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FIXED POINT THEORY FOR PERMISSIBLE MAPS VIA INDEX THEORY

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Abstract. New fixed point theorems for permissible maps between Fréchet spaces are presented. The proof relies on index theory developed by Dzedzej and on viewing a Fréchet space as the projective limit of a sequence of Banach spaces.

1. Introduction.

This paper presents new fixed point theorems for permissible compact maps (or more generally permissible maps which are of compact attraction) between Fréchet spaces. In the literature [1, 6], one usually assumes the map F is defined on a subset X of a Fréchet space E and its restriction (again called F) is well defined on $\overline{X_n}$ (see Section 2). In general of course for Volterra operators the restriction is always defined on X_n and in most applications it is in fact defined on $\overline{X_n}$ and usually even on E_n (see Section 2).

In this paper we make use of the fact that the restriction is well defined on X_n and we only assume it admits an extension (satisfying certain properties) on $\overline{X_n}$. It is also worth remarking here that one could write the results in this paper for compositions of maps with proximally ∞-connected values using the index theory in [2] (this has the advantage that no knowlege of homology theory is needed to construct the index).

Existence in Section 2 will rely on index theory and so we begin by discussing the maps we will consider in this paper.

Let X and Y be Hausdorff topological spaces. We say $F : X \to 2^Y$ (here 2^Y denotes the family of nonempty subsets of Y) is locally compact if for every $x \in X$ there exists a neighborhood U of x such that the restriction $F|_U : U \to 2^Y$ is compact. Now, if $F : X \to 2^X$, we let $F^n(x) = F(F^{n-1}(x))$.

Definition 1.1. Let $F : X \to 2^X$ be upper semicontinuous, $x \in X$ and $A \subseteq X$. We say that A *attracts* x if, for each neighborhood U of A, there is a $n \in \{1, 2, \cdots\}$ with $F^n(x) \subseteq U$. Also, we say that A is an *attractor* for

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F if it attracts all points in X. Now, we say that the map F is *of compact attraction* if it has a compact attractor and is locally compact.

Definition 1.2. A multivalued map $F : X \to 2^Y$ is in the class $\mathcal{A}_m(X,Y)$ if

(1) F is continuous,

(2) for each $x \in X$, the set $F(x)$ consists of one or m acyclic components; here m is a positive integer.

Definition 1.3. A decomposition $(F_1, \cdots, F_n)$ of a multivalued map $F : X \to 2^Y$ is a sequence of maps

$$X = X_0 \overset{F_1}{\to} X_1 \overset{F_2}{\to} X_2 \overset{F_3}{\to} \ \cdots \ \overset{F_{n-1}}{\to} X_{n-1} \overset{F_n}{\to} X_n = Y,$$

where $F_i \in \mathcal{A}_{m_i}(X_{i-1}, X_i)$, $F = F_n \circ \cdots \circ F_1$. One can say that the map F is determined by the decomposition $(F_1, \cdots, F_n)$. The number n is said to be the *length* of the decomposition $(F_1, \cdots, F_n)$. We will denote the class of decompositions by $\mathcal{D}(X,Y)$.

Definition 1.4. An upper semicontinuous map $F : X \to 2^Y$ is said to be *permissible* provided it admits a selector $G : X \to 2^Y$ which is determined by a decomposition $(G_1, \cdots, G_n) \in \mathcal{D}(X,Y)$. We denote the class of permissible maps from X into Y by $\mathcal{P}(X,Y)$.

Let X be a closed convex subset of a normed space E and let $F : X \to 2^X$ be a permissible map which is of compact attraction. Let U be an open subset of X with $Fix\,F \cap \partial U = \emptyset$. Then the index $i(X,F,U)$ is well defined (see [3, p. 42] or see [4, Sections 50-53]) and has the usual properties ([3, p. 43]).

Let (X,d) be a metric space and S a nonempty subset of X. For $x \in X$ let $d(x,S) = \inf_{y\in S} d(x,y)$. Also, $diam\,S = \sup\{d(x,y) : x,y \in S\}$. We let $B(x,r)$ denote the open ball in X centered at x of radius r and by $B(S,r)$ we denote $\cup_{x\in S} B(x,r)$. For two nonempty subsets S_1 and S_2 of X, we define the generalized Hausdorff distance H to be

$$H(S_1,S_2) = \inf\{\epsilon > 0 : \ S_1 \subseteq B(S_2,\epsilon), \ S_2 \subseteq B(S_1,\epsilon)\}.$$

Now, suppose $G : S \to 2^X$; here 2^X denotes the family of nonempty subsets of X. Then G is said to be *hemicompact* if each sequence $\{x_n\}_{n\in N}$ in S has a convergent subsequence whenever $d(x_n, G(x_n)) \to 0$ as $n \to \infty$.

Now, let I be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha\in I}$ be a family of locally convex spaces. For each $\alpha \in I$, $\beta \in I$ for which $\alpha \leq \beta$, let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha\in I} E_\alpha : \ x_\alpha = \pi_{\alpha,\beta}(x_\beta) \ \forall\, \alpha, \, \beta \in I, \, \alpha \leq \beta \right\}$$

is a closed subset of $\prod_{\alpha\in I} E_\alpha$ and is called the *projective limit* of $\{E_\alpha\}_{\alpha\in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow} \{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [5, p. 439] $\cap_{\alpha\in I} E_\alpha$.)

2. Fixed point theory in Fréchet spaces

Let $E = (E, \{|\cdot|_n\}_{n\in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$; here $N = \{1, 2, \cdots\}$. We assume that the family of seminorms satisfies

$$|x|_1 \le |x|_2 \le |x|_3 \le \cdots \quad \text{for every } x \in E. \tag{2.1}$$

A subset X of E is said to be *bounded* if, for every $n \in N$, there exists $r_n > 0$ such that $|x|_n \le r_n$ for all $x \in X$. For $r > 0$ and $x \in E$, we denote $B(x, r) = \{y \in E : |x - y|_n \le r, \ \forall n \in N\}$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$, we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \quad \text{if and onnly if} \quad |x - y|_n = 0. \tag{2.2}$$

We denote by $\mathbf{E}^n = (E / \sim_n, |\cdot|_n)$ the quotient space and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (2.1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \ge n$ (again, this seminorm is denoted by $|\cdot|_n$). Also, (2.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m / \sim_n$ can be regarded as a subset of $\mathbf{E}_n$. Now, $\mu_{n,m}\,\mu_{m,k} = \mu_{n,k}$ if $n \le m \le k$ and $\mu_n = \mu_{n,m}\,\mu_m$ if $n \le m$. We now assume the following condition holds:

$$\begin{cases} \text{for each } n \in N, \quad \text{there exists a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \to E_n. \end{cases} \tag{2.3}$$

Remark 2.1. (1) For convenience the norm on E_n is denoted by $|\cdot|_n$.

(2) In our applications $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.

(3) Note that, if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$), then $x \in E$. However, if $x \in E_n$, then x is not necessaily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example, if $E = C[0, \infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0, n]$ and $E_n = C[0, n]$.

Finally, we assume

$$\begin{cases} E_1 \supseteq E_2 \supseteq \cdots \quad \text{and, for each } n \in N, \\ |j_n\,\mu_{n,n+1}\,j_{n+1}^{-1}\,x|_n \le |x|_{n+1} \quad \text{for all } x \in E_{n+1} \end{cases} \tag{2.4}$$

(here we use the notation from [5], i.e., decreasing in the generalized sense). Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$ where $\cap_1^\infty$ is the generalized intersection [5]) denote the projective limit of $\{E_n\}_{n\in N}$ (note $\pi_{n,m} = j_n\,\mu_{n,m}\,j_m^{-1} : E_m \to E_n$ for $m \ge n$) and note $\lim_{\leftarrow} E_n \cong E$, so, for convenience, we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$, we set $X_n = j_n\,\mu_n(X)$ and we let $\overline{X_n}$, $int\,X_n$ and ∂X_n denote, respectively, the closure, the interior and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also, the *pseudo-interior* of X is defined by

$$pseudo-int\,(X)=\{x\in X:\ j_n\,\mu_n(x)\in\overline{X_n}\setminus\partial X_n\ \text{ for every }\ n\in N\}.$$

The set X is said to be *pseudo-open* if $X=pseudo-int\,(X)$. For $r>0$ and $x\in E_n$, we denote $B_n(x,r)=\{y\in E_n:\ |x-y|_n\le r\}$.

Let $M\subseteq E$ and consider the map $F:M\to 2^E$. Assume that, for each $n\in N$ and $x\in M$, that $j_n\,\mu_n\,F\,(x)$ is closed. Let $n\in N$ and $M_n=j_n\,\mu_n(M)$. Since we only consider Volterra type operators, we assume that

$$\text{if }\ x,\,y\in E\ \text{ with }\ |x-y|_n=0\ \text{ then }\ H_n(F\,x,F\,y)=0; \tag{2.5}$$

here H_n denotes the appropriate generalized Hausdorff distance (alternatively, we could assume that if $j_n\,\mu_n\,x=j_n\,\mu_n\,y$ for all $n\in N$ and $x,\,y\in M$, then $j_n\,\mu_n\,F\,x=j_n\,\mu_n\,F\,y$ and of course, here we do not need to assume that $j_n\,\mu_n\,F\,(x)$ is closed for each $n\in N$ and $x\in M$). Now, (2.5) guarantees that we can define (a well defined) F_n on M_n as follows:

For $y\in M_n$, there exists a $x\in M$ with $y=j_n\,\mu_n(x)$ and we let

$$F_n\,y=j_n\,\mu_n\,F\,x$$

(we could of course call it $F\,y$ since it is clear in the situation we use it); note that $F_n:M_n\to C(E_n)$ and note if there exists $z\in M$ with $y=j_n\,\mu_n(z)$, then $j_n\,\mu_n\,F\,x=j_n\,\mu_n\,F\,z$ from (2.5) (here $C(E_n)$ denotes the family of nonempty closed subsets of E_n). In this paper, we assume F_n will be defined on $\overline{M_n}$, i.e., we assume the F_n described above admits an extension (again we call it F_n) $F_n:\overline{M_n}\to 2^{E_n}$ (we will assume certain properties on the extension).

We now show how easily one can extend fixed point theory in Banach spaces to applicable fixed point theory in Fréchet spaces.

Theorem 2.1. *Let E and E_n be as described above, C a convex subset in E, U a pseudo-open bounded subset of E, $F:Y\to 2^E$ with $Y\subseteq E$ and $\overline{C_n}\subseteq Y_n$ for each $n\in N$. Also, assume, for each $n\in N$ and $x\in Y$, that $j_n\,\mu_n\,F\,(x)$ is closed and, in addition, for each $n\in N$, that $F_n:\overline{C_n}\to 2^{E_n}$ is as described above. Suppose that the following conditions are satisfied:*

$$\text{for each }\ n\in N,\ F_n\in\mathcal{P}(\overline{C_n},\overline{C_n})\ \text{ is a compact map,} \tag{2.6}$$

$$\begin{cases}\text{for each }\ n\in N,\ F_n\ \text{ has no fixed points in }\ \partial\,W_n;\ \text{here}\\ W_n=U_n\cap\overline{C_n}\ \text{ and }\partial W_n\ \text{ denotes the boundary of }\ W_n\ \text{ in }\ \overline{C_n},\end{cases} \tag{2.7}$$

$$\text{for each }\ n\in N,\ i(\overline{C_n},F_n,W_n)\neq\{0\}, \tag{2.8}$$

$$\begin{cases}\text{for each }\ n\in\{2,3,\cdots\},\ \text{ if }\ y\in W_n\ \text{ solves }\ y\in F_n\,y\\ \text{in }\ E_n,\ \text{ then }\ j_k\,\mu_{k,n}\,j_n^{-1}\,(y)\in W_k\ \text{ for }\ k\in\{1,\cdots,n-1\}.\end{cases} \tag{2.9}$$

Then F has a fixed point in E.

Remark 2.2. Note in Theorem 2.1 that if $x \in \overline{C_n}$, then $x \in Y_n$ and so there exists a $y \in Y$ with $x = j_n\, \mu_n\,(y)$. Thus $F_n\,(x) = j_n\, \mu_n\, F(y)$.

Proof. Fix $n \in N$. We now show

$$U_n = int\, U_n \quad \text{and} \quad \overline{C_n} \quad \text{is convex.} \tag{2.10}$$

We first show U_n is a open subset of E_n, so $int\, U_n = U_n$. To see this, note $U_n \subseteq \overline{U_n} \setminus \partial U_n$ since if $y \in U_n$ then there exists $x \in X$ with $y = j_n\mu_n(x)$ and this together with $U = pseudo - int\, U$ yields $j_n\mu_n(x) \in \overline{U_n} \setminus \partial U_n$, i.e., $y \in \overline{U_n} \setminus \partial U_n$. In addition, notice

$$\overline{U_n} \setminus \partial U_n = (int\, U_n \cup \partial U_n) \setminus \partial U_n = int\, U_n \setminus \partial U_n = int\, U_n$$

since $int\, U_n \cap \partial U_n = \emptyset$. Consequently,

$$U_n \subseteq \overline{U_n} \setminus \partial U_n = int\, U_n, \quad \text{so} \quad U_n = int\, U_n.$$

To show the second part of (2.10), let $\hat{x}$, $\hat{y} \in \mu_n(C)$ and $\lambda \in [0,1]$. Then, for every $x \in \mu_n^{-1}(\hat{x})$ and $y \in \mu_n^{-1}(\hat{y})$, we have $\lambda x + (1-\lambda)y \in C$ since C is convex and so $\lambda\hat{x} + (1-\lambda)\hat{y} = \lambda\mu_n(x) + (1-\lambda)\mu_n(y)$. It is easy to check that $\lambda\mu_n(x) + (1-\lambda)\mu_n(y) = \mu_n(\lambda x + (1-\lambda)y)$, so, as a result,

$$\lambda\hat{x} + (1-\lambda)\hat{y} = \mu_n(\lambda x + (1-\lambda)y) \in \mu_n(C)$$

and so $\mu_n(C)$ is convex. Now, since j_n is linear, we have $C_n = j_n(\mu_n(C))$ is convex and as, a result, $\overline{C_n}$ is convex. Thus (2.10) holds.

Now, (2.8) guarantees that there exists $y_n \in W_n = U_n \cap \overline{C_n}$ with $y_n \in F_n\, y_n$ in E_n. Let's look at $\{y_n\}_{n\in N}$. Notice $y_1 \in W_1$ and $j_1\, \mu_{1,k}\, j_k^{-1}\,(y_k) \in W_1$ for $k \in N \setminus \{1\}$ from (2.9). Note $j_1\, \mu_{1,n}\, j_n^{-1}(y_n) \in F_1\,(j_1\, \mu_{1,n}\, j_n^{-1}(y_n))$ in E_1; to see note that, for $n \in N$ fixed, there exists $x \in E$ with $y_n = j_n\, \mu_n\,(x)$, so $j_n\, \mu_n\,(x) \in F_n\,(y_n) = j_n\, \mu_n\, F(x)$ on E_n, so on E_1, we have

$$\begin{aligned} j_1\, \mu_{1,n}\, j_n^{-1}(y_n) &= j_1\, \mu_{1,n}\, j_n^{-1}\, j_n\, \mu_n\,(x) \in j_1\, \mu_{1,n}\, j_n^{-1}\, j_n\, \mu_n\, F(x) \\ &= j_1\, \mu_{1,n}\, \mu_n\, F(x) = j_1\, \mu_1\, F(x) = F_1(j_1\, \mu_1\,(x)) \\ &= F_1(j_1\, \mu_{1,n}\, j_n^{-1}\, j_n\, \mu_n\,(x)) = F_1\,(j_1\, \mu_{1,n}\, j_n^{-1}(y_n)). \end{aligned}$$

$j_1\, \mu_{1,n}\, j_n^{-1}(y_n) \in F_1\,(j_1\, \mu_{1,n}\, j_n^{-1}(y_n))$ in E_1, $j_1\, \mu_{1,n}\, j_n^{-1}\,(y_n) \in W_1$ for $n \in N$, together with (2.6) implies that there is a subsequence $N_1^\star$ of N and a $z_1 \in \overline{W_1}$ with $j_1\, \mu_{1,n}\, j_n^{-1}\,(y_n) \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$ and $z_1 \in F_1\, z_1$ since F_1 is upper semicontinuous. Also (2.7) implies $z_1 \in W_1$. Let $N_1 = N_1^\star \setminus \{1\}$. Now $j_2\, \mu_{2,n}\, j_n^{-1}\,(y_n) \in W_2$ for $n \in N_1$ together with (2.6) guarantees that there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in \overline{W_2}$ with $j_2\, \mu_{2,n}\, j_n^{-1}\,(y_n) \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$ and $z_2 \in F_2\, z_2$. Also, (2.7) implies $z_2 \in W_2$. Note from (2.4) and the uniqueness of limits that $j_1\, \mu_{1,2}\, j_2^{-1}\, z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1$ (note $j_1\, \mu_{1,n}\, j_n^{-1}\,(y_n) = j_1\, \mu_{1,2}\, j_2^{-1}\, j_2\, \mu_{2,n}\, j_n^{-1}\,(y_n)$ for $n \in N_2^\star$). Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \cdots, \quad N_k^\star \subseteq \{k, k+1, \cdots\}$$

and $z_k \in \overline{W_k}$ with $j_k\,\mu_{k,n}\,j_n^{-1}(y_n) \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$ and $z_k \in F_k\, z_k$. Also, (2.7) implies $z_k \in W_k$. Note $j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \cdots\}$. Also, let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Now $z_k \in F_k\, z_k$ in E_k. Note as well that

$$\begin{aligned} z_k &= j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,z_{k+1} = j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,j_{k+1}\,\mu_{k+1,k+2}\,j_{k+2}^{-1}\,z_{k+2} \\ &= j_k\,\mu_{k,k+2}\,j_{k+2}^{-1}\,z_{k+2} = \cdots = j_k\,\mu_{k,m}\,j_m^{-1}\,z_m = \pi_{k,m}\,z_m \end{aligned}$$

for every $m \geq k$. We can do this for each $k \in N$. As a result $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and also note $y \in Y$ since $z_k \in W_k \subseteq \overline{C_k} \subseteq Y_k$ for each $k \in N$. Thus, for each $k \in N$, we have

$$j_k\,\mu_k\,(y) = z_k \in F_k\,z_k = j_k\,\mu_k\,F\,y \quad \text{in } E_k$$

so $y \in F\,y$ in E. □

Remark 2.3. We can replace (2.9) in Theorem 2.1 with

$$\begin{cases} \text{for each } \ n \in \{2, 3, \cdots\}, \ \text{ if } \ y \in W_n \ \text{ solves } \ y \in F_n\,y \\ \text{in } \ E_n, \ \text{ then } \ j_k\,\mu_{k,n}\,j_n^{-1}(y) \in \overline{W_k} \ \text{ for } \ k \in \{1, \cdots, n-1\}. \end{cases}$$

Remark 2.4. In Theorem 2.1, it is possible to replace $\overline{C_n} \subseteq Y_n$ with $\overline{C_n}$ a subset of the closure of Y_n in E_n provided Y is a closed subset of E, so in this case, we could have $Y = C$ if C is closed. To see this note, from $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and $\pi_{k,m}(y_m) \to z_k$ in E_k as $m \to \infty$, we can conclude that $y \in \overline{Y} = Y$ (note $q \in \overline{Y}$ if and only if, for every $k \in N$, there exists $(x_{k,m}) \in Y$, $x_{k,m} = \pi_{k,n}(x_{n,m})$ for $n \geq k$ with $x_{k,m} \to j_k\,\mu_k\,(q)$ in E_k as $m \to \infty$). Thus $z_k = j_k\,\mu_k\,(y) \in Y_k$ and so $j_k\,\mu_k\,(y) \in j_k\,\mu_k\,F\,(y)$ in E_k as before.

Remark 2.5. One could write the result in Theorem 2.1 with $\mathcal{P}$ replaced by $J(\mathcal{A})^c$ using the index theory from [2] (this has the advantage that no knowlege of homology theory is needed to construct the index).

Essentially, the same reasoning as in Theorem 2.1 yields the following result (in addition, here we have the analogue of Remark 2.3 and Remark 2.4).

Theorem 2.2. *Let E and E_n be as described above, C a convex subset in E, U a pseudo-open bounded subset of E and $F : Y \to 2^E$ with $Y \subseteq E$, and $\overline{C_n} \subseteq Y_n$ for each $n \in N$. Also assume, for each $n \in N$ and $x \in Y$, that $j_n\,\mu_n\,F\,(x)$ is closed and, in addition, for each $n \in N$, that $F_n : \overline{C_n} \to 2^{E_n}$ is as described above. Suppose that the following condition is satisfied:*

$$\text{(2.11)} \qquad \begin{cases} \textit{for each } \ n \in N, \ F_n \in \mathcal{P}(\overline{C_n}, \overline{C_n}) \ \textit{ is of} \\ \textit{compact attraction and is a hemicompact map.} \end{cases}$$

Also, assume that (2.7), (2.8) and (2.9) hold. Then F has a fixed point in E.

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