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CONVERGENCE OF VISCOSITY APPROXIMATIONS TO FIXED POINTS OF NONEXPANSIVE NONSELF-MAPPINGS IN BANACH SPACES

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Abstract. Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to E$ a nonexpansive mapping satisfying the weak inwardness condition. Assume that every weakly compact convex subset of E has the fixed point property. For $f : C \to C$ a contraction and $t \in (0, 1)$, let x_t be a unique fixed point of a contraction $T_t : C \to E$, defined by $T_t x = tf(x) + (1 - t)Tx$, $x \in C$. It is proved that if $\{x_t\}$ is bounded, then x_t converges to a fixed point of T, which is the unique solution of certain variational inequality. Moreover, the strong convergence of other implicit and explicit iterative schemes involving the sunny nonexpansive retraction is also given in a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm.

1. Introduction

Let E be a real Banach space and C be a nonempty closed convex subset of E. Recall that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in (0, 1)$ such that $\|f(x) - f(y)\| \le k\|x - y\|,\ x,\ y \in C$. We use Σ_C to denote the collection of all contractions on C. That is, $\Sigma_C = \{f : f : C \to C \text{ a contraction}\}$. Note that each $f \in \Sigma_C$ has a unique fixed point in C.

Now let $T : C \to C$ be a nonexpansive mapping (recall that a mapping $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|,\ x,\ y \in C$) and $F(T)$ denote the set of fixed points of T; that is, $F(T) = \{x \in C : x = Tx\}$.

Given a real number $t \in (0, 1)$, a contraction $f \in \Sigma_C$ and a nonexpasive mapping T, let a contraction $T_t := T_t^f : C \to C$ be defined by

$$T_t x = tf(x) + (1 - t)Tx, \quad \forall x \in C. \tag{1.1}$$

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and let $x_t := x_t^f \in C$ be the unique fixed point of T_t. Then x_t is the unique solution of the fixed point equation

$$x_t = tf(x_t) + (1-t)Tx_t. \tag{1.2}$$

A special case of (1.2) has been considered by Browder [2] in a Hilbert space as follows. Fix $u \in C$ and define a contraction G_t on C by

$$G_t x = tu + (1-t)Tx, \quad \forall x \in C.$$

Let $z_t \in C$ be the unique fixed point of G_t. Thus

$$z_t = tu + (1-t)Tz_t. \tag{1.3}$$

(Such a sequence $\{z_t\}$ is said to be an approximating fixed point of T since it possesses the property that if $\{x_t\}$ is bounded, then $\lim_{t\to 0}\|Tz_t - z_t\| = 0$.) In 1967, the strong convergence of $\{z_t\}$ as $t \to 0$ for a self-mapping T of a bounded C was proved in a Hilbert space independently by Browder [2] and Halpern [7]. In 1980, Reich [18] extended the result of Browder [2] to a uniformly smooth Banach space. Ha and Jung [6] and Takahashi and Ueda [22] improved results of Reich [18] to a reflexive Banach space with a uniformly Gâteaux differentiable norm. Thereafter, Singh and Waston [19] extended result of Browder and Halpern to a nonexpansive nonself-mapping T satisfying Rothe's boundary condition : $T(\partial C) \subset C$ (here ∂C denotes the boundary of C). In 1995, Xu and Yin [27] proved that if C is a nonempty closed convex (not necessarily bounded) subset of Hilbert space H, if $T : C \to H$ is a nonexpansive nonself-mapping, and if $\{z_t\}$ is the sequence defined by (1.3) which is bounded, then $\{z_t\}$ converges strongly as $t \to 1$ to a fixed point of T. They also studied other schemes involving the nearest point projection P from H onto C, which were introduced by Marino and Trombetta [15]. Jung and Kim [10], Jung and Kim [11], Kim and Takahashi [13] and Xu [24] extended the corresponding results of Xu and Yin [27] to Banach space settings.

On the other hand, the viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi [16]. In 2004, in order to extend Theorem 2.2 of Moudafi [16] to a Banach space setting, Xu [26] consider the the following explicit iterative scheme: for $T : C \to C$ nonexpansive mappings, $f \in \Sigma_C$ and $\lambda_n \in (0,1)$,

$$x_{n+1} = \lambda_n f(x_n) + (1-\lambda_n)Tx_n, \quad \forall n \geq 0.$$

Moreover, in [26], he also studied the strong convergence of x_t defined by (1.2) as $t \to 0$ in either a Hilbert space or a uniformly smooth Banach space and showed that the strong $\lim_{t\to 0} x_t$ is the unique solution of certain variational inequality. This result of Xu [26] also improved Theorem 2.1 of Moudafi [16] as the continuous version. Very recently, using the sunny nonexpansive retraction Q from E onto C and $T : C \to E$ nonexpansive nonself-mappings, Song and Chen [20] considered the implicit iterative scheme

$$y_t = Q(tf(y_t) + (1-t)Ty_t)$$

and the explicit iterative scheme

$$y_{n+1} = Q(\lambda_n f(y_n) + (1-\lambda_n)Ty_n), \quad \forall n \geq 0,$$

and improved the results of Xu [26] to the case of nonself-mapping in a reflexive Banach space with a weakly sequentially continuous duality mapping.

In this paper, we establish the strong convergence of $\{x_t\}$ defined by (1.2) for $T : C \to E$ nonexpansive nonself-mapping in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, thus generalizing the results of Xu [26] (and Moudafi [16]) to the case of nonself-mappings. We also study the strong convergence of the implicit iterative scheme:

$$x_t = tf(x_t) + (1-t)QTx_t$$

and the explicit iterative scheme:

$$x_{n+1} = \lambda_n f(x_n) + (1-\lambda_n)QTx_n, \quad \forall n \geq 0$$

for the sunny nonexpansive retraction Q from E onto C and $T : C \to E$ nonexpansive nonself-mapping in a reflexive and strictly Banach space with a uniformly Gâteaux differentiable norm. Our results improve the corresponding results in Jung and Kim [10], Jung and Kim [11], Moudafi [16], Xu [24], Xu and Yin [27] and others.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $x^* \in E^*$ at $x \in E$ will be denoted by (x, x^*).

A Banach space E is called *strictly convex* if its unit sphere $U = \{x \in E : \|x\| = 1\}$ does not contain any linear segment. For every ε with $0 \leq \varepsilon \leq 2$, the modulus $\delta(\varepsilon)$ of convexity of E is defined by

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : \|x\| \leq 1, \quad \|y\| \leq 1, \quad \|x-y\| \geq \varepsilon\}.$$

E is said to be *uniformly convex* if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then E is reflexive and strictly convex.

The following lemma is well-known [1, p. 79].

Lemma 2.1. *Let C be a closed convex of a reflexive and strictly convex Banach space E. Then $C^o = \{x \in C : \|x\| = \inf\{\|y\| : y \in C\}\}$ is a singleton.*

The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x,\ y$ in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be *uniformly Gâteaux differentiable* if each $y \in U$, this limit is attained uniformly for $x \in U$. Finally, the norm is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit in (2.1) is attained uniformly for $(x, y) \in U \times U$. Since the dual E^* of E is uniformly convex if and only if the norm of E is uniformly Fréchet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gâteaux differentiable norm. The converse implication is false. A discussion of these and related concepts may be found in [3, 17].

The (normalized) *duality mapping* J from E into the family of nonempty (by Hahn-Banach theorem) weak-star compact subsets of its dual E^* is defined by

$$J(x) = \{f \in E^* : (x, f) = \|x\|^2 = \|f\|^2\}.$$

for each $x \in E$. It is single valued if and only if E is smooth. It is also well-known that if E has a uniformly Gâteaux differentiable norm, J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak-star topology of E^*. Suppose that J is single valued. Then J is said to be *weakly sequentially continuous* if, for each $\{x_n\} \in E$ with $x_n \rightharpoonup x$, $J(x_n) \overset{*}{\rightharpoonup} J(x)$ ([5]).

We need the following lemma for the proof of our main results, which was also given in Jung and Morales [12].

Lemma 2.2. *Let X be a Banach space and J the normalized duality mapping. Then, for any given $x,\ y \in X$, we have*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y).$$

Let μ be a continuous linear functional on l^∞ and $(a_0, a_1, \cdots) \in l^\infty$. We write $u_n(a_n)$ instead of $\mu((a_0, a_1, \cdots))$. μ is said to be *Banach limit* if μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $u_n(a_{n+1}) = \mu_n(a_n)$ for all $(a_0, a_1, \cdots) \in l^\infty$. We know that if μ is a Banach limit, then $\liminf_{n\to\infty} a_n \le \mu_n(a_n) \le \limsup_{n\to\infty} a_n$ for every $a = (a_1, a_2, ...) \in \ell^\infty$.

Let $\{x_n\}$ be a bounded sequence in E. Then we can define the real valued continuous convex function ϕ on E by

$$\phi(z) = \mu_n\|x_n - z\|^2, \quad \forall z \in E.$$

The following lemma which was given in [6, 22] is, in fact, a variant of Lemma 1.3 in [17].

Lemma 2.3. *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm and $\{x_n\}$ a bounded sequence in E. Let μ be a Banach limit and $u \in C$. Then*

$$\mu_n\|x_n - u\|^2 = \min_{y\in C} \mu_n\|x_n - y\|^2$$

if and only if $\mu_n(x - u, J(x_n - u)) \le 0$ for all $x \in C$.

We also need the following result which was essentially proved by Takahashi and Jeong [21] (see also [10, 11]).

Lemma 2.4. *Let E be a uniformly convex Banach space, C a nonempty closed convex subset of E, and $\{x_n\}$ a bounded sequence of E. Then the set*

$$M = \{u \in C : \mu_n \|x_n - u\|^2 = \min_{z \in C} \mu_n \|x_n - z\|^2\}$$

consists of one point.

Recall that a closed convex subset C of E is said to have the *fixed point property* for nonexpansive self-mappings (FPP for short) if every nonexpansive mapping $T : C \to C$ has a fixed point, that is, there is a point $p \in C$ such that $Tp = p$. It is well-known that every bounded closed convex subset of a uniformly convex Banach space has the FPP (cf. [4, p. 22]).

Let D be a subset of C. Then a mapping $Q : C \to D$ is said to be *retraction* from C onto D if $Qx = x$ for all $x \in D$. A retraction $Q : C \to D$ is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $t \geq 0$ and $Qx + t(x - Qx) \in C$. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive. Sunny nonexpansive retractions are characterized as follows [4, p. 48]: If E is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction in and only if the following inequality

$$\langle x - Qx, J(z - Qx)\rangle \leq o, \quad \forall x \in C,\ z \in D. \tag{2.2}$$

Let $I_C(x)$ be the inward set of a closed convex subset C of E at x given by

$$I_C(x) = \{z \in E : z = x + \lambda(y - x) \ \text{ for some } y \in C,\ \lambda \geq 0\}.$$

A nonself-mapping $T : C \to E$ is said to satisfy *the inwardness condition* if $Tx \in I_C(x)$ for all $x \in C$ and respectively, to satisfy *the weak inwardness condition* if $Tx \in \overline{I_C(x)}$ for all $x \in C$, where $\overline{I_C(x)}$ is the closure of $I_C(x)$ in the norm topology. Every self-mapping is trivially weakly inward.

Using the proof of Theorem 2 in Jung and Kim [10], we prove the following lemma.

Lemma 2.5. *Let E be a smooth Banach space and C a nonempty closed convex subset of E which is also a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Let $T : C \to E$ be a mapping satisfying the weak inwardness condition. Then $F(T) = F(QT)$.*

Proof. It is clear that $F(T) \subset F(QT)$. To show $F(QT) \subset F(T)$, let $z \in F(QT)$. Since Q is a suuny nonexpansive retraction, by (2.2), we have

$$\langle z - Tz, J(z - y)\rangle \leq 0, \quad \forall y \in C$$

On the other hand, $Tz \in \overline{I_C(z)}$ by the weak inwardness condition. Hence for each integer $n \geq 1$, there exists $z_n \in C$ and $\alpha_n \geq 0$ such that

$$y_n := z + \alpha_n(z_n - z) \to Tz \quad (n \to \infty).$$

Since the duality mapping J in smooth space is norm to weak* continuous, it follows that

$$\begin{aligned} 0 &\le \alpha_n \langle Tz - z, J(z_n - z)\rangle = \langle Tz - z, J(\alpha_n(z_n - z))\rangle \\ &= \langle Tz - z, J(y_n - z)\rangle \to \langle Tz - z, J(Tz - z)\rangle = -\|Tz - z\|^2. \end{aligned}$$

Hence $Tz = z$. The proof is complete. □

Finally, we need the following lemma, which is essentially Lemma 2 of Liu [14].

Lemma 2.6. *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \le (1 - \lambda_n)s_n + \lambda_n\beta_n + \gamma_n, \quad \forall n \ge 0,$$

where $\{\lambda_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfying the condition:

(i) $\{\lambda_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$ *or, equivalently,* $\prod_{n=0}^{\infty}(1 - \lambda_n) = 0$,

(ii) $\limsup_{n\to\infty} \beta_n \le 0$ *or* $\sum_{n=1}^{\infty} \lambda_n\beta_n < \infty$,

(iii) $\gamma_n \ge 0$ $(n \ge 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

3. Main results

First, we study the strong convergence of $\{x_t\}$ defined by (1.2).

Theorem 3.1. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to E$ a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that for $f \in \Sigma_C$ and $t \in (0, 1)$, the contraction $T_t := tf + (1 - t)T$ has a (unique) fixed point $x_t \in C$. If the fixed point set $F(T)$ of T is nonempty, then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point of T. If we define $Q : \Sigma_C \to F(T)$ by*

$$Q(f) := \lim_{t\to 0} x_t, \quad \forall f \in \Sigma_C, \tag{3.1}$$

then $Q(f)$ solves the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - p)\rangle \le 0, \quad \forall f \in \Sigma_C,\ p \in F(T).$$

Proof. Let $p \in F(T)$. First, we prove that $\{x_t\}$ is bounded. In fact, for $p \in F(T)$, we have

$$\begin{aligned} \|x_t - p\| &\le (1 - t)\|Tx_t - p\| + t\|f(x_t) - p\| \\ &\le (1 - t)\|x_t - p\| + t\|f(x_t) - p\| \end{aligned}$$

and so

$$\begin{aligned} \|x_t - p\| &\le \|f(x_t) - p\| \le \|f(x_t) - f(p)\| + \|f(p) - p\| \\ &\le k\|x_t - p\| + \|f(p) - p\|. \end{aligned}$$

Hence

$$\|x_t - p\| \leq \frac{1}{1-k}\|f(p) - p\|$$

and $\{x_t\}$ is bounded, so are $\{Tx_t\}$ and $\{f(x_t)\}$. As a result, it also follows that

$$\|x_t - Tx_t\| = t\|Tx_t - f(x_t)\| \to 0 \quad (\text{as } t \to 0). \tag{3.2}$$

We now show that $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point of T. To this end, let $t_n \to 0$ and $x_n = x_{t_n}$. Define $\phi : C \to [0, \infty)$ by $\phi(z) = \mu_n\|x_n - z\|^2$. Since ϕ is continuous and convex, $\phi(z) \to \infty$ as $\|z\| \to \infty$, and E is reflexive, ϕ attains its infimum over C (cf. [1, p. 79]). Let $z \in C$ be such that

$$\mu_n\|x_n - z\|^2 = \min_{y \in C} \mu_n\|x_n - y\|^2$$

and let

$$M = \{u \in C : \mu_n\|x_n - u\|^2 = \min_{y \in C} \mu_n\|x_n - y\|^2\}.$$

Then, by Lemma 2.4, we know that M consists of one point, say z. We must show that this z is a fixed point of T. Since T satisfies the weak inwardness condition, there are some $v_n \in C$ and $\lambda_n \geq 0$ such that

$$w_n ;= z + \lambda_n(v_n - z) \to Tz.$$

If $\lambda_n \leq 1$ for infinitely many n and these n, then we have $w_n \in C$ and hence $Tz \in C$. Since $\|x_n - Tz\| \leq \|x_n - Tx_n\| + \|x_n - z\|$, by (3.2) we have $\phi(Tz) \leq \phi(z)$ and $Tz = z$. So, we may assume $\lambda_n > 1$ for all sufficiently large n. We then write

$$v_n = r_n w_n + (1 - r_n)z,$$

where $r_n = \lambda_n^{-1}$. If $\{\lambda_n\}$ is bounded, then we have $v = rTz + (1-r)z$ for some cluster point r of $\{r_n\}$ and some weak cluster point $v \in C$ of $\{v_n\}$. By the convexity of ϕ, we obtain $\phi(v) \leq r\phi(Tz) + (1-r)\phi(z) \leq \phi(z)$ and hence $v = z$ and $Tz = z$. So assume that $\lambda_n \to \infty$. Then we must have $v_n \to z$ strongly. By Theorem 2 of [23], $\|\cdot\|^2$ is uniformly convex on any bounded subset of E; especially, we have a continuous increasing function $g = g_r : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$, with $g(0) = 0$, such that

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x - y\|)$$

for $0 \leq \lambda \leq 1$ and $x,\ y \in B_r$, where B_r is the closed ball centered at 0 and with radius r that is big enough so that B_r contains z and $\{w_n\}$. It follows that

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y) - \lambda(1-\lambda)g(\|x - y\|)$$

for $0 \leq \lambda \leq 1$ and $x,\ y \in B_r$. Noting $v_n \in C$, we derive that

$$\phi(z) \leq \phi(v_n) \leq r_n\phi(w_n) + (1 - r_n)\phi(z) - r_n(1 - r_n)g(\|w_n - z\|)$$

and hence $(1 - r_n)g(\|w_n - z\|) \leq \phi(w_n) - \phi(z)$. Taking limit as $n \to \infty$, by $\phi(Tz) \leq \phi(z)$, we obtain

$$g(\|Tz - z\|) \leq \phi(Tz) - \phi(z) \leq 0.$$

Therefore, $Tz = z$, that is, z is a fixed point of T. Moreover, by Lemma 2.3, we have

$$\mu_n\langle x - z, J(x_n - z)\rangle \le 0. \tag{3.3}$$

On the other hand, since

$$\begin{aligned} x_t - z &= t(f(x_t) - z) + (1-t)(Tx_t - z), \\ \|x_t - z\|^2 &= t\langle f(x_t) - z, J(x_t - z)\rangle + (1-t)\langle Tx_t - z, J(x_t - z)\rangle \\ &\le t\langle f(x_t) - z, J(x_t - z)\rangle + (1-t)\|x_t - z\|^2, \end{aligned}$$

we have

$$\begin{aligned} \|x_t - z\|^2 &\le \langle f(x_t) - z, J(x_t - z)\rangle \\ &= \langle f(x_t) - x, J(x_t - z)\rangle + \langle x - z, J(x_t - z)\rangle. \end{aligned}$$

Hence by (3.3), for $x \in C$,

$$\begin{aligned} \mu_n\|x_n - z\|^2 &\le \mu_n\langle f(x_n) - x, J(x_n - z)\rangle + \mu_n\langle x - z, J(x_n - z)\rangle \\ &\le \mu_n\langle f(x_n) - x, J(x_n - z)\rangle \\ &\le \mu_n\|f(x_n) - x\|\|x_n - z\|. \end{aligned}$$

In particular,

$$\mu_n\|x_n - z\|^2 \le \mu_n\|f(x_n) - f(z)\|\|x_n - z\| \le k\mu_n\|x_n - z\|^2.$$

Since $k \in (0,1)$, we have

$$\mu_n\|x_n - z\|^2 = 0.$$

Hence there exists a subsequence, which is still denoted $\{x_n\}$, such that $x_n \to z$.

Now suppose that there is another subsequence $\{x_j\}$ of $\{x_t^n\}$ such that $x_j \to q$. Then q is a fixed point of T by (3.2), that is, $q \in F(T)$. It follows from (3.3) that

$$\|q - z\|^2 \le f(q) - z, J(q - z)\rangle, \quad \text{and} \quad \|z - q\|^2 \le \langle f(z) - q, J(z - q)\rangle. \tag{3.4}$$

Adding two inequality in (3.4) yields

$$2\|z - q\|^2 \le \|z - q\|^2 + \langle f(z) - f(q), J(z - q)\rangle \le (1 + k)\|z - q\|^2.$$

Since $k \in (0,1)$, this implies that $z = q$. Hence $x_t \to q$ as $t \to 0$.

Define $Q : \Sigma_C \to F(T)$ by

$$Q(f) = \lim_{t\to 0} x_t.$$

Since $x_t = tf(x_t) + (1-t)Tx_t$, we have $(I - f)x_t = -\frac{1-t}{t}(I - T)x_t$. Hence for $p \in F(T)$,

$$\langle (I - f)x_t, J(x_t - p)\rangle = -\frac{1-t}{t}\langle (I - T)x_t - (I - T)p, J(x_t - p)\rangle \le 0.$$

Letting $t \to 0$ yields $\langle (I - f)Q(f), J(Q(f) - p)\rangle \le 0$. This completes the proof. □

Remark 3.1. In Theorem 3.1, if $f(x) = u$, $x \in C$, is a constant, then

$$\langle Q(u) - u, J(Q(u) - p)\rangle \leq 0, \quad \forall u \in C,\ p \in F(T).$$

Hence by (2.2), Q defined by (3.1) reduces to the sunny nonexpansive retraction from C to $F(T)$

Remark 3.2. (1) Theorem 3.1 generalizes the corresponding results of Moudafi [16] and Xu [26] to the case of nonself-mappings.

(2) Theorem 3.1 improves the corresponding results of Jung and Kim [11], Xu [24] and Reich [18] to the viscosity method.

(3) To guarantee the existence of a fixed point of the contraction T_t defined by (1.1), the weak inwardness condition upon the mapping T is used. In fact, it is well-known (cf. [4]) that if C, a bounded closed convex subset of a Banach space E, has the FPP and a nonexpansive $T : C \to E$ is weakly inward, then the contraction T_t does have a fixed point for every $t \in (0, 1)$. Hence we have the following corollary.

Corollary 3.1. *Let E, C, T be as in Theorem 3.1. Suppose in addition that C is bounded. For each $f \in \Sigma_C$ and $t \in (0, 1)$, let x_t be a unique element of C which satisfies*

$$x_t = tf(x_t) + (1 - t)Tx_t.$$

Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point of T. If we define $Q : \Sigma_C \to F(T)$ by

$$Q(f) := \lim_{t\to 0} x_t, \quad \forall f \in \Sigma_C,$$

then $Q(f)$ solves the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - p)\rangle \leq 0, \quad \forall f \in \Sigma_C,\ p \in F(T).$$

Remark 3.3. (1) Corollary 3.1 generalizes Corollary 1 of Xu and Yin [27] to the viscosity method in a Banach space.

(2) Since Rothe's boundary condition : $T(\partial C) \subset C$ implies the weak inwardness condition, Corollary 3.1 also improves upon Theorem of Singh and Waston [19] in the case of $f = u$ a constant.

Next, we denote by Q the sunny and nonexpansive retraction of E onto C. Now let $T : C \to E$ be nonexpansive and $f \in \Sigma_C$. Following Marino and Trombetta [15], we define the contraction $U_t := U_t^f$ and $S_t := S_t^f$ from C into itself by

$$U_t x = tf(x) + (1 - t)QTx, \quad \forall x \in C,\ t \in (0, 1).$$

Then Banach's contraction principle yields a unique point $x_t \in C$ (resp. $y_t \in C$) that is fixed by U_t, that is, we have

$$x_t = tf(x_t) + (1 - t)QTx_t, \quad \forall t \in (0, 1).$$

Theorem 3.2. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset* *of E, and $T : C \to E$ a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retact of E with Q as the sunny nonexpansive retraction, and that for each $t \in (0,1)$ and $f \in \Sigma_C$, x_t is a (unique) fixed point of the contraction $U_t := tf + (1-t)QT$. If the fixed point set $F(T)$ of T is nonempty, then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point of T. If we define $R : \Sigma_C \to F(T)$ by*

$$R(f) := \lim_{t \to 0} x_t, \quad \forall f \in \Sigma_C,$$

then $R(f)$ solves the variational inequality

$$\langle (I-f)R(f), J(R(f)-p)\rangle \le 0, \quad \forall f \in \Sigma_C,\ p \in F(T).$$

Proof. If the fixed point set $F(T)$ of T is nonempty, then $\{x_t\}$ is bounded. In fact, for $p \in F(T)$, we have

$$\begin{aligned}\|x_t - p\| &\le (1-t)\|QTx_t - QTp\| + t\|f(x_t) - p\| \\ &\le (1-t)\|x_t - p\| + t\|f(x_t) - p\|.\end{aligned}$$

Hence, as in the proof of Theorem 3.1, $\|x_t - p\| \le \frac{1}{1-k}\|f(p) - p\|$ and so $\{x_t\}$ is bounded.

Now, let $t_n \to 0$ and $x_n = x_{t_n}$. As in the proof of Theorem 3.1, we define the same function $\phi : C \to [0,\infty)$ by $\phi(z) = \mu_n\|x_n - z\|^2$ and let

$$M = \{x \in C : \mu_n\|x_n - x\|^2 = \min_{y \in C} \mu_n\|x_n - y\|^2\}.$$

Then M is invariant under QT. In fact, since

$$\|x_t - QTx_t\| = t\|QTx_t - x\| \to 0 \ \text{ as } t \to 0,$$

we have, for all $x \in M$,

$$\phi(QTx) = \mu_n\|x_n - QTx\|^2 = \mu_n\|QTx_n - QTx\|^2 \le \mu_n\|x_n - x\|^2 = \phi(x),$$

and hence $QTx \in M$ because $QTx \in C$. Furthermore, M contains a fixed point of QT. To this end, define

$$M^o = \{v \in M : \|v - w\| = \min_{y \in M} \|w - y\|\}.$$

Then, by Lemma 2.1, M^o is a singleton. Denote such a singleton by z. Then we have

$$\|QTz - w\| = \|QTz - QTw\| \le \|z - w\|$$

and hence $QTz = z$. Applying the method of the proof of Theorem 3.1 to the nonexpansive mapping QT, we have that $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point z of QT. By Lemma 2.5, z is a fixed point of T. The remainder of the proof follows from that in the proof of Theorem 3.1. □

Remark 3.4. Theorem 3.2 improves the corresponding results of Jung and Kim [10], Jung and Kim [11], and Xu and Yin [27] to the viscosity method.

Now we consider the explicit iterative scheme: for Q the sunny and nonexpansive retraction of E onto C, $T : C \to E$ nonexpansive nonself-mapping, $f \in \Sigma_C$ and $\lambda_n \in (0, 1)$,

$$\begin{cases} x_0 \in C \\ x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n)QTx_n \quad \forall n \geq 0. \end{cases} \tag{3.5}$$

Theorem 3.3. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to E$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$ satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction, and that for each $t \in (0, 1)$ and $f \in \Sigma_C$, $\{x_n\}$ is the sequence defined by* (3.5)*. Let $\{\lambda_n\}$ be a sequence in* $(0, 1)$ *which satisfies the conditions:*

(C1) $\lim_{n\to\infty} \lambda_n = 0$; (C2) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(C3) $|\lambda_{n+1} - \lambda_n| \leq \circ(\lambda_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$.

Then $\{x_n\}$ converges strongly to $R(f) \in F(T)$, where $R(f)$ which solves the variational inequality

$$\langle (I - f)R(f), J(R(f) - p)\rangle \leq 0, \quad \forall f \in \Sigma_C,\ p \in F(T).$$

Proof. We proceed with the following steps:

Step 1: $\|x_n - z\| \leq \max\{\|x_0 - z\|, \frac{1}{1-k}\|f(z) - z\|\}$ for all $n \geq 0$ and all $z \in Fix(T)$ and so $\{x_n\}$ is bounded.

We use an inductive argument. Indeed, let $z \in Fix(T)$ and $d = \max\{\|x_0 - z\|, \frac{1}{1-k}\|f(z) - z\|\}$. Then by the nonexpansivity of T and $f \in \Sigma_C$,

$$\begin{aligned} \|x_1 - z\| &\leq (1 - \lambda_0)\|QTx_0 - QTz\| + \lambda_0\|f(x_0) - z\| \\ &\leq (1 - \lambda_0)\|x_0 - z\| + \lambda_0(\|f(x_0) - f(z)\| + \|f(z) - z\|) \\ &\leq (1 - (1 - k)\lambda_0)\|x_0 - z\| + \lambda_0\|f(z) - z\| \\ &\leq (1 - (1 - k)\lambda_0)d + \lambda_0(1 - k)d = d. \end{aligned}$$

Using an induction, we obtain

$$\|x_{n+1} - z\| \leq d, \quad \forall n \geq 0.$$

Hence, it follows that $\{x_n\}$ is bounded, and so are $\{QTx_n\}$ and $\{f(x_n)\}$.

Step 2: $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. By Step 1 above, there exists a constant $L > 0$ such that for all $n \geq 0$, $\|f(x_n)\| + \|QTx_n\| \leq L$. Since for all $n \geq 0$, we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad =\|(1 - \lambda_n)(QTx_n - QTx_{n-1}) \\
&\qquad + (\lambda_n - \lambda_{n-1})(f(x_{n-1}) - QTx_{n-1}) + \lambda_n(f(x_n) - f(x_{n-1}))\| \\
&\quad \le (1 - \lambda_n)\|x_n - x_{n-1}\| + L|\lambda_n - \lambda_{n-1}| + k\lambda_n\|x_n - x_{n-1}\| \\
&\quad \le (1 - (1 - k)\lambda_n)\|x_n - x_{n-1}\| + (\circ(\lambda_n) + \sigma_{n-1})L.
\end{aligned}$$

By taking $s_{n+1} = \|x_{n+1} - x_n\|$, $\alpha_n = (1 - k)\lambda_n$, $\alpha_n\beta_n = \circ(\lambda_n)L$ and $\gamma_n = \sigma_{n-1}L$, we have

$$s_{n+1} \le (1 - \alpha_n)s_n + \alpha_n\beta_n + \gamma_n,$$

and, by Lemma 2.6,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Step 3: $\lim_{n\to\infty} \|x_{n+1} - QTx_n\| = 0$. Indeed, since

$$\|x_{n+1} - QTx_n\| = \lambda_n\|QTx_n - f(x_n)\| \le L\lambda_n$$

for some L, by (C1), we have $\lim_{n\to\infty} \|x_{n+1} - QTx_n\| = 0$.

Step 4: $\lim_{n\to\infty} \|x_n - QTx_n\| = 0$. Indeed, by Step 2 and Step 3,

$$\|x_n - QTx_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - QTx_n\| \to 0.$$

Step 5: $\limsup_{n\to\infty}\langle(I-f)R(f), J(R(f)-x_n)\rangle \le 0$, where $R(f) = \lim_{t\to 0} x_t$, $(x_t = tf(x_t) + (1 - t)QTx_t)$, solves the variational inequality

$$\langle(I - f)R(f), J(R(f) - p)\rangle \le 0. \quad \forall f \in \Sigma_C,\ p \in Fix(T).$$

Indeed we can write

$$x_t - x_n = t(f(x_t) - x_n) + (1 - t)(QTx_t - x_n).$$

Putting $a_n(t) = \|QTx_n - x_n\|(2\|x_t - x_n\| + \|QTx_n - x_n\|) \to 0$ $(n \to \infty)$ and using Lemma 2.2, we obtain

$$\begin{aligned}
\|x_t - x_n\|^2 &\le (1 - t)^2\|QTx_t - x_n\|^2 + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\
&\le (1 - t)^2(\|QTx_t - QTx_n\| + \|QTx_n - x_n\|)^2 \\
&\qquad + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2 \\
&\le (1 - t)^2\|x_t - x_n\|^2 + a_n(t) \\
&\qquad + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2.
\end{aligned}$$

The last inequality implies

$$\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2} + \frac{1}{2t}a_n(t).$$

It follows that

$$\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le M\frac{t}{2}, \tag{3.6}$$

where $M > 0$ is a constant such that $M \ge \|x_t - x_n\|^2$ for all $n \ge 0$ and $t \in (0, 1)$. Taking the lim sup as $t \to 0$ in (3.6) and noting the fact that J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak-star topology of E^*, we have

$$\limsup_{n\to\infty}\langle (I-f)R(f), J(R(f)-x_n)\rangle \le 0.$$

Step 6: $\lim_{n\to\infty}\|x_n - R(f)\| = 0$. By using (3.5), we have

$$x_{n+1} - R(f) = \lambda_n(f(x_n) - R(f)) + (1-\lambda_n)(QTx_n - R(f)).$$

Applying Lemma 1, we obtain

$$\begin{aligned}
&\|x_{n+1} - R(f)\|^2\\
&\le (1-\lambda_n)^2\|QTx_n - R(f)\|^2 + 2\lambda_n\langle f(x_n) - R(f), J(x_{n+1} - R(f))\rangle\\
&\le (1-\lambda_n)^2\|x_n - R(f)\|^2 + 2\lambda_n\langle f(x_n) - f(R(f)), J(x_{n+1} - R(f))\rangle\\
&\quad + 2\lambda_n\langle f(R(f)) - R(f), J(x_{n+1} - R(f))\rangle\\
&\le (1-\lambda_n)^2\|x_n - R(f)\|^2 + 2k\lambda_n\|x_n - R(f)\|\|x_{n+1} - R(f)\|\\
&\quad + 2\lambda_n\langle f(R(f)) - R(f), J(x_{n+1} - R(f))\rangle\\
&\le (1-\lambda_n)^2\|x_n - R(f)\|^2 + k\lambda_n(\|x_n - R(f)\|^2 + \|x_{n+1} - R(f)\|^2)\\
&\quad + 2\lambda_n\langle f(R(f)) - R(f), J(x_{n+1} - R(f))\rangle.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\|x_{n+1} - R(f)\|^2 &\le \frac{1-(2-k)\lambda_n + \lambda_n^2}{1-k\lambda_n}\|x_n - R(f)\|^2\\
&\quad + \frac{2\lambda_n}{1-k\lambda_n}\langle f(R(f)) - R(f), J(x_{n+1} - R(f))\rangle\\
&\le \frac{1-(2-k)\lambda_n}{1-k\lambda_n}\|x_n - R(f)\|^2 + \frac{\lambda_n^2}{1-k\lambda_n}M\\
&\quad + \frac{2\lambda_n}{1-k\lambda_n}\langle f(R(f)) - R(f), J(x_{n+1} - R(f))\rangle,
\end{aligned} \tag{3.7}$$

where $M = \sup_{n\ge 0}\|x_n - R(f)\|^2$. Put

$$\begin{aligned}
\alpha_n &= \frac{2(1-k)\lambda_n}{1-k\lambda_n},\\
\beta_n &= \frac{M\lambda_n}{2(1-k)} + \frac{1}{1-k}\langle f(R(f)) - R(f), J(x_{n+1} - R(f))\rangle.
\end{aligned}$$

From (C1), (C2) and Step 5, it follows that $\alpha_n \to 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$ and

$$\limsup_{n\to\infty}\beta_n \le 0.$$

Since (3.7) reduces to

$$\|x_{n+1} - R(f)\|^2 \le (1-\alpha_n)\|x_n - R(f)\|^2 + \alpha_n\beta_n,$$

from Lemma 2.6, we conclude that $\lim_{n\to\infty}\|x_n - R(f)\| = 0$. This completes the proof. □

Remark 3.5. (1) Theorem 3.3 improves Theorem 4.2 of Xu [26] (and Theorem 2.2 of Moudafi [16]) to the case of nonself-mappings.

(2) Condition (C3) on $\{\lambda_n\}$ in Theorem 3.3 is independent of Xu's condition [25, 26]:

$$\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty \quad \text{or} \quad \lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1.$$

For this fact, see [8, 9].

(3) Our results apply to all uniformly convex and uniformly smooth Banach spaces and in particular, to all L^p spaces, $1 < p < \infty$.

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