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WEAK CONVERGENCE OF MANN ITERATIVE SEQUENCE FOR NONEXPANSIVE MAPPINGS IN PROBABILISTIC HILBERT SPACES

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Abstract. The purpose of this paper is to establish the weak convergence theorem of Mann iterative sequence for nonexpansive mappings in probabilistic Hilbert spaces. In order to establish the weak convergence theorem, a new method was presented in this paper, that is *method of mathematical expectation*.

1. Introduction

Menger [1] introduced the notion of a probabilistic metric space in 1942 and, since then, the theory of probabilistic metric spaces has developed in many directions [2]. The idea of Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to the situations when we do not know exactly the distance between two points, we know only probabilities of possible values of this distance. Such a probabilistic generalization of metric spaces appears to be well adapted for the investigation of physiological thresholds and physical quantities particularly in connections with both string and E-infinity theory (see [3-6]). It is also of fundamental importance in probabilistic functional analysis, nonlinear analysis and applications [7-10]. In the same idea, probabilistic normed spaces and probabilistic inner product spaces were studied by numerous authors [9, 11-13].

In 1994, Chang [9] introduced the definition of a probabilistic inner product space. In 2001, Yongfu Su [11] modified Chang's definition and introduced the following definition of a probabilistic inner product space.

Definition 1.1. A mapping $F : (-\infty, +\infty) \to [0, 1]$ is called a *distribution function* if it is nondecreasing and left-continuous with $\inf_{t\in R} F(t) = 0$, $\sup_{t\in R} F(t) = 1$, where $R = (-\infty, +\infty)$.

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The set of all distribution functions is denoted by D. A special element of D is the function which defined by

$$H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Definition 1.2. ([9,11]) A *probabilistic inner product space* (briefly, PI-space) is a triple $(E, F, *)$ where E is a real linear space and mapping $F : E \times E \to D$ is denoted by $F_{x,y}(t)$ for every $(x, y) \in E \times E$ satisfies the following conditions:
(PI-1) $F_{x,x}(0) = 0$ for all $x \in E$;
(PI-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in E$;
(PI-3) $F_{x,x}(t) = H(t)$ if and only if $x = \theta$;
(PI-4) For all $x, y \in E$ and $\lambda \in R$,

$$F_{\lambda x,y}(t) = \begin{cases} F_{x,y}(\frac{t}{\lambda}), & \lambda > 0, \\ H(t), & \lambda = 0, \\ 1 - F_{x,y}(\frac{t}{\lambda}+), & \lambda < 0, \end{cases}$$

where $F_{x,y}(t+) = \lim_{t' \to t^+} F_{x,y}(t')$.

(PI-5) If x with y is linearly independent, then

$$F_{x+y,z}(t) = (F_{x,z} * F_{z,y})(t),$$

where

$$(F_{x,z} * F_{z,y})(t) = \int_{-\infty}^{+\infty} F_{x,z}(t-u) dF_{z,y}(u).$$

Definition 1.3. Let $(E, F, *)$ be a probabilistic inner product space.

(1) A sequence $\{x_n\}$ in E is said to be *probabilistic converges* to a point $x \in E$ if, for any given $\varepsilon > 0$ and $\lambda > 0$, there must exists a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{x_n - x, x_n - x}(\varepsilon) > 1 - \lambda$ for all $n > N$.

(2) A linear functional f defined on E is called *continuous* if $\{x_n\}$ probabilistic converges to a point $x \in E$ implies that $f(x_n) \to f(x)$ as $n \to \infty$ for any sequence $\{x_n\}$ in E.

(3) A sequence $\{x_n\}$ in E is said to be *weak probabilistic converges* to a point $x \in E$ if, for any given linear continuous functional $f(x)$, $f(x_n) \to f(x)$ as $n \to \infty$.

It is obvious that, if a sequence $\{x_n\}$ probabilistic converges to $x \in E$, then $\{x_n\}$ weak probabilistic converges to a point $x \in E$. However, we do not know whether weak probabilistic convergence can implies probabilistic convergence.

Definition 1.4. Let $(E, F, *)$ be a probabilistic inner product space and $T : D(T) \to R(T)$ be a nonlinear operator with domain $D(T)$ and range $R(T)$. T

is said to be *Lipschitz* if there exists a constant $L > 0$ such that

$$F_{Tx-Ty,Tx-Ty}(t) \geq F_{x-y,x-y}\Big(\frac{t}{L}\Big), \quad \forall\, x, y \in E,\, t \in R.$$

If $L < 1$, then T is said to be *contraction* and, if $L = 1$, then T is said to be *nonexpansive*.

Definition 1.5. A probabilistic inner product space $(E, F, *)$ is called *with mathematical expectation* if

$$\int_{-\infty}^{+\infty} t dF_{x,y}(t)$$

is convergent for all $x, y \in E$.

2. The mathematical expectation of probabilistic inner product spaces

Theorem 2.1. *Let $(E, F, *)$ be a probabilistic inner product space with mathematical expectation. Let*

$$\langle x, y\rangle = \int_{-\infty}^{+\infty} t dF_{x,y}(t)$$

for all $x, y \in E$. Then $(E, \langle\cdot,\cdot\rangle)$ is a inner product space, so that $(E, \|\cdot\|)$ is a normed space, where $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in E$.

Proof. We verify respectively the conditions of Hilbert space by using conditions (PI-1)-(PI-5) of probabilistic inner product space (see Definition 1.2).

(I-1) It follows from the condition (PI-1) that

$$\langle x, x\rangle = \int_{0}^{+\infty} t dF_{x,x}(t) \geq 0, \quad \forall\, x \in E.$$

(I-2) It is obvious from (PI-2) that $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in E$.

(I-3) It follows from (PI-3) that $\langle x, x\rangle = 0$ if and only if $x = \theta$.

(I-4) For any $x, y \in E$ and $\lambda \in R$, we need to consider three possible cases:

If $\lambda > 0$, from (PI-4) we have

$$\begin{aligned}\langle \lambda x, y\rangle &= \int_{-\infty}^{+\infty} t dF_{\lambda x,x}(t) = \int_{-\infty}^{+\infty} t dF_{x,x}\Big(\frac{t}{\lambda}\Big)\\ &= \lambda \int_{-\infty}^{+\infty} \frac{t}{\lambda} dF_{x,x}\Big(\frac{t}{\lambda}\Big) = \lambda \int_{-\infty}^{+\infty} t dF_{x,x}(t)\\ &= \lambda\langle x, y\rangle.\end{aligned}$$

If $\lambda = 0$, from (PI-4), we have

$$\langle \lambda x, y\rangle = \langle 0, y\rangle = \int_{-\infty}^{+\infty} t dH(t) = 0 = \lambda\langle x, y\rangle.$$

If $\lambda < 0$, from (PI-4) and Lemma 2.2, we have

$$\begin{aligned}\langle \lambda x, y\rangle &= \int_{-\infty}^{+\infty} tdF_{\lambda x,y}(t) = \int_{-\infty}^{+\infty} td(1 - F_{x,y}\Big(\frac{t}{\lambda}+\Big)\Big)\\ &= |\lambda| \int_{-\infty}^{+\infty} \frac{t}{\lambda} d\Big(1 - F_{x,y}\Big(\frac{-t}{|\lambda|}+\Big)\Big) = |\lambda| \int_{-\infty}^{+\infty} \frac{-t}{|\lambda|} dF_{x,y}\Big(\frac{-t}{|\lambda|}\Big)\\ &= -|\lambda| \int_{-\infty}^{+\infty} tdF_{x,y}(t+) = \lambda \int_{-\infty}^{+\infty} tdF_{x,y}(t+)\\ &= \lambda \int_{-\infty}^{+\infty} tdF_{x,y}(t) = \lambda\langle x, y\rangle.\end{aligned}$$

(I-5). Now, we prove that, for any $x, y, z \in E$,

$$\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle.$$

We also need to consider two possible cases.

If x and y are linearly dependent, without loss of generality, we assume $x + y = \lambda x$. In this case, we have

$$\begin{aligned}\langle x + y, z\rangle &= \lambda\langle x, z\rangle = (1 + \lambda - 1)\langle x, z\rangle\\ &= \langle x, z\rangle + (\lambda - 1)\langle x, z\rangle\\ &= \langle x, z\rangle + \langle(\lambda - 1)x, z\rangle\\ &= \langle x, z\rangle + \langle y, z\rangle.\end{aligned}$$

If x and y are linearly independent, from (PI-5), we have

$$F_{x+y,z}(t) = \int_{-\infty}^{+\infty} F_{x,z}(t - u)dF_{z,y}(u) = F_{x,z}(t) * F_{z,y}(t).$$

Therefore, we have

$$\int_{-\infty}^{+\infty} tdF_{x+y,z}(t) = \int_{-\infty}^{+\infty} tdF_{x,z}(t) + \int_{-\infty}^{+\infty} tdF_{z,y}(t),$$

which implies

$$\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle.$$

This completes the proof. □

Definition 2.2. Let $(E, F, *)$ be a probabilistic inner product space with mathematical expectation. If E is complete in the $\|\cdot\|$, then E is called *probabilistic Hilbert space*, where $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in E$.

Theorem 2.3. *Let $(E, F, *)$ be a probabilistic inner product space with mathematical expectation. For a sequence $\{x_n\} \subset E$, m-convergence (in norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$) implies probabilistic convergence.*

Proof. Suppose that $\{x_n\}$ m-converges to a point x. Then we have

$$\lim_{n\to\infty}\langle x_n - x, x_n - x\rangle = 0,$$

which leads to

$$\lim_{n\to\infty}\int_0^{+\infty} t dF_{x_n-x,x_n-x}(t) = 0. \tag{9}$$

Hence, for any $\varepsilon > 0$ and $0 < \lambda < 1$,

$$\begin{aligned}
&\int_0^{+\infty} t dF_{x_n-x,x_n-x}(t)\\
&= \int_0^{\varepsilon} t dF_{x_n-x,x_n-x}(t) + \int_{\varepsilon}^{+\infty} t dF_{x_n-x,x_n-x}(t)\\
&\geq \int_{\varepsilon}^{+\infty} \varepsilon dF_{x_n-x,x_n-x}(t)\\
&= \varepsilon(1 - F_{x_n-x,x_n-x}(\varepsilon)).
\end{aligned} \tag{10}$$

It follows from (9) and (10) that there exists a positive integer N such that

$$F_{x_n-x,x_n-n}(\varepsilon) > 1-\lambda, \quad \forall n \geq N,$$

so that $\{x_n\}$ probabilistic converges to a point x. This completes the proof. □

3. Weak convergence theorem

Theorem 3.1. *Let $(E, F, *)$ be a probabilistic Hilbert space and $T : E \to E$ be a nonexpansive mapping with nonempty fixed points set $F(T)$. For any guess $x_0 \in E$, define the Mann iterative sequence $\{x_n\}$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences which was assumed to satisfy the conditions

$$\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n) = \infty.$$

Then $\{x_n\}$ weak probabilistic converges to a fixed point of T.

Proof. By the definition of nonexpansive mappings in the form of probabilistic inner product spaces, we have

$$\begin{aligned}
\|Tx - Ty\|^2 = \langle Tx - Ty, Tx - Ty\rangle &= \int_{-\infty}^{+\infty} t dF_{Tx-Ty,Tx-Ty}(t)\\
&\leq \int_{-\infty}^{+\infty} t dF_{x-y,x-y}(t) = \|x-y\|^2, \quad \forall\, x, y \in E,
\end{aligned}$$

so that T is also a nonexpansive mapping in the form of Hilbert space $(E, \|\cdot\|)$. Therefore, by using the Mann result [14] we know, that $\{x_n\}$ converges weakly to a fixed point of T in the Hilbert space $(E, \|\cdot\|)$. Since $(E, F, *)$ is a probabilistic Hilbert space. Then $(E, \|\cdot\|)$ is namely a Hilbert space with norm $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in E$. We denote by $(E, \|\cdot\|)^*$ the dual space of $(E, \|\cdot\|)$ and also denote by $(E, F, *)^*$ the dual space of $(E, F, *)$, that is, $(E, F, *)^*$ is the set of all linear continuous (from probabilistic convergent to the topology of real numbers) functional. Since m-convergence (in norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$) implies probabilistic convergenve, then

$$(E, F, *)^* \subset (E, \|\cdot\|)^*.$$

Hence, if $\{x_n\}$ converges weakly in norm $\|\cdot\|$ to a fixed point $p \in F(T)$, then $\{x_n\}$ converges weakly in probabilistic to the same fixed point $p \in F(T)$. This completes the proof. □

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