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

Shih-sen Chang, Yeol Je Cho and Haiyun Zhou

Abstract. In this paper, we first introduce some iterative sequences of Halpern type for asymptotically nonexpansive mappings and nonexpansive mappings in Banach spaces and then we discuss strong convergence for the iterative processes. The results presented in this paper extend, supplement and improve the correspoding main results of Reich [11], Shimizu and Takahashi [13], Shioji and Takahashi [15], [16] and Wittmann [18].

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E, D is a nonempty subset of E and $J : E \to 2^{E^*}$ is the *normalized duality mapping* defined by

$$J(x) = \{f \in E^*, \langle x, f\rangle = \|x\|\|f\|, \quad \|f\| = \|x\|\}, \quad \forall x \in E. \tag{1}$$

Definition 1. Let $T : D \to D$ be a mapping.

(1) The mapping T is said to be *asymptotically nonexpansive* ([7]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\| \tag{2}$$

for all $x, y \in D$ and $n \in N$.

(2) The mapping T is said to be *nonexpansive* if the sequence $\{k_n\}$ appeared in (2) is a constant sequence $\{1\}$, i.e.,

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in D.$$

Definition 2. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *uniformly Gâteaux differentiable* if, for each $y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exits uniformly for all $x \in U$.

It is well-known that the following proposition is true:

Received October 7, 2007.

2000 *Mathematics Subject Classification.* 47H09, 49M05.

Key words and phrases. Asymptotically nonexpansive mapping, nonexpansive mapping, the sequences of Halpern type, fixed point.

Proposition 3. ([6]) *Let E be a Banach space with a uniformly Gâteaux differentiable norm. Then the normalized duality mapping $J : E \to 2^{E^*}$ defined by* (1) *is single-valued and uniformly continuous on each bounded subset of E from the norm topology of E to the weak* topology of E^*.*

Let D be a nonempty closed convex subset of X, $T : D \to D$ be a nonexpansive mapping and $F(T)$ denote the set of fixed points of T. For a fixed $u \in D$ and each $t \in (0, 1)$, we can define a contractive mapping $T_t : D \to D$ by

$$T_t x = tu + (1-t)Tx \tag{3}$$

for all $x \in D$. Then, by Banach's contraction principle, there exists a unique fixed point $z_t \in D$ of T_t, that is, z_t is the unique solution of the equation

$$z_t = tu + (1-t)Tz_t. \tag{4}$$

In [3], Browder proved that, if X is a Hilbert space, then z_t converges strongly to a fixed point of T as $t \to 0$ and, in [11], Reich extended Browder's result to the setting of uniformly smooth Banach spaces.

The fixed point z_t of T_t in (4) is defined implicitly, but we can devise explicitly an iterative method which converges in norm to a fixed point of T.

In [8], Halpern studied initially such a method as follows:

Let $\{\alpha_n\}$ be a sequence in (0,1] and $x_0 \in D$ be any initial value. Define a sequence $\{x_n\} \subset D$ in an explicit and iterative way by

$$x_n = \alpha_n u + (1-\alpha_n)Tx_n, \quad \forall n \geq 0. \tag{5}$$

Then, under some additional conditions, the sequence $\{x_n\}$ converges strongly to a fixed point of T if $\{\alpha_n\}$ satisfies the following control conditions:

$$\alpha_n \to 0 \quad (n \to \infty), \tag{C1}$$

$$\sum_{n=0}^{\infty} \alpha_n = \infty \text{ or, equivalently, } \prod_{n=0}^{\infty}(1-\alpha_n) = 0. \tag{C2}$$

In [9], Lions improves Halpern's control conditions by showing the strong convergence of the sequence $\{x_n\}$ if $\{\alpha_n\}$ satisfies $(C1)$, $(C2)$ and the following condition:

$$\frac{\alpha_{n+1} - \alpha_n}{\alpha_{n+1}^2} \to 0 \quad (n \to \infty). \tag{$C3$}$$

Note that, for the natural and important choice $\{\frac{1}{n}\}$ of $\{\alpha_n\}$, the results of both Halpern and Lions don't work.

In [18], Wittmann overcame the problem mentioned above by proving the strong convergence of $\{x_n\}$ in Hilbert spaces if $\{\alpha_n\}$ satisfies $(C1)$, $(C2)$ and the following conditon:

$$\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \tag{$C4$}$$

In [19], Xu suggested the following control condition instead of the conditions $(C3)$ or $(C4)$ and proved the strong convergence of the sequence $\{x_n\}$:

$$(C5) \qquad \frac{\alpha_{n+1} - \alpha_n}{\alpha_{n+1}} \to 0 \quad \text{or, equivalently,} \quad \frac{\alpha_n}{\alpha_{n+1}} \to 1 \quad (n \to \infty).$$

Recently, in [5], Cho, Kang and Zhou considered the new control condition

$$(C6) \qquad |\alpha_{n+1} - \alpha_n| \le \circ(\alpha_{n+1}) + \sigma_n,$$

where $\sum_{n=0}^{\infty} \sigma_n < \infty$, and proved some strong convergence theorems of the sequence $\{x_n\}$ for nonexpansive mappings in uniformly smooth Banach spaces. They improved the corresponding results of Lions [9], Wittmann [18], Xu [19], [20] and others. For more results and examples on the control conditions of Halpern's iteration, see [5].

Now, we are in a position to introduce the following new iterative sequences of Halpern type for asymptotically nonexpansive or nonexpansive mappings in Banach spaces:

Let D be a nonempty closed convex subset of E, $x \in D$ be a given point and $T : D \to D$ be a mapping.

If T is an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$, then we consider the sequence $\{x_n\}$ of Halpern type defined by

$$(6) \qquad \begin{cases} x_0 \in D, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) \dfrac{1}{n+1} \displaystyle\sum_{j=0}^{n} T^j y_n, \\ y_n = \beta_n x_n + (1 - \beta_n) \dfrac{1}{n+1} \displaystyle\sum_{j=0}^{n} T^j x_n, \quad \forall n \ge 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$.

If $\beta_n = 1$ for all $n \ge 0$ in (6), then $y_n = x_n$, and so we have the sequence $\{x_n\}$ defined by

$$(7) \qquad \begin{cases} x_0 \in D, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) \dfrac{1}{n+1} \displaystyle\sum_{j=0}^{n} T^j x_n, \quad \forall n \ge 0. \end{cases}$$

If T is a nonexpansive mapping, then we can consider also the sequence $\{x_n\}$ of Haplern type defined by

$$\begin{cases} x_0 \in D, \\ x_{n+1} = \alpha_n x + (1-\alpha_n)\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} T^j y_n, \\ y_n = \beta_n x_n + (1-\beta_n)\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} T^j x_n, \quad \forall n \geq 0, \end{cases} \tag{8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$.

If $\beta_n = 1$ for all $n \geq 0$ in (8), then we have the sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in D, \\ x_{n+1} = \alpha_n x + (1-\alpha_n)\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} T^j x_n, \quad \forall n \geq 0. \end{cases} \tag{9}$$

Now we consider some special cases of the sequences defined by (6)$\sim$(9) as follows:

(I) If $E = H$ is a Hilbert space and $T : D \to D$ is a nonexpansive mapping, then the sequence $\{x_n\}$ defined by (9) was introduced and studied in Shimizu and Takahashi [13]. They showed that, if $F(T) \neq \emptyset$ and $\{\alpha_n\}$ satisfies $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ converges strongly to some point in $F(T)$ which is nea! rest to x in $F(T)$.

(II) If E is a uniformly smooth Banach space, D is a weakly compact convex subset of E, then the sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in D, \\ x_{n+1} = \alpha_n x + (1-\alpha_n) T x_n, \quad \forall n \geq 0, \end{cases} \tag{10}$$

was introduced and studied in Reich [11], [12]. He showed that, if $\alpha_n = n^{-a}$ with $0 < a < 1$, then the sequence $\{x_n\}$ defined by (10) converges strongly to a fixed point of T in D.

(III) In [15], Shioji and Takahashi extended Wittmann's result to the case of Banach space whose norm is uniformly Gâteaux differentiable.

The purpose of this paper is to study the strong convergence problem of the sequences of Halper type defined by (6)$\sim$(9), respectively. The results presented in this paper extend, supplement and improve the corresponding results of Reich [11], Shimizu and Takahashi [13], [14], Shioji and Takahashi [15], [16] and Wittmann [18].

The following lemmas play an important role in proving our main results:

Lemma 4. ([4]) *Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$, the following hold:*

$$\text{(a)} \qquad \|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle$$

for all $j(x+y) \in J(x+y)$ and

$$\text{(b)} \qquad \|x+y\|^2 \ge \|x\|^2 + 2\langle y, j(x)\rangle, \quad \forall j(x) \in J(x).$$

Lemma 5. ([10]) *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative sequences satisfying the following condition: There exists a positive integer n_0 such that*

$$a_{n+1} \le (1-\lambda_n)a_n + b_n + c_n, \quad \forall n \ge n_0,$$

where $\{\lambda_n\}$ is a sequence in $[0,1]$, $\sum_{n=0}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$, $\sum_{n=0}^{\infty} c_n < \infty$. Then $a_n \to 0$ as $n \to \infty$

Now we give our main results in this paper as follows. Let $F(T)$ denote the set of all fixed points of the mapping T.

Theorem 6. *Let E be a real Banach space whose norm is uniformly Gâteaux differentiable, D be a nonempty closed convex subset of E and $T : D \to D$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$ and $\sum_{n=0}^{\infty}(e_n - 1) < \infty$, where*

$$e_n = \frac{1}{n+1}\sum_{j=0}^{n} k_j \ge 1, \quad \forall n \ge 0, \tag{11}$$

and let $F(T) \ne \emptyset$ in D. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ satisfying the following conditions:

$$\sum_{n=0}^{\infty} \alpha_n = \infty. \tag{12}$$

For any given $x \in D$ and $n \ge 1$, define a contractive mapping $S_n : D \to D$ by

$$S_n(z) = (1-d_n)x + d_n T^n z, \tag{13}$$

where

$$k_n^2 - 1 \le (1-d_n)^2, \quad n \ge 0, \quad d_n \in (0,1), \quad d_n \to 1 \quad (n \to \infty). \tag{14}$$

Let z_n be the unique fixed point of S_n, i.e., z_n satisfies the following:

$$z_n = S_n z_n = (1-d_n)x + d_n T^n z_n, \quad \forall n \ge 1. \tag{15}$$

If the sequence $\{z_n\}$ converges strongly to a point $z \in F(T)$ and

$$\|x_n - Tx_n\| \to 0 \quad (n \to \infty),$$

where $\{x_n\}$ is the sequence of Halpern type defined by (6), then $\{x_n\}$ converges strongly to the fixed point z if and only if $\{y_n\}$ defined by (6) is bounded.

Proof. Necessity: If the sequence $\{x_n\}$ converges strongly to $z \in F(T)$, it follows from (6) and the asymptotically nonexpansive property of T that

$$\begin{aligned}
&\|y_n - z\| \\
&= \left\| \beta_n(x_n - z) + (1-\beta_n)\frac{1}{n+1}\sum_{j=0!}^{n}(T^j x_n - z) \right\| \\
&\le \beta_n\|x_n - z\| + (1-\beta_n)\frac{1}{n+1}\sum_{j=0}^{n} k_j\|x_n - z\| \\
&= \beta_n\|x_n - z\| + (1-\beta_n)e_n\|x_n - z\| \\
&\le e_n\|x_n - z\|, \quad \forall n \ge 0,
\end{aligned} \tag{16}$$

where e_n $(n \ge 0)$ is defined by (11). By the assumption that $k_n \to 1$ as $n \to \infty$, the sequence $\{k_n\}$ is bounded and so

$$1 \le e_n \le \sup_{n\ge 0} k_n := M_1 < \infty. \tag{17}$$

Since $x_n \to z$, it follows from (16) that $y_n \to z$ as $n \to \infty$ and hence $\{y_n\}$ is a bounded sequence.

Sufficiency: Let $\{y_n\}$ be a bounded sequence. From (6), we have

$$\begin{aligned}
&\|x_{n+1} - z\| \\
&= \left\| \alpha_n(x - z) + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(T^j y_n - z) \right\| \\
&\le \alpha_n\|x - z\| + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} k_j\|y_n - z\| \\
&= \alpha_n\|x - z\| + (1-\alpha_n)e_n\|y_n - z\| \\
&\le M_1, \quad \forall n \ge 0,
\end{aligned} \tag{18}$$

where $M_1 = \max\{\|x - z\|, \sup_{n\ge 0} e_n \cdot \|y_n - z\|\}$. Besides, we also have

$$\|T^n x_n - z\| \le k_n\|x_n - z\|, \quad \forall n \ge 0, \tag{19}$$

$$\|T^n y_n - z\| \le k_n\|y_n - z\|, \quad \forall n \ge 0, \tag{20}$$

which show that $\{x_n\}$, $\{T^n x_n\}$ and $\{T^n y_n\}$ all are bounded sequences. On the other hands, by the assumption that the norm of E is uniformly Gâteaux differentiable, it follows from Proposition 3 that the normalized duality mapping $J : E \to E^*$ is uniformly continuous on each bounded subset of E from the norm topology of E to the weak* topology of E^*. Hence, for any $n \ge 0$, it follows from (6) and Lemma 4 (a) that

$$\begin{aligned}
&\|x_{n+1}-z\|^2 \\
&= \left\|(1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(T^j y_n - z) + \alpha_n(x-z)\right\|^2 \\
&\le (1-\alpha_n)^2\Big\{\frac{1}{n+1}\sum_{j=0}^{n}k_j\Big\}^2 \cdot \|y_n - z\|^2 \\
&\quad + 2\alpha_n\langle x-z, J(x_{n+1}-z)\rangle \\
&= (1-\alpha_n)^2 e_n^2\|y_n-z\|^2 + 2\alpha_n\langle x-z, J(x_{n+1}-z)\rangle.
\end{aligned} \tag{21}$$

Next, we consider the first term on the right side of (21). From (16), we have

$$\begin{aligned}
&(1-\alpha_n)^2 e_n^2\|y_n-z\|^2 \\
&\le (1-\alpha_n)e_n^4\|x_n-z\|^2 \\
&= (1-\alpha_n)\|x_n-z\|^2 + (1-\alpha_n)(e_n^4-1)\|x_n-z\|^2 \\
&\le (1-\alpha_n)\|x_n-z\|^2 + (e_n-1)\cdot M_2,
\end{aligned} \tag{22}$$

where $M_2 = \sup_{n\ge 0}\{(e_n^3+e_n^2+e_n+1)\}\cdot \sup_{n\ge 0}\|x_n-z\|^2$.

Now, we consider the second term on the right side of (21). From (15), we have

$$x_n - z_m = (1-d_m)(x_n-x) + d_m(x_n - T^m z_m), \quad \forall n\ge 0,\ m\ge 1. \tag{23}$$

Hence, from Lemma 4 (b) and (23), we have

$$\begin{aligned}
&d_m^2\|x_n - T^m z_m\|^2 \\
&= \|(x_n-z_m) - (1-d_m)(x_n-x)\|^2 \\
&\ge \|x_n-z_m\|^2 - 2(1-d_m)\langle x_n - x, J(x_n-z_m)\rangle \\
&= \|x_n-z_m\|^2 - 2(1-d_m)\langle x_n - z_m + z_m - x, J(x_n-z_m)\rangle \\
&= (1-2(1-d_m))\|x_n-z_m\|^2 + 2(1-d_m)\langle x - z_m, J(x_n-z_m)\rangle.
\end{aligned}$$

Simplifying the above inequality, we have

$$\begin{aligned}
&\frac{d_m^2}{1-d_m}\|x_n - T^m z_m\|^2 + \frac{1-2d_m}{1-d_m}\|x_n-z_m\|^2 \\
&\ge 2\langle x - z_m, J(x_n-z_m)\rangle.
\end{aligned} \tag{24}$$

Since $\|x_n - Tx_n\| \to 0$ as $n\to\infty$, it follows that $\|x_n - T^m x_n\|\to 0$ as $n\to\infty$ for $m\ge 1$ and so $\{x_n - T^m x_n\}$ is bounded by a positive constant M_3. Also, by the assumption, $z_m \to z \in F(T)$ and hence we have

$$\lim_{m\to\infty}\|T^m z_m - z\| \le \lim_{m\to\infty} k_m\|z_m - z\| = 0,$$

i.e., $T^m z_m \to z$ as $m \to \infty$ and so

$$\begin{aligned}\lim_{m\to\infty} \|x_n - z_m\| &= \lim_{m\to\infty!} \|x_n - T^m z_m\| \\ &= \|x_n - z\|,\end{aligned} \tag{25}$$

which shows that the sequences $\{x_n - z_m\}$ and $\{x_n - T^m z_m\}$ are bounded sequences by a positive constant M_4.

On the other hand, since $d_m \to 1$, $k_m \to 1$ as $m \to \infty$, we can assume that $d_m > \frac{1}{2}$ and $1 \le k_m \le 2$ for all $m \ge 1$ and, from (14), also we have

$$\begin{aligned}\Delta_{m,n} &= \frac{d_m^2}{1-d_m}\|x_n - T^m z_m\|^2 + \frac{1-2d_m}{1-d_m}\|x_n - z_m\|^2 \\ &= \frac{d_m^2}{1-d_m}\|x_n - T^m z_m\|^2 - \frac{2d_m-1}{1-d_m}\|x_n - z_m\|^2 \\ &= \frac{2d_m-1}{1-d_m}(\|x_n - T^m z_m\|^2 - \|x_n - z_m\|^2) \\ &\quad + (1-d_m)\|x_n - T^m z_m\|^2 \\ &\le \frac{1}{1-d_m}[(\|T^m x_n - T^m z_m\| + \|T^m x_n - x_n\|)^2 - \|x_n - z_m\|^2] \\ &\quad + M_4^2(1-d_m) \\ &= \frac{1}{1-d_m}(\|T^m z_m - T^m x_n\|^2 + 2\|T^m x_n - x_n\|\|T^m z_m - T^m x_n\| \\ &\quad + \|T^m x_n - x_n\|^2 - \|x_n - z_m\|^2) + M_4^2(1-d_m) \\ &\le \frac{1}{1-d_m}[(k_m^2-1)\|z_m - x_n\|^2 + 2k_m\|T^m x_n - x_n\|\|z_m - x_n\| \\ &\quad + M_3\|T^m x_n - x_n\|] + M_4^2(1-d_m) \\ &\le \frac{1}{1-d_m}[(1-d_m)^2 M_4^2 + M_5\|T^m x_n - x_n\|] + M_4^2(1-d_m) \\ &\le (1-d_m)M_4^2 + \frac{M_5}{1-d_m}\|T^m x_n - x_n\| + M_4^2(1-d_m),\end{aligned} \tag{26}$$

where $M_5 = 4M_4 + M_3$. For fixed $m \ge 1$, letting $n \to \infty$ in (26), we have $\|T^m x_n - x_n\| \to 0$ as $n \to \infty$ by our assumption ($\|Tx_n - x_n\| \to 0$ as $n \to \infty$) and so

$$\frac{M_5}{1-d_m}\|T^m x_n - x_n\| \to 0 \quad (n \to \infty).$$

Thus, from (26), it follows that

$$\limsup_{n\to\infty} \Delta_{m,n} \le 2M_4^2(1-d_m)$$

and so letting $m \to \infty$ yields

$$\limsup_{m\to\infty}\limsup_{n\to\infty} \Delta_{m,n} \le 0.$$

Thus it follows from (24) that

$$\Delta_{m,n} \geq 2\langle x - z_m, J(x_n - z_m)\rangle$$

and

$$\begin{aligned} 0 &\geq \limsup_{m\to\infty} \limsup_{n\to\infty} \Delta_{m,n} \\ &\geq 2 \limsup_{m\to\infty} \limsup_{n\to\infty} \langle x - z_m, J(x_n - z_m)\rangle. \end{aligned} \tag{27}$$

Next, we prove that

$$\limsup_{n\to\infty} \langle x - z, J(x_n - z)\rangle \leq 0.$$

In fact, it follows from (27) that, for any $\epsilon > 0$, there exists a positive integer N such that

$$\limsup_{n\to\infty} \langle x - z_m, J(x_n - z_m)\rangle \leq \epsilon \tag{28}$$

for all $m \geq N$. Since $J : E \to E^*$ is uniformly continuous from the norm topology of E to the weak* topology of E^* on each bounded subset of E and $z_m \to z$ as $m \to \infty$, we have

$$\limsup_{n\to\infty} \langle x - z, J(x_n - z)\rangle \leq \epsilon.$$

Since ϵ is arbitrary, we have

$$\limsup_{n\to\infty} \langle x - z, J(x_n - z)\rangle \leq 0. \tag{29}$$

Next, set $\tau_n = \max\{\langle x - z, J(x_n - z)\rangle, 0\}$. Then $\tau_n \to 0$ as $n \to \infty$.

Substituting (22) into (21), we have

$$\begin{aligned} &\|x_{n+1} - z\|^2 \\ &\leq (1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle x - z, J(x_{n+1} - z)\rangle + (e_n - 1)M_2 \\ &\leq (1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n\tau_n + (e_n - 1)M_2, \quad \forall n \geq 0. \end{aligned} \tag{30}$$

Taking $a_n = \|x_n - z\|^2$, $\lambda_n = \alpha_n$, $c_n = (e_n - 1)M_2$ and $b_n = 2\lambda_n\tau_n$ in (30), we have

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n + c_n, \quad \forall n \geq 0,$$

since $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}(e_n - 1)M_2 < \infty$, $b_n = \circ(\lambda_n)$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$. By Lemma 5, we know that $a_n \to 0$ as $n \to \infty$, i.e., $x_n \to z$ as $n \to \infty$. This completes the proof. □

Remark 7. We can consider an example of the sequence $\{d_n\}$ defined by $d_n = 1 - (k_n^2 - 1)^{\frac{1}{2}}$ for $1 < k_n < 2^{\frac{1}{2}}$ $(n \geq 0)$ and $d_n = \frac{n}{n+1}$ for $k_n = 1$ $(n \geq 0)$ which satisfies the condition (14) in Theorem 6.

From Theorem 6, we have the following:

Theorem 8. *If all the conditions in Theorem 6 are satisfied, then the sequence* $\{x_n\}$ *defined by* (7) *converges strongly to a fixed point* z *of* T *if and only if* $\{x_n\}$ *is bounded.*

Proof. In (6), taking $\beta_n = 1$ for all $n \geq 0$, we have $y_n = x_n$ for all $n \geq 0$. Therefore, the sequence $\{x_n\}$ defined by (6) deduces to the sequence $\{x_n\}$ defined by (7). The conclusion of Theorem 8 can be obtained from Theorem 6 immediately. □

Finally, we give a convergence theorem of the iterative sequences defined by (8) or (9) for nonexpansive mappings.

Theorem 9. *Let* E *be a real Banach space whose norm is uniformly Gâteaux differentiable and* D *be a nonempty closed convex subset of* E *and* $T : D \to D$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be two sequences in* $[0, 1]$ *satisfying the condition* (12). *Let* $x \in D$ *and define a contractive mapping* $S_t : D \to D$ *by*

$$S_t(z) = (1 - t)x + tTz, \quad \forall z \in D, \tag{31}$$

where

$$0 < t < 1, \quad t \to 1^-. \tag{32}$$

Let z_t *be the unique fixed point of* S_t. *If* $\{z_t\}$ *converges strongly to a point* $z \in F(T)$ *as* $t \to 1^-$ *and* $\|x_n - Tx_n\| \to 0$ *as* $n \to \infty$, *where* $\{x_n\}$ *is the sequence defined by* (8) *or* (9), *then* $\{x_n\}$ *converges to the fixed point* z *of* T.

Proof. Since $T : D \to D$ is a nonexpansive mapping, T is an asymptotically nonexpansive mapping with a constant sequence $\{k_n\} = \{1\}$ for $n \geq 0$. This implies that

$$e_n = \frac{1}{n+1} \sum_{j=0}^{n} k_j = k_n = 1, \quad \forall n \geq 0,$$

and so

$$\sum_{n=1}^{\infty} (e_n - 1) = \sum_{n=1}^{\infty} (k_n - 1) = 0 < \infty.$$

(I) In order to prove Theorem 9, we first prove that the sequences $\{x_n\}$, $\{y_n\}$ and $\{x_n\}$ defined by (8) and (9), respectively, all are bounded. For the sake of simplicity, we only prove the sequences $\{x_n\}$, and $\{y_n\}$ defined by (8) are bounded.

For the purpose, we denote by

$$M = \max\{\|x_1 - z\|, \|x - z\|\}, \tag{33}$$

where $x \in D$, $z \in F(T)$ is the points appeared in Theorem 9. Now, by induction, we can prove the following inequalities:

$$\begin{cases} \|x_n - z\| \leq M, \\ \|y_n - z\| \leq M, \quad \forall n \geq 1. \end{cases} \tag{34}$$

In fact, for $n = 1$, from (33), we have

$$\|x_1 - z\| \le M$$

and

$$\begin{aligned}\|y_1 - z\| &= \|\beta_1(x_1 - z) + (1-\beta_1)\frac{1}{2}\sum_{j=0}^{1}(T^j x_1 - z)\| \\ &\le \beta_1\|x_1 - z\| + (1-\beta_1)\|x_1 - z\| \\ &\le M.\end{aligned}$$

Hence the conclusion of (34) is true for $n = 1$. Suppose the conclusion of (34) is true for $n = k$. Now, we prove that the conclusion of (34) is also true for the case of $n = k + 1$. In fact, it follows from (8) that

$$\begin{aligned}&\|x_{k+1} - z\| \\ &= \left\|\alpha_k(x - z) + (1-\alpha_k)\frac{1}{k+1}\sum_{j=0}^{k}(T^j y_k - z)\right\| \\ &\le \alpha_k\|x - z\| + (1-\alpha_k)\|y_k - z\| \\ &\le M\end{aligned}$$

and

$$\begin{aligned}&\|y_{k+1} - z\| \\ &= \left\|\beta_{k+1}(x_{k+1} - z) + (1-\beta_{k+1})\frac{1}{k+2}\sum_{j=0}^{k+1}(T^j x_{k+1} - z)\right\| \\ &\le \beta_{k+1}\|x_{k+1} - z\| + (1-\beta_{k+1})\|x_{k+1} - z\| \\ &\le M.\end{aligned}$$

This implies that the conclusion of (34) is true and so the sequences $\{x_n\}$ and $\{y_n\}$ both are bounded.

(II) Taking $e_n = 1$, $n \ge 0$, $d_m = t$, $t \in (0,1)$, and $t \to 1^-$, $z_m = z_t$, $T^m z_m = T z_t$ in (16) and (21)$\sim$(27), we can prove similarly that the sequence $\{x_n\}$ defined by (8) satisfies the following inequalities:

$$\limsup_{n\to\infty}\langle x - z, J(x_n - z)\rangle \le 0$$

and

$$\|x_{n+1} - z\|^2 \le (1-\alpha_n)\|x_n - z\|^2 + \circ(\alpha_n), \quad \forall n \ge 0.$$

Therefore, by Lemma 5, it follows that $x_n \to z$ as $n \to \infty$. This completes the proof. $\square$

Remark 10. It is pointed out by Reich [12], Takahashi and Ueda [17] that, in Theorem 9, if D satisfies additional condition that "D is a weak compact convex subset of E", then the sequence $\{z_t\}$ satisfying

$$z_t = tx + (1-t)Tz_t, \quad \forall t \in (0,1)$$

converges strongly to a fixed point $z \in D$ of T.

Remark 11. Theorem 9 extends, supplements and improves the corresponding results of Wittmann [18] and Shioji and Takahashi [15], [16], Shimizu and Takahashi [13], [14]. Besides, the method of proof is also quitely different from that that given in [11], [13]-[16] and [18].

Remark 12. Theorem 6 and Theorem 8 are two new results which establish some necessary and sufficient conditions for the strong convergence of the sequences defined by (6) and (7), respectively.

Remark 13. Under some conditions of Theorem 6 and Theorem 8, we can obtain the same results which establish some necessary and sufficient conditions for the strong convergence of the sequences (of Halpern type) with errors, respectively.

References

[1] E. Asplund, *Positivity of duality mappings*, Bull. Amer. Math. Soc. **73** (1967), 200–203.
[2] J. B. Baillon, *Un theoreme de type ergodic pour les contractions non lineares dans un espace de Hilbert*, C. R. Acad. Sci. Paris, Sér. A **280** (1975), 1511–1514.
[3] F. E. Browder, *Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces*, Arch. Rational Mech. Anal. **24** (1967), 82–90.
[4] S. S. Chang *Some problems and results in the study of nonlinear analysis* Nonlinear Anal. TMA **30** (1997), 4197–4208.
[5] Y. J. Cho, S. M. Kang and H. Y. Zhou *Some control conditions on the iterative methods*, Commun. on Appl. Nonlinear Anal. **12** (2005), 27–34.
[6] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.
[7] K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972), 171–174.
[8] B. Halpern, *Fixed points of nonexpansive maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.
[9] P. L. Lions, *Approximation de points fixes de contractions*, C. R. Acad. Sci. Paris Sèr. A **284** (1977), 1357–1359.
[10] L. S. Liu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl. **194** (1995), 114–125.
[11] S. Reich, *Some problems and results in fixed point theory*, Contem. Math. **21** (1983), 179–187.
[12] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), 128–292.
[13] T. Shimizu and W. Takahashi, *Strong convergence theorem for asymptotically nonexpansive mappings*, Nonlinear Anal. TMA **26** (1996), 265–272.
[14] T. Shimizu and W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Anal **211** (1997), 71–83.
[15] N. Shioji and W. Takahashi, *Strong convergence of approximated sequence for nonexpansive mappings*, Proc. Amer. Math. Soc. **125** (1997), 3641–3645.
[16] N. Shioji and W. Takahashi, *Strong convergence theorems for asymptotically nonexpansive semigroups in Hilbert spaces*, Nonlinear Anal. TMA **34** (1998), 87–99.
[17] W. Takahashi and Y. Ueda, *On Reich's strong convergence theorems for resolvents of accretive operators*, J. Math. Anal. Appl. **104** (1984), 546–543.
[18] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. **58** (1992), 486–491.

[19] H. K. Xu, *Another control condition in an iteraive method for nonexpansive mappings*, Bull. Austral. Math. Soc. **65** (2002), 109–113.

[20] H. K. Xu, *Remarks on an iteraive method for nonexpansive mappings*, Commun. on Appl. Nonlinear Anal. **10** (2003), 67–75.

Shih-sen Chang: Department of Mathematics, Yibin University, Yibin, Sichuan 644007, Sichuan University, Chengdu, Sichuan 610064, People's Republic of China
E-mail address: sszhang_1@yahoo.com.cn

Yeol Je Cho: Department of Mathematics Education, The Research Institute of Natural Sciences, College of Education, Gyeongsang National University, Chinju 660-701, Korea
E-mail address: yjcho@gsnu.ac.kr

Haiyun Zhou: Department of Mathematics, Shijizhuang Mechanical Engineering College, Shijizhuang 050003, People's Republic of China
E-mail address: witman66@yahoo.com.cn