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GENERALIZED PROJECTION AND APPROXIMATION FOR GENERALIZED VARIATIONAL INEQUALITIES SYSTEM IN BANACH SPACES

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Abstract. The approximate solvability of a generalized system for nonlinear variational inequality in Hilbert spaces was studied, based on the convergence of projection methods. But little research was done in Banach space. The primary reason was that projection mapping lacked preferably property in Banach space. In this paper, we introduced the generalized projection methods. By using these methods, the results presented in this paper extended the main results of S. S. Chang [3] from Hilbert spaces to Banach space.

1. Introduction and Preliminary

Let B be a Banach space with dual space B^*. As usually, $\langle \varphi, x\rangle$ denotes the duality pairing of B^* and B, where $\varphi \in B^*$ and $x \in B$. (If B is a Hilbert space, $\langle \varphi, x\rangle$ denotes an inner product in it.) Let K be a nonempty, closed and convex subset of B and $T : K \times K \to B^*$ a mapping. We consider a system of nonlinear variational inequality (SNVI) problem as follows: to find $x^*, y^* \in K$ such that

$$\langle \rho T(y^*, x^*) + Jx^* - Jy^*, x - x^*\rangle \geq 0, \quad \forall x \in K,\ \rho > 0; \tag{1.1}$$

$$\langle \eta T(x^*, y^*) + Jy^* - Jx^*, x - y^*\rangle \geq 0, \quad \forall x \in K,\ \eta > 0, \tag{1.2}$$

where the $J : B \to 2^{B^*}$ is normalized duality mapping defined by

$$J(x) \in \langle J(x), x\rangle = \|J(x)\|\|x\| = \|x\|^2 = \|J(x)\|^2, \quad \forall x \in B,$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing of B^* and B. Without confusion, one understands that $\|J(x)\|$ is the B^* norm and $\|x\|$ is the B-norm. Take a functional $V : B^* \times B \to R$, which is called Lyapunov functional,

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is defined by the formula:

$$V(\varphi, x) = \|\varphi\|^2 - 2\langle \varphi, x\rangle + \|x\|^2,$$

where $\varphi \in B^*$ and $x \in B$. It is easy to see that $V(\varphi, x) \geq (\|\varphi\| - \|x\|)^2$. Thus the functional $V : B^* \times B \to R^+$ is nonnegative.

Definition. ([1, Definition 6.2]) Operator $\pi_K : B^* \to K$ is called the *generalized projection operator* if it associates with an arbitrary fixed point $\varphi \in B^*$ the minimum point of the functional $V(\varphi, x)$, i.e., a solution to the minimization problem

$$V(\varphi, \pi_K \varphi) = \inf_{y\in K} V(\varphi, y). \tag{1.3}$$

$\pi_K \varphi \in K \subset B$ is then called a *generalized projection* of the point φ.

Applying the definitions of V and J, a functional $V_2 : B \times B \to R$ is defined by the formula:

$$V_2(x, y) = V(Jx, y), \quad \forall x, y \in B.$$

The following properties of the operators J, V and π_K are useful for our paper (see [1] or [2]).

(i) J is a monotone and bounded operator in arbitrary Banach spaces.

(ii) J is a strictly monotone operator in strictly convex Banach spaces.

(iii) J is a continuous operator in smooth Banach paces.

(iv) J is an uniformly continuous operator on each bounded set in uniformly smooth Banach spaces.

(v) J is the identity operator in Hilbert spaces, i.e., $J = I_H$.

(vi) $V(\phi, x)$ is continuous.

(vii) $V(\varphi, x)$ is convex with respect to φ when x is fixed and with respect to x when ϕ is fixed.

(viii) $(\|\varphi\| - \|x\|)^2 \leq V(\varphi, x) \leq (\|\varphi\| + \|x\|)^2$.

(ix) $V(\varphi, x) = 0$ if and only if $\varphi = Jx$.

If B is a reflexive Banach space with dual space B^* and K is a nonempty, closed and convex subset of B, then the following properties hold:

(x) $V(J\pi_K\varphi, x) \leq V(\varphi, x)$ for all $\varphi \in B^*$ and $x \in B$.

(xi) The operator π_K is J fixed in each point $x \in K$, i.e., $\pi_K Jx = x$.

(xii) π_K is monotone in B^*, i.e., for all $\varphi_1, \varphi_2 \in B^*$,

$$\langle \pi_K\varphi_1 - \pi_K\varphi_2, \varphi_1 - \varphi_2\rangle \geq 0.$$

(xiii) If the Banach space B is uniformly smooth, then π_K is continuous and, for all $\varphi_1, \varphi_2 \in B^*$, we have

$$\|\pi_K\varphi_1 - \pi_K\varphi_2\| \leq 2R_1 g_B^{-1}(\|\varphi_1 - \varphi_2\|/R_1),$$

where $R_1 = (\|\pi_K\varphi_1\|^2 + \|\pi_K\varphi_2\|^2)^{1/2}$ and g_B^{-1} is the inverse function to g_B that is defined by the modulus of smoothness for a uniformly smooth Banach space.

(xiv) If B is also smooth, then operator $\pi_K : B^* \to K$ is single valued and, for any given $\varphi \in B^*$,

$$\langle \varphi - J\widetilde{\varphi}, \widetilde{\varphi} - x\rangle \geq 0, \quad \forall x \in K \quad \Longleftrightarrow \quad \widetilde{\varphi} = \pi_K \varphi.$$

Once the generalized projection operator $\pi_K : B^* \to K$ is introduced, solving the variational inequalities (1.1) and (1.2) is equivalent to finding a fixed point of a special operator from K to K. That is described by the following theorem.

Theorem A. *Let B be an smooth Banach space, T be an arbitrary operator from $B \times B$ to B^* and ρ, η be two arbitrary fixed positive number. Then the points $(x^*, y^*) \in K \times K$ is a solution of the variational inequality system* (1.1) *and* (1.2) *if and only if $(x^*, y^*) \in K \times K$ is a solution of the following operator equations in $B \times B$*

$$x^* = \pi_K[Jy^* - \rho T(y^*, x^*)], \quad \forall \rho > 0, \tag{1.4}$$

$$y^* = \pi_K[Jx^* - \eta T(x^*, y^*)], \quad \forall \eta > 0. \tag{1.5}$$

Proof. The variational inequality (1.1) can write to

$$\langle Jy^* - \rho T(y^*, x^*) - Jx^*, x^* - x\rangle \geq 0.$$

By the property (xiv), the above formula is equivalent to

$$x^* = \pi_K[Jy^* - \rho T(y^*, x^*)], \quad \forall \rho > 0.$$

Similarly, the variational inequality (1.2) is equivalent to the following projection formula:

$$y^* = \pi_K[Jx^* - \eta T(x^*, y^*)], \quad \forall \eta > 0.$$

This completes the proof. □

Next, we consider some special cases of the SNVI problems (1.1) and (1.2):

(I) If $\eta = 0$, then, applying the properties (xi), (xiv) and Theorem A, the SNVI problems (1.1) and (1.2) reduces to the following nonlinear variational inequality (NVI) problem: to find $x^* \in K$ such that

$$\langle T(x^*, x^*.), x - x^*\rangle \geq 0, \quad \forall x \in K. \tag{1.6}$$

(II) If K is a closed convex cone of B, then the SNVI problems (1.1) and (1.2) is equivalent to the following system of nonlinear complementarity (SNC) problems: to find $x^*, y^* \in K$ such that $T(y^*, x^*) \in K^*, T(x^*, y^*) \in K^*$ and

$$\langle \rho T(y^*, x^*) + Jx^* - Jy^*, x^*\rangle = 0, \quad \forall \rho > 0; \tag{1.7}$$

$$\langle \eta T(x^*, y^*) + Jy^* - Jx^*, y^*\rangle = 0, \quad \forall \eta > 0, \tag{1.8}$$

where K^* is the polar cone to K defined by

$$K^* = \{f \in B^* : \langle f, x\rangle \geq 0, \quad \forall x \in K\}.$$

(III) If $T : K \to B^*$ is a univariate mapping, then the SNVI problems (1.1) and (1.2) are reduced to the following SNVI problem: to find $x^*, y^* \in K$ such that

$$\langle \rho T(y^*) + Jx^* - Jy^*, x - x^* \rangle \geq 0, \quad \forall x \in K,\ \rho > 0; \tag{1.9}$$

$$\langle \eta T(x^*) + Jy^* - Jx^*, x - y^* \rangle \geq 0, \quad \forall x \in K,\ \eta > 0. \tag{1.10}$$

2. Algorithms

In this section, we deal with an introduction of general two-step models for generalized projection methods and its special form can be applied to the convergence analysis for generalized projection methods in the context of the approximation solvability of the SNVI problems (1.1), (1.2) and (1.9), (1.10), etc.

Algorithm 2.1. (see [3]) For arbitrarily chosen initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n \pi_K[Jy_n - \rho T(y_n, x_n)] \\ y_n = (1-\beta_n)x_n + \beta_n \pi_K[Jx_n - \eta T(x_n, y_n)], \end{cases} \tag{2.1}$$

where π_K is the generalized projection of B^* onto K, ρ and $\eta > 0$ are constants and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$.

If $T : K \to B^*$ is a univariate mapping, then Algorithm 2.1 is reduced to the following:

Algorithm 2.2. For arbitrarily chosen initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n \pi_K[Jy_n - \rho T(y_n)], \\ y_n = (1-\beta_n)x_n + \beta_n \pi_K[Jx_n - \eta T(x_n)], \end{cases} \tag{2.2}$$

where π_K is the generalized projection of B^* onto K, ρ, $\eta > 0$ are constants and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$.

For $\beta_n = 1$ in Algorithm 2.1, we have the following:

Algorithm 2.3. For arbitrarily chosen initial points $x_0, y_0 \in K$ compute the sequence $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n \pi_K[Jy_n - \rho T(y_n, x_n)], \\ y_n = \pi_K[Jx_n - \eta T(x_n, y_n)], \end{cases} \tag{2.3}$$

where $\{\alpha_n\} \subset [0, 1]$ for all $n \geq 0$.

For $\eta = 0$ and $\beta_n = 1$ in Algorithm 2.1, we have the following:

Algorithm 2.4. For arbitrarily chosen initial point $x_0 \in K$, compute the sequence $\{x_n\}$ such that

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \pi_K[Jx_n - \rho T(x_n, x_n)], \tag{2.4}$$

where $\{\alpha_n\} \subset [0, 1]$ for all $n \geq 0$.

3. An approximation of the solutions of the variational inequality systems (1.1) and (1.2)

In this section, we study the approximation of the solution of the variational inequality systems (1.1) and (1.2) by the Ishikawa sequence. The techniques used in this section have been used by many authors (see [5]-[8]). The following lemma given by Chidume and Li [4] is useful for the proof of the theorem in this section.

Lemma 3.1. ([4, Lemma 3.4]) *Let B be a uniformly convex Banach space. Then for arbitrary $r > 0$, there exists a continuous, strictly increasing convex function $g : R^+ \to R^+$, $g(0) = 0$, such that, for all $x_1, x_2,\ y \in B_r(0) := \{x \in B : \|x\| \leq r\}$ and $\alpha \in [0, 1]$, the following inequality holds:*

$$\begin{aligned} &V_2(\alpha x_1 + (1 - \alpha)x_2, y) \\ &\leq \alpha V_2(x_1, y) + (1 - \alpha)V_2(x_2, y) - \alpha(1 - \alpha)g(\|x_1 - x_2\|). \end{aligned} \tag{3.1}$$

Lemma 3.2. *Let B be a real Banach space and $J : B \to B^*$ be the normalized duality mapping, then for any $x, y \in B$ the following holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y).$$

Definition 3.1 A two-variable mapping $T : K \times K \to B^*$ is said to be *completely continuous mapping,* if $x_n \rightharpoonup x, y_n \rightharpoonup y$ implies that $T(x_n, y_n) \to T(x, y)$.

Theorem 3.1. *Let B be an uniformly convex and uniformly smooth Banach space and let K be a closed and convex subset of B. Let $T : K \times K \to B^*$ be a mapping on $K \times K$ such that $J - rT$ $(r > 0)$ be a completely continuous and*

$$\langle T(x, y), J^*[Jx - rT(x, y)]\rangle \geq 0, \quad \forall x, y \in K, \tag{3.2}$$

where $J^* = J^{-1}$ is the normalized duality mapping on B^*. Suppose that $(x^*, y^*) \in K \times K$ is a solution to the SNVI problems (1.1), (1.2) and $\{x_n\}, \{y_n\}$ are the sequences generated by Algorithm 2.1. If $\{\alpha_n\}$ and $\{\beta_n\}$ satisfies conditions $0 < a \leq \alpha_n \leq b < 1, 0 < c \leq \beta_n \leq d < 1$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to x^* and y^*, respectively.

Proof. For any $z \in B$, from Algorithm 2.1 and the inequality (3.1) of Lemma 3.1, we have

$$\begin{aligned} V_2(x_{n+1}, z) \leq{}& (1 - \alpha_n)V_2(x_n, z) + \alpha_n V_2(\pi_K[Jy_n - \rho T(y_n, x_n)], z) \\ &- \alpha_n(1 - \alpha_n)g(\|\pi_K[Jy_n - \rho T(y_n, x_n)] - x_n\|), \end{aligned} \tag{3.3}$$

$$\begin{aligned} V_2(y_n,z) \le{}& (1-\beta_n)V_2(x_n,z) + \beta_n V_2(\pi_K[Jx_n - \eta T(x_n,y_n)], z) \\ & - \beta_n(1-\beta_n)g(\|\pi_K[Jx_n - \eta T(x_n,y_n)] - x_n\|). \end{aligned} \tag{3.4}$$

From the definition of the functional V_2, Lemma 3.2 and the convexity property of the Lyapunov functional V, we obtain

$$\begin{aligned} & V_2(\pi_K[Jx_n - \eta T(x_n,y_n)], z) \\ & = V(J\pi_K[Jx_n - \eta T(x_n,y_n)], z) \\ & \le V(Jx_n - \eta T(x_n,y_n), z) \\ & = \|Jx_n - \eta T(x_n,y_n)\|^2 + \|z\|^2 - 2\langle Jx_n - \eta T(x_n,y_n), z\rangle \\ & \le \|x_n\|^2 - 2\eta\langle T(x_n,y_n), J^*[Jx_n - \eta T(x_n,y_n)]\rangle \\ & \quad + \|z\|^2 - 2\langle Jx_n - \eta T(x_n,y_n), z\rangle \\ & = \|x_n\|^2 - 2\langle Jx_n, z\rangle + \|z\|^2 - 2\eta\langle T(x_n,y_n), J^*[Jx_n - \eta T(x_n,y_n)] - z\rangle \\ & \le V(Jx_n, z) - 2\eta\langle T(x_n,y_n), J^*[Jx_n - \eta T(x_n,y_n)] - z\rangle \\ & = V_2(x_n,z) - 2\eta\langle T(x_n,y_n), J^*[Jx_n - \eta T(x_n,y_n)] - z\rangle, \end{aligned} \tag{3.5}$$

$$\begin{aligned} & V_2(\pi_K[Jy_n - \rho T(y_n,x_n)], z) \\ & \quad = V(J\pi_K[Jy_n - \rho T(y_n,x_n)], z) \\ & \quad \le V(Jy_n - \rho T(y_n,x_n), z) \\ & \quad \le V_2(y_n,z) - 2\rho\langle T(y_n,x_n), J^*[Jy_n - \rho T(y_n,x_n)] - z\rangle. \end{aligned} \tag{3.6}$$

Substituting the above inequality into inequality (3.3), (3.4) and applying the condition (3.2), we have

$$\begin{aligned} V_2(y_n,z) \le{}& (1-\beta_n)V_2(x_n,z) + \beta_n V_2(\pi_K[Jx_n - \eta T(x_n,y_n)], z) \\ & - \beta_n(1-\beta_n)g(\|\pi_K[Jx_n - \eta T(x_n,y_n)] - x_n\|) \\ \le{}& (1-\beta_n)V_2(x_n,z) + \beta_n V_2(x_n,z) \\ & - 2\eta\beta_n\langle T(x_n,y_n), J^*[Jx_n - \eta T(x_n,y_n)] - z\rangle \\ & - \beta_n(1-\beta_n)g(\|\pi_K[Jx_n - \eta T(x_n,y_n)] - x_n\|) \\ \le{}& V_2(x_n,z) - 2\eta\beta_n\langle T(x_n,y_n), J^*[Jx_n - \eta T(x_n,y_n)] - z\rangle \\ & - \beta_n(1-\beta_n)g(\|\pi_K[Jx_n - \eta T(x_n,y_n)] - x_n\|), \end{aligned} \tag{3.7}$$

$$\begin{aligned} V_2(x_{n+1},z) \le{}& (1-\alpha_n)V_2(x_n,z) + \alpha_n V_2(y_n,z) \\ & - 2\rho\alpha_n\langle T(y_n,x_n), J^*[Jy_n - \rho T(y_n,x_n)] - z\rangle \\ & - \alpha_n(1-\alpha_n)g(\|\pi_K[Jy_n - \rho T(y_n,x_n)] - x_n\|) \\ \le{}& V_2(x_n,z) + 2\rho\alpha_n\langle T(y_n,x_n), z\rangle + 2\eta\alpha_n\beta_n\langle T(x_n,y_n), z\rangle \\ & - \alpha_n\beta_n(1-\beta_n)g(\|\pi_K[Jx_n - \eta T(x_n,y_n)] - x_n\|) \\ & - \alpha_n(1-\alpha_n)g(\|\pi_K[Jy_n - \rho T(y_n,x_n)] - x_n\|). \end{aligned} \tag{3.8}$$

Taking $z = \theta$ in the above inequality and taking the sum for $i = 1, 2, \ldots, n$, we obtain

$$\begin{aligned} &\alpha_n\beta_n(1-\beta_n)g(\|\pi_K[Jx_n - \eta T(x_n, y_n)] - x_n\|) \\ &\qquad + \alpha_n(1-\alpha_n)g(\|\pi_K[Jy_n - \rho T(y_n, x_n)] - x_n\|) \\ &\quad \le V_2(x_n, \theta) - V_2(x_{n+1}, \theta), \end{aligned} \tag{3.9}$$

$$ac(1-d)\sum_{i=1}^{n} g(\|\pi_K[Jx_i - \eta T(x_i, y_i)] - x_i\|) \le V_2(x_0, \theta) - V_2(x_{n+1}, \theta),$$

$$a(1-b)\sum_{i=1}^{n} g(\|\pi_K[Jy_i - \rho T(y_i, x_i)] - x_i\|) \le V_2(x_0, \theta) - V_2(x_{n+1}, \theta).$$

Since $V_2 : B \times B \to R^+$ is nonnegative and $V_2(x_0, \theta) < \infty$, the sum becomes

$$\sum_{i=1}^{\infty} g(\|\pi_K[Jx_i - \eta T(x_i, y_i)] - x_i\|) < \infty,$$

$$\sum_{i=1}^{\infty} g(\|\pi_K[Jy_i - \rho T(y_i, x_i)] - x_i\|) < \infty.$$

Thus we have

$$g(\|\pi_K[Jx_n - \eta T(x_n, y_n)] - x_n\|) \to 0 \quad \text{as } n \to \infty,$$

$$g(\|\pi_K[Jy_n - \rho T(y_n, x_n)] - x_n\|) \to 0 \quad \text{as } n \to \infty.$$

Applying the properties of g, we have

$$\begin{aligned} &\|\pi_K[Jx_n - \eta T(x_n, y_n)] - x_n\| \to 0 \quad \text{as } n \to \infty, \\ &\|\pi_K[Jy_n - \rho T(y_n, x_n)] - x_n\| \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.10}$$

But $\|y_n - x_n\| = \beta_n\|\pi_K[Jx_n - \eta T(x_n, y_n)] - x_n\| \to 0$ and hence we also have

$$\begin{aligned} &\|\pi_K[Jx_n - \eta T(x_n, y_n)] - y_n\| \\ &\le \|\pi_K[Jx_n - \eta T(x_n, y_n)] - x_n\| + \|y_n - x_n\| \to 0. \end{aligned} \tag{3.11}$$

On the other hand, by the formula (3.9), we obtain $V_2(x_{n+1}, \theta) \le V_2(x_n, \theta)$, i.e., $\|x_{n+1}\| \le \|x_n\|$, $\{x_n\}$ is bounded sequence and so $\{y_n\}$ is bounded. Since $J - rT$ is completely continuous, there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{Jy_{n_k} - \rho T(y_{n_k}, x_{n_k})\}$ and $\{Jx_{n_k} - \eta T(x_{n_k}, y_{n_k})\}$ converge. Therefore, from (3.10), (3.11) and the continuity of π_K, $\{x_{n_k}\}$ and $\{y_{n_k}\}$ converge. Let $\lim_{k\to\infty} x_{n_k} = x^*$ and $\lim_{k\to\infty} y_{n_k} = y^*$. In virtue of arbitrary subsequence of $\{x_n\}$, $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$. Using the continuity properties of the operators π_K, $J - rT$ and combing (3.10), (3.11), we obtain

$$x^* = \pi_K[Jy^* - \rho T(y^*, x^*)], \quad \forall \rho > 0,$$

$$y^* = \pi_K[Jx^* - \eta T(x^*, y^*)], \quad \forall \eta > 0.$$

This completes the proof. $\square$

Remark 1. Theorem 3.1 extends and improves the main results of Chang [3] from Hilbert spaces to Banach spaces.

The following theorems can be obtained from Theorem 3.1 immediately:

Theorem 3.2. *Let B be an uniformly convex and uniformly smooth Banach space and K be a closed and convex subset of B. Let $T : K \to B^*$ be a univariate mapping on K such that $J - rT$ $(r > 0)$ is completely continuous and*

$$\langle T(x), J^*[Jx - rT(x)]\rangle \geq 0, \quad \forall x \in K, \tag{3.12}$$

where $J^* = J^{-1}$ is the normalized duality mapping on B^*. Suppose that $(x^*, y^*) \in K \times K$ is a solution to the SNVI problems (1.9), (1.10) *and $\{x_n\}$, $\{y_n\}$ are the sequences generated by Algorithm 2.2. If $\{\alpha_n\}$ and $\{\beta_n\}$ satisfies the conditions $0 < a \leq \alpha_n \leq b < 1$ and $0 < c \leq \beta_n \leq d < 1$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to x^* and y^*, respectively.*

Theorem 3.3. *Let B be an uniformly convex and uniformly smooth Banach space and K be a closed and convex subset of B. Let $T : K \times K \to B^*$ be a mapping on $K \times K$ such that $J - rT$ $(r > 0)$ is completely continuous and*

$$\langle T(x,y), J^*[Jx - rT(x,y)]\rangle \geq 0, \quad \forall x, y \in K, \tag{3.13}$$

where $J^* = J^{-1}$ is the normalized duality mapping on B^*. Suppose that $(x^*, y^*) \in K \times K$ is a solution to the SNVI problems (1.1), (1.2) *and $\{x_n\}$, $\{y_n\}$ are the sequences generated by Algorithm 2.3. If $\{\alpha_n\}$ satisfies the conditions $0 < a \leq \alpha_n \leq b < 1$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to x^* and y^*, respectively.*

Proof. Observe that (3.7) becomes

$$\begin{aligned} V_2(y_n, z) &= V_2(\pi_K[Jx_n - \eta T(x_n, y_n)], z) \\ &\leq V_2(x_n, z) - 2\eta\langle T(x_n, y_n), J^*[Jx_n - \eta T(x_n, y_n)] - z\rangle \\ &\leq V_2(x_n, z) + 2\eta\langle T(x_n, y_n), z\rangle. \end{aligned}$$

Taking $z = \theta$ in the above inequality we obtain $V_2(y_n, \theta) \leq V_2(x_n, \theta)$, i.e., $\|y_n\| \leq \|x_n\|$. Since $\{x_n\}$ is bounded, $\{y_n\}$ is also bounded. Similarly, using the proof of Theorem 3.1 and applying the completely continuity of $J - rT$ $(r > 0)$, we can obtain correspond results. This completes the proof. □

Theorem 3.4. *Let B be an uniformly convex and uniformly smooth Banach space and K be a closed and convex subset of B. Let $T : K \times K \to B^*$ be a mapping on $K \times K$ such that $J - \rho T$ $(\rho > 0)$ is completely continuous and*

$$\langle T(x,x), J^*[Jx - \rho T(x,x)]\rangle \geq 0, \quad \forall x, y \in K, \tag{3.14}$$

where $J^* = J^{-1}$ is the normalized duality mapping on B^*. Suppose that $x^* \in K$ is a solution to the NVI problem (1.6) *and $\{x_n\}$ is the sequence generated by Algorithm 2.4. If $\{\alpha_n\}$ satisfies the condition $0 < a \leq \alpha_n \leq b < 1$. Then the sequence $\{x_n\}$ converges strongly to x^*.*

Remark 2. When B is a Hilbert space, the condition (3.2) becomes

$$\langle T(x,y), x - rT(x,y)\rangle \geq 0.$$

This implies that $\langle T(x,y), x\rangle \geq r\|T(x,y)\|^2$, in other words, the mapping $T(x,y)$ is coercive in the first variable. Under the above conditions, all the results in Chang [3] follow.

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