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ON CHAOTIC OPERATOR ORDER $A \gg C \gg B$ IN HILBERT SPACES

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Abstract. In this paper, we characterize the chaotic operator order $A \gg C \gg B$. Consequently all other possible characterizations follow easily. Some satellite theorems of the Furuta inequality are naturally given. And finally, using results of characterizing $A \gg C \gg B$, and by the Douglas's majorization and factorization theorem we are able to characterize the chaotic operator order $A \gg B$ in terms of operator equalities.

1. Introduction

We shall use the capital letters throughout this paper to denote bounded linear operators on a Hilbert space and I the identity operator. That $T \geq O$ means T is a positive operator and a positive and invertible operator A is denoted by $A > O$. For $A, B > O$, we write $\log A \geq \log B$ by $A \gg B$ in short, which is a standard chaotic operator order and is weaker than the operator order $A \geq B$. Let us recall the well-known classical Löwner-Heinz inequality as it is used frequently in this paper: If $S \geq T \geq O$, then $S^\alpha \geq T^\alpha$ for $\alpha \in [0, 1]$. Essentially, the Löwner-Heinz inequality does not hold in general if $\alpha > 1$. It is easily seen that $A \gg B$ does not imply $A \geq B$ in general.

The contents of this paper has been arranged as follows. In section 2, we characterize the chaotic operator order $A \gg C \gg B$ due to Theorem FM below by Fujii et al. [4], and we provide as many equivalent statements as possible. In section 3, using the same method as in section 2 it follows that all other possible characterizations of $A \gg C \gg B$ are obtained. In section 4, we construct two formulas, and use them to prove some satellite theorems of Furuta. Finally, in section 5, due to the Douglas's theorem [1] we present characterizations of $A \gg B$ in terms of operator equalities.

Since we are considering the logarithm of operators, in what follows we assume that $A, B, C, X, Y > O$. First we require three known results as follows.

Theorem FC. ([3, 12, Chaotic Furuta inequality]) *$A \gg B$ if and only if*

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(a$_0$) $A^r \geq (A^{r/2}B^pA^{r/2})^{\frac{r}{p+r}}$ *hollds for all* $p \geq 0$ *and* $r \geq 0$.

The Furuta question about characterization of the chaotic operator order $A \gg B$ [9, (Q)2] was completely solved by Fujii et al. in [4]. Instead of $A \gg B$ in the Furuta's question, it is true that $A \geq B$ if and only if $A^{r-t} \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{r-t}{(p-t)s+r}}$ holds for all $p \geq 1$, $r \geq t$, $s \geq 1$ and $t \in [0,1]$ [4, Theorem 1]. In the same paper they also claimed and proved the next result.

Theorem FM. ([4, Theorem 5]) $A \gg B$ *if and only if*

(b$_0$) $A^{r-t} \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{r-t}{(p-t)s+r}}$ *hollds for all* $p \geq 0$, $r \geq 0$, $s \in [1,2]$ *and* $t \leq 0$.

Obviously, Theorem FM is a generalization of Theorem FC, and the proof of Theorem FM requires some nontrivial results including Theorem FC. Since $A \gg B$ if and only if $B^{-1} \gg A^{-1}$, we remark that (a$_0$) and (b$_0$) are equivalent, respectively, to the operator inequalities

(a$_1$) $(B^{r/2}A^pB^{r/2})^{\frac{r}{p+r}} \geq B^r$ *hollds for all* $p \geq 0$ *and* $r \geq 0$;

(b$_1$) $[B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{r-t}{(p-t)s+r}} \geq B^{r-t}$ *hollds for all* $p \geq 0$, $r \geq 0$, $s \in [1,2]$ *and* $t \leq 0$.

Theorem D. ([1, Douglas's majorization and factorization theorem]) *For any operators A and B the following statements are equivalent.*

(c$_1$) *range*$(B) \subseteq$ *range*(A);

(c$_2$) A^* *majorizes* B^*, *i.e.,* $BB^* \leq \lambda^2 AA^*$, *i.e.,* $\| B^*x \| \leq \lambda \| A^*x \|$ *for some* $\lambda \geq 0$ *and all* $x \in H$ (*majorization*);

(c$_3$) *There exists C such that* $B = AC$ (*factorization*).

Moreover, $\| C \|^2 = \inf \{\mu \mid BB^* \leq \mu AA^*\}$ (*due to the equivalence of* (c$_2$) *and* (c$_3$)).

2. Characterizations of chaotic operator order $A \gg C \gg B$ by Theorem FM

We shall use Theorem FM to give different characterizations of the chaotic operator order $A \gg C \gg B$ in this section.

Theorem 2.1. *For all* $p \geq 0$, $r \geq 0$, $s \in [1,2]$ *and* $t \leq 0$, *the following are equivalent to one another.*

(2.1) $A \gg C \gg B$;

(b$_0$) $A^{r-t} \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{r-t}{(p-t)s+r}}$;

(b$_1$) $[B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{r-t}{(p-t)s+r}} \geq B^{r-t}$;

(2.2) $A^{\frac{(p-t)s+r}{q}} \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{1/q}$ *for all* $q \geq 1$ *such that*

$(r-t)q \geq (p-t)s+r$;

(2.3) $[B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{1/q} \geq B^{\frac{(p-t)s+r}{q}}$ *for all* $q \geq 1$ *such that* $(r-t)q \geq (p-t)s+r$;

(2.4) $[C^{r/2}(C^{-t/2}A^pC^{-t/2})^sC^{r/2}]^{\frac{r-t}{(p-t)s+r}} \geq [C^{r/2}(C^{-t/2}C^pC^{-t/2})^sC^{r/2}]^{\frac{r-t}{(p-t)s+r}}$
$\geq [C^{r/2}(C^{-t/2}B^pC^{-t/2})^sC^{r/2}]^{\frac{r-t}{(p-t)s+r}}$;

(2.5) $C^{r/2}(C^{-t/2}A^pC^{-t/2})^sC^{r/2} \gg C^{r/2}(C^{-t/2}C^pC^{-t/2})^sC^{r/2}$
$\gg C^{r/2}(C^{-t/2}B^pC^{-t/2})^sC^{r/2}$;

(2.6) $[C^{r/2}(C^{-t/2}A^pC^{-t/2})^sC^{r/2}]^{1/q} \geq [C^{r/2}(C^{-t/2}C^pC^{-t/2})^sC^{r/2}]^{1/q}$
$\geq [C^{r/2}(C^{-t/2}B^pC^{-t/2})^sC^{r/2}]^{1/q}$ *for all* $q \geq 1$ *such that* $(r-t)q \geq (p-t)s+r$.

Proof. That (2.1)⇔(b_0)⇔(b_1) is due to Theorem FM and a remark.

(b_0)⇒(2.2). By (b_0), we have

$$A^{\alpha(r-t)} \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{\alpha(r-t)}{(p-t)s+r}} \text{ for all } \alpha \in [0,1].$$

Letting $\frac{\alpha(r-t)}{(p-t)s+r} = \frac{1}{q}$ in above yields (2.2). Since $\frac{r-t}{(p-t)s+r} = \frac{1}{\alpha q} \geq \frac{1}{q}$, $(r-t)q \geq (p-t)s+r \geq r-t$ and $q \geq 1$. (2.2)⇒(b_0). Let $(r-t)q = (p-t)s+r$ in (2.2). Similarly we have (b_1)⇔(2.3) and (2.2)⇔(2.3). (b_0) and (b_1)⇒(2.4). Let $B = C$ in (b_1) and $A = C$ in (b_0). (2.4)⇒(2.5). Since the chaotic operator order is weaker than the operator order, (2.5)⇒(2.1). Let $r = t$ (i.e., $r = t = 0$) and $p = s = 1$ in (2.5). Implications (2.2) and (2.3)⇒(2.6) and (2.6)⇒(2.5) are easy now. The proof is thus completed. □

Let $s = 1$ and $t = 0$ in Theorem 2.1. Then we have the next result. Notice that equivalence of statements (2.1), (2.9) and (2.10) below appeared in [7, Theorem 1].

Corollary 2.2. *For all* $p \geq 0$ *and* $r \geq 0$, *the following are equivalent to one another.*

(2.1) $A \gg C \gg B$;

(a_0) $A^r \geq (A^{r/2}B^pA^{r/2})^{\frac{r}{p+r}}$;

(a_1) $(B^{r/2}A^pB^{r/2})^{\frac{r}{p+r}} \geq B^r$;

(2.7) $A^{\frac{p+r}{q}} \geq (A^{r/2}B^pA^{r/2})^{1/q}$ *for all* $q \geq 1$ *such that* $rq \geq p+r$;

(2.8) $(B^{r/2}A^pB^{r/2})^{1/q} \geq B^{\frac{p+r}{q}}$ *for all* $q \geq 1$ *such that* $rq \geq p+r$;

(2.9) $(C^{r/2}A^pC^{r/2})^{\frac{r}{p+r}} \geq (C^{r/2}C^pC^{r/2})^{\frac{r}{p+r}} \geq (C^{r/2}B^pC^{r/2})^{\frac{r}{p+r}}$;

(2.10) $C^{r/2}A^pC^{r/2} \gg C^{r/2}C^pC^{r/2} \gg C^{r/2}B^pC^{r/2}$;

(2.11) $(C^{r/2}A^pC^{r/2})^{1/q} \geq (C^{r/2}C^pC^{r/2})^{1/q} \geq (C^{r/2}B^pC^{r/2})^{1/q}$ *for all* $q \geq 1$ *such that* $rq \geq p+r$.

In relation, to Corollary 2.2, we mention that it was proved in [3, Theorem 1] that, if $A \gg B$, then

$$(B^rA^pB^r)^{\frac{2r}{p+2r}} \geq B^{2r}, \text{ equivalently, } A^{2r} \geq (A^rB^pA^r)^{\frac{2r}{p+2r}}$$

hold for all $p \geq 0$ and $r \geq 0$. The original proof was relatively long and complex.

3. Other possible characterizations of chaotic operator order $A \gg C \gg B$ due to Theorem 2.1

Reformations of operator inequalities (b_0) and (b_1) in Theorem 2.1 are the keynotes in this section. The proofs of the following three results are similar to the proof of Theorem 2.1 and we should provid for each one the outline only.

Theorem 3.1. *For all $p \geq 0$, $r \geq 0$, $s \in [1, 2]$ and $t \leq 0$, the following are equivalent to one another.*

(2.1) $A \gg C \gg B$;

(3.1) $A^r \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{r}{(p-t)s+r}}$;

(3.2) $[B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{r}{(p-t)s+r}} \geq B^r$;

(3.3) $A^{\frac{(p-t)s+r}{q}} \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{1/q}$ *for all $q \geq 1$ such that* $rq \geq (p-t)s + r$;

(3.4) $[B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{1/q} \geq B^{\frac{(p-t)s+r}{q}}$ *for all $q \geq 1$ such that* $rq \geq (p-t)s + r$;

(3.5) $[C^{r/2}(C^{-t/2}A^pC^{-t/2})^sC^{r/2}]^{\frac{r}{(p-t)s+r}} \geq [C^{r/2}(C^{-t/2}C^pC^{-t/2})^sC^{r/2}]^{\frac{r}{(p-t)s+r}}$ $\geq [C^{r/2}(C^{-t/2}B^pC^{-t/2})^sC^{r/2}]^{\frac{r}{(p-t)s+r}}$;

(3.6) $C^{r/2}(C^{-t/2}A^pC^{-t/2})^sC^{r/2} \gg C^{r/2}(C^{-t/2}C^pC^{-t/2})^sC^{r/2}$ $\gg C^{r/2}(C^{-t/2}B^pC^{-t/2})^sC^{r/2}$;

(3.7) $[C^{r/2}(C^{-t/2}A^pC^{-t/2})^sC^{r/2}]^{1/q} \geq [C^{r/2}(C^{-t/2}C^pC^{-t/2})^sC^{r/2}]^{1/q}$ $\geq [C^{r/2}(C^{-t/2}B^pC^{-t/2})^sC^{r/2}]^{1/q}$ *for all $q \geq 1$ such that $rq \geq (p-t)s+r$.*

Proof. By (b_0) in Theorem 2.1, since $\frac{r}{r-t} \leq 1$, we have (3.1) and so (2.1)⇒(3.1) ⇔(3.2). (3.1)⇒(3.3). By (3.1) we have

$$A^{\alpha r} \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{\alpha r}{(p-t)s+r}} \text{ for all } \alpha \in [0, 1].$$

Let $\frac{\alpha r}{(p-t)s+r} = \frac{1}{q}$ etc. (3.3)⇒(3.1). Let $rq = (p-t)s + r$ in (3.3) and so the proof is finished. □

Note that Theorem 3.1 also implies Corollary 2.2 by letting $s = 1$ and $t = 0$ in Theorem 3.1.

Theorem 3.2. *For all $p \geq 0$, $r \geq 0$, $s \in [1, 2]$ and $t \leq 0$, the following are equivalent to one another.*

(2.1) $A \gg C \gg B$;

(3.8) $A^{-t} \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{-t}{(p-t)s+r}}$;

(3.9) $[B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{-t}{(p-t)s+r}} \geq B^{-t}$;

(3.10) $A^{\frac{(p-t)s+r}{q}} \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{1/q}$ *for all* $q \geq 1$ *such that* $-tq \geq (p-t)s+r$;

(3.11) $[B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{1/q} \geq B^{\frac{(p-t)s+r}{q}}$ *for all* $q \geq 1$ *such that* $-tq \geq (p-t)s+r$;

(3.12) $[C^{r/2}(C^{-t/2}A^pC^{-t/2})^sC^{r/2}]^{\frac{-t}{(p-t)s+r}} \geq [C^{r/2}(C^{-t/2}C^pC^{-t/2})^sC^{r/2}]^{\frac{-t}{(p-t)s+r}}$
$\geq [C^{r/2}(C^{-t/2}B^pC^{-t/2})^sC^{r/2}]^{\frac{-t}{(p-t)s+r}}$;

(3.13) $C^{r/2}(C^{-t/2}A^pC^{-t/2})^sC^{r/2} \gg C^{r/2}(C^{-t/2}C^pC^{-t/2})^sC^{r/2}$
$\gg C^{r/2}(C^{-t/2}B^pC^{-t/2})^sC^{r/2}$;

(3.14) $[C^{r/2}(C^{-t/2}A^pC^{-t/2})^sC^{r/2}]^{1/q} \geq [C^{r/2}(C^{-t/2}C^pC^{-t/2})^sC^{r/2}]^{1/q}$
$\geq [C^{r/2}(C^{-t/2}B^pC^{-t/2})^sC^{r/2}]^{1/q}$ *for all* $q \geq 1$ *such that* $-tq \geq (p-t)s+r$.

Proof. By (b_0) in Theorem 2.1, since $\frac{-t}{r-t} \leq 1$, we have (3.8) and so (2.1)⇒(3.8) ⇔(3.9). (3.8)⇒(3.10). By (3.8) we have

$$A^{-\alpha t} \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{-\alpha t}{(p-t)s+r}} \text{ for all } \alpha \in [0,1].$$

Let $\frac{-\alpha t}{(p-t)s+r} = \frac{1}{q}$, etc. (3.10)⇒(3.8). Let $-tq = (p-t)s+r$ in (3.10). □

In order to satisfy both conditions $\frac{1}{r-t} \leq 1$ and $q \geq 1$ in the proof of the next result, we have to assume $r \geq 1$ (instead of $r \geq 0$). Nevertheless, statemants in Theorem 3.3 below (similar to statements in previous results) are generally not all equivalent to one another.

Theorem 3.3. *For all* $p \geq 0$, $r \geq 1$, $s \in [1,2]$ *and* $t \leq 0$, *consider the following statements.*

(2.1) $A \gg C \gg B$;

(3.15) $A \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1}{(p-t)s+r}}$;

(3.16) $[B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{1}{(p-t)s+r}} \geq B$;

(3.17) $A^{\frac{(p-t)s+r}{q}} \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{1/q}$ *for all* $q \geq 1$ *such that* $q \geq (p-t)s+r$;

(3.18) $[B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{1/q} \geq B^{\frac{(p-t)s+r}{q}}$ *for all* $q \geq 1$ *such that* $q \geq (p-t)s+r$;

(3.19) $[C^{r/2}(C^{-t/2}A^pC^{-t/2})^sC^{r/2}]^{\frac{1}{(p-t)s+r}}$
$\geq [C^{r/2}(C^{-t/2}C^pC^{-t/2})^sC^{r/2}]^{\frac{1}{(p-t)s+r}}$
$\geq [C^{r/2}(C^{-t/2}B^pC^{-t/2})^sC^{r/2}]^{\frac{1}{(p-t)s+r}}$;

(3.20) $C^{r/2}(C^{-t/2}A^pC^{-t/2})^sC^{r/2} \gg C^{r/2}(C^{-t/2}C^pC^{-t/2})^sC^{r/2}$
$\gg C^{r/2}(C^{-t/2}B^pC^{-t/2})^sC^{r/2}$;

(3.21) $[C^{r/2}(C^{-t/2}A^pC^{-t/2})^sC^{r/2}]^{1/q} \geq [C^{r/2}(C^{-t/2}C^pC^{-t/2})^sC^{r/2}]^{1/q}$

$$\geq [C^{r/2}(C^{-t/2}B^pC^{-t/2})^sC^{r/2}]^{1/q} \text{ for all } q \geq 1 \text{ such that } q \geq (p-t)s+r.$$

Then we conclude that (2.1)⇒(3.15), *and statements* (3.15), (3.16), (3.17) *and* (3.18) *are equivalent to one another. Moreover, the following implications hold:* (3.15)⇒(3.19)⇒(3.20) *and* (3.18)⇒(3.21)⇒(3.20).

Proof. By (b_0) in Theorem 2.1, since $\frac{1}{r-t} \leq 1$, we have (3.15) and so (2.1)⇒ (3.15)⇔(3.16). (3.15)⇒(3.17). By (3.15) we have

$$A^\alpha \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{\alpha}{(p-t)s+r}} \text{ for all } \alpha \in [0,1].$$

Let $\frac{\alpha}{(p-t)s+r} = \frac{1}{q}$, etc. Then $q \geq (p-t)s+r \geq 1$ as $r \geq 1$. (3.17)⇒(3.15). Let $q = (p-t)s+r$ in (3.17).

All other implications can be proved as before. This completes the proof. □

The next result was proved in [2, Lemma 1] by making use of the induction process. In fact, the result is a simple special case of Theorem 3.3.

Corollary 3.4. ([2, Lemma 1]) *If* $A \gg B$ *and for all possitive integers* n, *we have*

$$(B^{n/2}A^2B^{n/2})^{\frac{1}{n+2}} \geq B.$$

Proof. Let $p = 2$, $r = n$, $s = 1$ and $t = 0$ in (3.16) of Theorem 3.3. □

4. Some satellite theorems of Furuta

The α-geometric mean of operators A and B introduced by Kubo-Ando [13] is given by

$$A\sharp_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2} \text{ for all } \alpha \in [0,1].$$

The binary operation $\natural_\beta$ for A and B is defined the same as $\sharp_\alpha$ for any real number β, i.e., $A \natural_\beta B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$. Due to the α-Geometric mean of two operators there are two types of the so called satellite theorem of Furuta, and both are indicated as follows.

(j) ([11, Satellite theorm of Furuta inequality]) *If* $A \geq B \geq O$, *then*

$$A^{-r}\sharp_{\frac{1+r}{p+r}}B^p \leq B \leq A \leq B^{-r} \natural_{\frac{1+r}{p+r}} A^p \text{ holds for all } r \geq 0 \text{ and } p \geq 1.$$

(jj) ([12, Satellite theorem of chaotic Furuta inequality]) *If* $A \gg B$ *for* $A, B > O$, *then*

$$A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \ll A \leq B^{-r} \natural_{\frac{1+r}{p+r}} A^p \text{ holds for all } r \geq 0 \text{ and } p \geq 1.$$

It is clear now that in the above satellite theorems, if (jj) holds true, then so does (j). It follows that Theorem FC in section 1 may be expressed as follows: For $p \geq 0$ and $r \geq 0$, and if $A \gg B$, then

$$A^{-r}\sharp_{\frac{r}{p+r}}B^p \leq I \leq B^{-r} \natural_{\frac{r}{p+r}} A^p.$$

In this section, we construct two formulas, Theorem 4.1 and 4.3 below, and use them to prove some satellite theorems. First, the following well-known results are required.

Theorem F. ([Furuta Inequality]) *If* $A \geq B \geq O$, *then, for each* $r \geq 0$, *both inequalities*

(a) $A^{\frac{p+r}{q}} \geq (A^{r/2}B^pA^{r/2})^{1/q}$:

(b) $(B^{r/2}A^pB^{r/2})^{1/q} \geq B^{\frac{p+r}{q}}$ *hold for* $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p+r$.

Lemma F. ([3, Furuta Lemma]) *For any real number* r, *the next equality holds*

$(BAB)^r = BA^{1/2}(A^{1/2}B^2A^{1/2})^{r-1}A^{1/2}B.$

Theorem GF. ([8, Grand Furuta inequality]) *If* $A \geq B$, *then, for each* $t \in [0,1]$,

$A^{1-t+r} \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1-t+r}{(p-t)s+r}}$ t *holds for all* $s \geq 1$, $p \geq 1$ *and* $r \geq t$.

Theorem 4.1. *Given operators* A, B *and for, any real numbers* a, b, c *and* d, *the following hold.*

(4.1) *If* $(B^{b/2}A^{-a}B^{b/2})^{c-1} \leq B^d$, *then* $A^a \natural_c B^b \leq B^{b+d}$.

(4.2) *If* $(A^{b/2}B^{-a}A^{b/2})^{c-1} \geq A^d$, *then* $A^{b+d} \leq B^a \natural_c A^b$.

Proof. (4.1) By assumption, we have

$$\begin{aligned} & A^a \natural_c B^b \\ &= A^{a/2}(A^{-a/2}B^bA^{-a/2})^cA^{a/2} \\ &= A^{a/2}A^{-a/2}B^{b/2}(B^{b/2}A^{-a}B^{b/2})^{c-1}B^{b/2}A^{-a/2}A^{a/2} \text{ by Lemma F} \\ &= B^{b/2}(B^{b/2}A^{-a}B^{b/2})^{c-1}B^{b/2} \\ &\leq B^{b+d}. \end{aligned}$$

That (4.2) follows similarly. □

Now, let us give alternative proofs of satellite theorems (j) and (jj) in above.

(j) If $A \geq B$, then $(B^{r/2}A^pB^{r/2})^{\frac{1+r}{p+r}} \geq B^{1+r}$ for all $p \geq 1$ and $r \geq 0$ by (b) in Theorem F, i.e., $B^{-(1+r)} \geq (B^{r/2}A^pB^{r/2})^{-\frac{1+r}{p+r}}$. Then $B^{1-r} \geq (B^{r/2}A^pB^{r/2})^{\frac{1-r}{p+r}}$ as $0 \leq \frac{r-1}{1+r} < 1$ for $r \geq 1$. Now, let $d = 1-r$, $b = r$, $-a = p$ and $c-1 = \frac{1-r}{p+r}$ in Theorem 4.1. Then $A^{-p} \sharp_{\frac{1+p}{p+r}} B^r \leq B \leq A \leq B^{-p} \sharp_{\frac{1+p}{p+r}} A^r$. The first inequality in (j) follows by interchanging of r with p.

(jj) By (a_1) in Theorem FC, $(B^{r/2}A^pB^{r/2})^{\frac{r}{p+r}} \geq B^r$ for all $p \geq 0$ and $r \geq 0$. Then $B^{-r} \geq (B^{r/2}A^pB^{r/2})^{\frac{-r}{p+r}}$, so that $B^{1-r} \geq (B^{r/2}A^pB^{r/2})^{\frac{1-r}{p+r}}$ as $0 \leq \frac{r-1}{r} < 1$ for all $r \geq 1$. the rest of the proof is the same as (j) above.

It is worth noting from the proofs above that the idea is to make $b+d=1$ by applying the Löwner-Heinz inequality to the existing operator inequality.

Corollary 4.2. *If $A \gg B$, then, for all $p \geq 0$, $r \geq 1$ and $t \leq 0$,*

$$A^{-p} \natural_{\frac{1+p}{p-t+r}} B^{r-t} \leq B \ll A \leq B^{-p} \natural_{\frac{1+p}{p-t+r}} A^{r-t}.$$

Moreover, the inequality holds, too, if $A \geq B$ and for all $p \geq 0$, $r \geq 1$ and $t \leq 0$.

Proof. In (b_1) of Theorem FM, let $s=1$, then $(B^{\frac{r-t}{2}} A^p B^{\frac{r-t}{2}})^{\frac{r-t}{p-t+r}} \geq B^{r-t}$ for all $p \geq 0$, $r \geq 0$ and $t \leq 0$, i.e., $B^{-(r-t)} \geq (B^{\frac{r-t}{2}} A^p B^{\frac{r-t}{2}})^{\frac{-(r-t)}{p-t+r}}$, which yields $B^{1+t-r} \geq (B^{\frac{r-t}{2}} A^p B^{\frac{r-t}{2}})^{\frac{1+t-r}{p-t+r}}$ as $0 \leq \frac{1+t-r}{t-r} < 1$ for all $r \geq 1$. Now, let $d = 1+t-r$, $b = r-t$, $-a = p$ and $c-1 = \frac{1+t-r}{p-t+r}$ in Theorem 4.1 to get the first inequality. □

Theorem 4.3. *Given operators A, B, X, Y and, for any real numbers a, b, c and d, the following hold.*

(4.3) *If $(B^{b/2} X^{-a} B^{b/2})^{c-1} \leq B^d$, then $X^a \natural_c B^b \leq B^{b+d}$.*

(4.4) *If $(A^{b/2} Y^{-a} A^{b/2})^{c-1} \geq A^d$, then $A^{b+d} \leq Y^a \natural_c A^b$.*

Proof. The proof is similar to Theorem 4.1 and should be omitted. □

Using Theorem GF, we have the following.

Corollary 4.4. *If $A \geq B$, then, for all $p \geq 1$, $r, s \geq 1$ and $t \in [0,1]$,*

$$(B^{-t/2} A^p B^{-t/2})^{-s} \natural_{\frac{1+(p-t)s}{(p-t)s+r}} B^r \leq B \leq A \leq (A^{-t/2} B^p A^{-t/2})^{-s} \natural_{\frac{1+(p-t)s}{(p-t)s+r}} A^r.$$

Proof. The grand Furuta inequality is obviously equivalent to

$$[B^{r/2}(B^{-t/2} A^p B^{-t/2})^s B^{r/2}]^{\frac{1-t+r}{(p-t)s+r}} \geq B^{1-t+r}$$

for all $s \geq 1$, $p \geq 1$, $r \geq t \in [0,1]$,

$$B^{-(1-t+r)} \geq [B^{r/2}(B^{-t/2} A^p B^{-t/2})^s B^{r/2}]^{-\frac{1-t+r}{(p-t)s+r}}.$$

It follows that

$$B^{1-r} \geq [B^{r/2}(B^{-t/2} A^p B^{-t/2})^s B^{r/2}]^{\frac{1-r}{(p-t)s+r}}$$

as $0 \leq \frac{r-1}{1-t+r} < 1$ for $r \geq 1$. Let $X = B^{-t/2} A^p B^{-t/2}$, $-a = s$, $b = r$, $d = 1-r$ and $c - 1 = \frac{1-r}{(p-t)s+r}$ in Theorem 4.3. Then the first inequality in question follows. The last inequality could be done similarly with $Y = A^{-t/2} B^p A^{-t/2}$.

Using Theorem FM, we have the following.

Corollary 4.5. *If $A \gg B$, then, for all $p \geq 0$, $r \geq 1$, $s \in [1,2]$, and $t \leq 1$,*

$$(B^{-t/2} A^p B^{-t/2})^{-s} \natural_{\frac{1+(p-t)s}{(p-t)s+r}} B^r \leq B \leq A \leq (A^{-t/2} B^p A^{-t/2})^{-s} \natural_{\frac{1+(p-t)s}{(p-t)s+r}} A^r.$$

Moreover, the inequality holds, too, if $A \geq B$ *and for all* $p \geq 0$, $r \geq 1$, $s \in [1,2]$ *and* $t \leq 1$.

Proof. Since by (b_1) in Theorem FM,

$$[B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{r-t}{(p-t)s+r}} \geq B^{r-t}$$

for $p \geq 0$, $r \geq 0$, $s \in [1,2]$ and $t \leq 0$, so that

$$B^{-(r-t)} \geq [B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{-\frac{r-t}{(p-t)s+r}}.$$

Hence $B^{1-r} \geq [B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{1-r}{(p-t)s+r}}$ as $0 \leq \frac{1-r}{t-r} \leq 1$ for all $r \geq 1$ and $t \leq 1$. Let $X = B^{-t/2}A^pB^{-t/2}$, $-a = s$, $b = r$, $d = 1-r$ and $c - 1 = \frac{1-r}{(p-t)s+r}$ in Theorem 4.3. Then we have the required inequalities as in Corollary 4.4. □

We may apply results in the section 3 to obtain more satellite theorems of Furuta, and we shall leave it to the reader. To conclude this section it is worth noting that some different satellite theorems of the chaotic Furuta inequality may be found in a recent paper in [5]; and the relationship between the operator order and the chaotic operator order are given and explained there.

5. Characterizations of chaotic operator order $A \gg B$ by Theorem D

In the literature the chaotic operator order $A \gg B$ was characterized in terms of operator inequalities, including our section two and three. In this section it will be done in terms of operator equalities due to Theorem D and previous results in sections 2 and 3.

Theorem 5.1. *For all* $p \geq 0$, $r \geq 0$, $s \in [1,2]$, $t \leq 0$ *and a nonnegative integer* $n \geq 0$ *such that* $(r-t)(n+1) = (p-t)s+r$, *the following are equivalent to one another.*

(5.1) $A \gg B$;

(5.2) $A^{r-t} \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1}{n+1}}$;

(5.3) $[B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{1}{n+1}} \geq B^{r-t}$;

(5.4) *There exists a unique* $S_1 > O$ *with* $\| S_1 \| \leq 1$ *such that*

$A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2} = (A^{\frac{r-t}{2}}S_1A^{\frac{r-t}{2}})^{n+1}$, *i.e.*,

$(A^{-t/2}B^pA^{-t/2})^s = A^{-t/2}S_1(A^{r-t}S_1)^nA^{-t/2} = A^{-t/2}(S_1A^{r-t})^nS_1A^{-t/2}$;

(5.5) *There exists a unique* $S_1 > O$ *with* $\| S_1 \| \leq 1$ *such that*

$B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2} = (B^{\frac{r-t}{2}}S_1^{-1}B^{\frac{r-t}{2}})^{n+1}$, *i.e.*,

$(B^{-t/2}A^pB^{-t/2})^s$

$= B^{-t/2}S_1^{-1}(B^{r-t}S_1^{-1})^nB^{-t/2} = B^{-t/2}(S_1^{-1}B^{r-t})^nS_1^{-1}B^{-t/2}$.

Proof. For all $p \geq 0$, $r \geq 0$, $s \in [1,2]$, $t \leq 0$ and a nonnegative ingeter $n \geq 0$ such that $(r-t)(n+1) = (p-t)s + r$, we let $q = n+1 \geq 1$. Then conditions (2.2) and (2.3) in Theorem 2.1 are obviously satisfied and so (5.1), (5.2) and (5.3) are equivalent to one another due to Theorem 2.1.

(5.2)⇒(5.4). By Theorem D there exists D with $\| D \| \leq 1$ such that

$$[A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1}{2(n+1)}} = A^{\frac{r-t}{2}}D = D^*A^{\frac{r-t}{2}}.$$

Let $S_1 = DD^*$ and so $\| S_1 \| = \| D \|^2 \leq 1$. Then

$$[A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1}{n+1}} = A^{\frac{r-t}{2}}S_1A^{\frac{r-t}{2}}$$

and S_1 is clearly unique, here we may assume $S_1 > O$ without loss of generality. Now, (5.4) follows immediately since

$$A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2} = (A^{\frac{r-t}{2}}S_1A^{\frac{r-t}{2}})^{n+1}$$
$$= A^{\frac{r-t}{2}}S_1(A^{r-t}S_1)^nA^{\frac{r-t}{2}} = A^{\frac{r-t}{2}}(S_1A^{r-t})^nS_1A^{\frac{r-t}{2}}.$$

(5.4)⇒(5.2). By (5.4),

$$[A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1}{n+1}} = A^{\frac{r-t}{2}}S_1A^{\frac{r-t}{2}} \leq A^{r-t},$$

the inequality is due to the fact that $S_1 \leq \| S_1 \| I \leq I$ as S_1 is Hermitian, and we have (5.2).

Next, since $A \gg B$ if and only if $B^{-1} \gg A^{-1}$, (5.4) is equivalent to (5.5) and so this completes the proof. □

Now, letting $s = 1$ and $t = 0$ in Theorem 5.1 yields the next result, which was originally proved in [12, Theorem 2.2].

Corollary 5.2. *For all $p \geq 0$, $r \geq 0$ and a nonnegative integer $n \geq 0$ such that $r(n+1) = p + r$, the following are equivalent to one another.*

(5.1) $A \gg B$;

(5.6) $A^r \geq (A^{r/2}B^pA^{r/2})^{\frac{1}{n+1}}$;

(5.7) $(B^{r/2}A^pB^{r/2})^{\frac{1}{n+1}} \geq B^r$;

(5.8) *There exists a unique $S_2 > O$ with $\| S_2 \| \leq 1$ such that*
$A^{r/2}B^pA^{r/2} = (A^{r/2}S_2A^{r/2})^{n+1}$, *i.e.,* $B^p = S_2(A^rS_2)^n = (S_2A^r)^nS_2$;

(5.9) *There exists a unique $S_2 > O$ with $\| S_2 \| \leq 1$ such that*
$B^{r/2}A^pB^{r/2} = (B^{r/2}S_2^{-1}B^{r/2})^{n+1}$, *i.e.,*
$A^p = S_2^{-1}(B^rS_2^{-1})^n = (S_2^{-1}B^r)^nS_2^{-1}$.

The proofs of the following three results are similar to the proof of Theorem 5.1, and we should provid for each one the outline only.

Theorem 5.3. *For all $p \geq 0$, $r \geq 0$, $s \in [1,2]$, $t \leq 0$ and a nonnegative integer $n \geq 0$ such that $r(n+1) = (p-t)s + r$, the following are equivalent to one another.*

(5.1) $A \gg B$;

(5.10) $A^r \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1}{n+1}}$;

(5.11) $[B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{1}{n+1}} \geq B^r$;

(5.12) *There exists a unique* $S_3 > O$ *with* $\| S_3 \| \leq 1$ *such that*
$A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2} = (A^{r/2}S_3A^{r/2})^{n+1}$, *i.e.,*
$(A^{-t/2}B^pA^{-t/2})^s = S_3(A^rS_3)^n = (S_3A^r)^nS_3$;

(5.13) *There exists a unique* $S_3 > O$ *with* $\| S_3 \| \leq 1$ *such that*
$B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2} = (B^{r/2}S_3^{-1}B^{r/2})^{n+1}$, *i.e.,*
$(B^{-t/2}A^pB^{-t/2})^s = S_3^{-1}(B^rS_3^{-1})^n = (S_3^{-1}B^r)^nS_3^{-1}$.

Proof. For all $p \geq 0$, $r \geq 0$, $s \in [1, 2]$, $t \leq 0$ and a nonnegative ingeter $n \geq 0$ such that $r(n+1) = (p-t)s + r$, we let $q = n + 1 \geq 1$. Then conditions (3.3) and (3.4) in Theorem 3.1 are obviously satisfied and so (5.1), (5.10) and (5.11) are equivalent to one another due to Theorem 3.1.

(5.10)⇒(5.12). By Theorem D there exists E with $\| E \| \leq 1$ such that

$$[A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1}{2(n+1)}} = A^{r/2}E = E^*A^{r/2}.$$

Let $S_3 = EE^*$ and so $\| S_3 \| = \| E \|^2 \leq 1$. Then

$$[A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1}{n+1}} = A^{r/2}S_3A^{r/2}$$

and S_3 is clearly unique, here we may assume $S_3 > O$ without loss of generality. Now, (5.12) follows immediately since

$$A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2} = (A^{r/2}S_3A^{r/2})^{n+1}$$
$$= A^{r/2}S_3(A^rS_3)^nA^{r/2} = A^{r/2}(S_3A^r)^nS_3A^{r/2}.$$ □

Theorem 5.4. *For all* $p \geq 0$, $r \geq 0$, $s \in [1, 2]$, $t \leq 0$ *and a nonnegative integer* $n \geq 0$ *such that* $-t(n+1) = (p-t)s + r$, *the following are equivalent to one another.*

(5.1) $A \gg B$;

(5.14) $A^{-t} \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1}{n+1}}$;

(5.15) $[B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{1}{n+1}} \geq B^{-t}$;

(5.16) *There exists a unique* $S_4 > O$ *with* $\| S_4 \| \leq 1$ *such that*
$A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2} = (A^{-t/2}S_4A^{-t/2})^{n+1}$, *i.e.,*
$(A^{-t/2}B^pA^{-t/2})^s = A^{\frac{r+t}{-2}}S_4(A^{-t}S_4)^nA^{\frac{r+t}{-2}} = A^{\frac{r+t}{-2}}(S_4A^{-t})^nS_4A^{\frac{r+t}{-2}}$;

(5.17) *There exists a unique* $S_4 > O$ *with* $\| S_4 \| \leq 1$ *such that*
$B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2} = (B^{-t/2}S_4^{-1}B^{-t/2})^{n+1}$, *i.e.,*
$(B^{-t/2}A^pB^{-t/2})^s$
$= B^{\frac{r+t}{-2}}S_4^{-1}(B^{-t}S_4^{-1})^nB^{\frac{r+t}{-2}} = B^{\frac{r+t}{-2}}(S_4^{-1}B^{-t})^nS_4^{-1}B^{\frac{r+t}{-2}}$.

Proof. For all $p \geq 0$, $r \geq 0$, $s \in [1,2]$, $t \leq 0$ and a nonnegative ingeter $n \geq 0$ such that $-t(n+1) = (p-t)s + r$, we let $q = n+1 \geq 1$. Then conditions (3.8) and (3.9) in Theorem 3.2 are obviously satisfied and so (5.1), (5.14) and (5.15) are equivalent to one another due to Theorem 3.2. □

Theorem 5.5. *For all $p \geq 0$, $r \geq 1$, $s \in [1,2]$, $t \leq 0$ and a nonnegative integer $n \geq 0$ such that $n+1 = (p-t)s + r$, consider the following statements.*

(5.1) $A \gg B$;

(5.18) $A \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1}{n+1}}$;

(5.19) $[B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{1}{n+1}} \geq B$;

(5.20) *There exists a unique $S_5 > O$ with $\| S_5 \| \leq 1$ such that*
$A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2} = (A^{1/2}S_5A^{1/2})^{n+1}$, *i.e.,*
$(A^{-t/2}B^pA^{-t/2})^s = A^{\frac{1-r}{2}}S_4(AS_4)^nA^{\frac{1-r}{2}} = A^{\frac{1-r}{2}}(S_4A)^nS_4A^{\frac{1-r}{2}}$;

(5.21) *There exists a unique $S_4 > O$ with $\| S_4 \| \leq 1$ such that*
$B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2} = (B^{1/2}S_4^{-1}B^{1/2})^{n+1}$, *i.e.,*
$(B^{-t/2}A^pB^{-t/2})^s = B^{\frac{1-r}{2}}S_4^{-1}(BS_4^{-1})^nB^{\frac{1-r}{2}} = B^{\frac{1-r}{2}}(S_4^{-1}B)^nS_4^{-1}B^{\frac{1-r}{2}}$.

Then we conclude that implication (5.1)⇒(5.18) *holds, and statements* (5.18), (5.19), (5.20) *and* (5.21) *are equivalent to one another.*

Proof. We let $q = n+1 \geq 1$. Then conditions (3.17) and (3.18) in Theorem 3.3 are satisfied, and so we have implications (5.1)⇒(5.18)⇔(5.19) due to Theorem 3.3. The rest of the proof should be omitted. □

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