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LOCAL CONVERGENCE OF THE SECANT METHOD UPPER HÖLDER CONTINUOUS DIVIDED DIFFERENCES

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Abstract. The semilocal convergence of the secant method under Hölder continuous divided differences in a Banach space setting for solving nonlinear equations has been examined by us in [3]. The local convergence was recently examined in [4]. Motivated by optimization considerations and using the same hypotheses but more precise estimates than in [4] we provide a local convergence analysis with the following advantages: larger radius of convergence and finer error estimates on the distances involved. The results can be used for projection methods, to develop the cheapest possible mesh refinement strategies and to solve equations involving autonomous differential equations [1], [4], [7], [8].

1. Introduction

In this study, we are concerned with the problem of approximating locally unique solution x^* of the equation

$$F(x) = 0 \tag{1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space Y. The secant method

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \quad \forall n \geq 0,\ x_{-1}, x_0 \in D \tag{2}$$

has been used by several authors to generate a sequence $\{x_n\}$ approximating x^* [2], [4], [8]. Here $[x, y; F] \in L(X, Y)$ the space of bounded linear operators from X into Y is called a divided difference of first order for the operator F at the points x and y $(x \neq y)$ if the following holds:

$$[x, y; F](x - y) = F(x) - F(y). \tag{3}$$

A survey on local as well as semilocal convergence theorems for method (2) can be found in [3], [5] and the references there.

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In particular in [3] we provided a semilocal convergence we analysis for method (2) under the general condition

$$\left\|[x_{-1}, x_0, F]^{-1}([x, y; F] - [v, w; F])\right\| \leq \overline{\ell}\, \|x - v\|^p + \overline{\overline{\ell}}\, \|y - w\|^p \tag{4}$$

for all $x, y, v, w \in D$ and some $p \in (0, 1]$ and $\overline{\ell}, \overline{\overline{\ell}} \geq 0$.

Hongwin and Qingbiao [6] used the special case of condition (4)

$$\left\|F'(x^*)^{-1}([x, y; F] - F'(z))\right\| \leq \ell(\|x - z\|^p + \|y - z\|^p) \tag{5}$$

to provide a local convergence analysis of method (2).

In our study motivated by optimization considerations we use the combination of (5) with condition

$$\left\|F'(x^*)^{-1}([x, y; F] - F'(x^*))\right\| \leq \ell_0\, \|(x - x^*\|^p + \|y - x^*\|^p) \tag{6}$$

to provide a new local convergence analysis of method (2) with the following advantages over the corresponding one in [6] for $\ell_0 < \ell$:

(a) a larger radius of convergence,

(b) finer error estimates on the distances $\|x_n - x^*\|$.

In view of (5) and (6),

$$\ell_0 \leq \ell \tag{7}$$

holds in general and $\frac{\ell}{\ell_0}$ can be arbitrarily large [4], [5]. Note also that the above stated advantages hold under the same hypotheses and computational cost since in practice the evaluation of ℓ requires includes the evaluation of ℓ_0.

2. Local convergence analysis of secant method (2)

We can show the main local convergence results for secant method (2):

Theorem 1. *Let $x^* \in D$ be a simple zero of F in the sense that $F'(x^*)^{-1} \in L(Y, X)$.*

Under conditions (5), (6), *further, assume*

$$U = U\Big(x^*, \Big(\frac{1}{\ell_0}\Big)^{\frac{1}{p}}\Big) = \Big\{x \in X\, \|x - x^*\| < \Big(\frac{1}{\ell_0}\Big)^{\frac{1}{p}}\Big\} \subseteq D. \tag{8}$$

Then sequence $\{x_n\}$ $(n \geq 0)$ generated by secant method (2) *converges to the unique solution $x^* \in U$ provided the initial points x_{-1}, x_0 are in smaller ball $U_0 = U(x^*, \tau)$, where*

$$r = \left\{\frac{1 + p}{2[\ell_0(1 + p) + \ell 2^p]}\right\}^{\frac{1}{p}}. \tag{9}$$

Moreover, the speed of convergence of the secant method (2) *is determined by*

$$\|x_n - x^*\| \leq r\left[\frac{1}{r}(\max(\|x_{-1} - x^*\|, \|x_0 - x^*\|))\right]^{\alpha}, \tag{10}$$

where,

$$\alpha = (p+1)^{\left[\frac{n+1}{2}\right]} \tag{11}$$

and $[y]$ *denotes the largest integer that is not larger than* y.

Proof. We shall first show that for all $u, v \in U_0$ linear operator $[u, v; F]$ is invertible on U_0. Indeed using (6) we obtain

$$\begin{aligned} &\left\| F'(x^*)^{-1}([u, v; F] - F'(x^*)) \right\| \\ &\leq \ell_0(\|u - x^*\|^p + \|v - x^*\|^p) < \ell_0 r^p < 1, \end{aligned} \tag{12}$$

by the choice of r. It follows from (12), and the Banach Lemma on invertible operators [5] that $[u, v; F']^{-1}F'(x^*)$ exists and

$$\left\| [u, v; F]^{-1}F'(x^*) \right\| \leq \frac{1}{1 - \ell_0(\|u - x^*\|^p + \|v - x^*\|^p)} \tag{13}$$

By hypothesis x_{-1}, x_0 are in U. Let us assume $x_k \in U$ for all integer values $k \leq n$. We shall show $x_{k+1} \in U$.

Using (2), (3), (5), (6) (for $u = x_{n-1}$, $v = x_n$), we can obtain in turn:

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&= \left\| x_n - x^* - [x_{n-1}, x_n; F]^{-1}(F(x_n) - F(x^*)) \right\| \\
&\leq \left\| [x_{n-1}, x_n; F]^{-1}F'(x^*) \right\| \\
&\quad \times \left\| \int_0^1 F'(x^*)^{-1}([x_{n-1}, x_n; F] - F'(x^* + t(x_n - x^*)) \right\| dt \, \|x_n - x^*\| \\
&\leq \frac{\|x_n - x^*\|}{1 - \ell_0(\|x_{n-1} - x^*\|^p + \|x_n - x^*\|^p)} \\
&\quad \times \ell \left[\int_0^1 (\|x_{n-1} - tx_n - (1-t)x^*\|^p + \|x_n - tx_n - (1-t)x^*\|^p \, dt \right] \\
&\leq \frac{\|x_n - x^*\|}{1 - \ell_0(\|x_{n-1} - x^*\|^p + \|x_n - x^*\|^p)} \\
&\quad \times \ell \left[\int_0^1 (t \|x_{n-1} - x_n\| + (1-t) \|x_{n-1} - x^*\|)^p dt \right. \\
&\quad \left. + \int_0^1 (1-t)^p \|x_n - x^*\|^p \, dt \right] \\
&< \frac{\ell r}{1 - 2\ell_0 r^p} \int_0^1 [(2rt + (1-t)r)^p + (1-t)^p r^p] dt \\
&= \frac{\ell r^{1+p}}{1 - 2\ell_0 r^p} \int_0^1 [(1+t)^p + (1-t)^p] dt \\
&= \frac{\ell r^{1+p}}{1 - 2\ell_0 r^p} \times \frac{2^{1+p}}{1+p} = r,
\end{aligned} \tag{14}$$

by the choice of r. It follows that $x_{n+1} \in U_0$ and the induction is completed.

Using (14) and (9), we get

$$\|x_{n+1} - x^*\| < \|x_n - x^*\|, \tag{15}$$

from which it follows that $\lim_{n\to\infty} x_n = x^*$. It also follows from (14) that, for all $n \geq 0$,

$$\|x_{n+1} - x^*\| \leq \frac{\ell[(\|x_{n-1} - x^*\| + \|x_n - x^*\|)^{1+p} - \|x_{n-1} - x^*\|^{1+p} + \|x_n - x^*\|^{1+p}]}{(1+p)(1-2\ell_0 r^p)}. \tag{16}$$

To show uniqueness, let y^* be a solution of equation (1) in

$$U\Big((x^*\Big(\frac{1}{\ell_0}\Big)^{\frac{1}{p}}\Big).$$

Define the linear operator $L = [y^*, x^*; f]$. Using (6), we obtain

$$\left\|F'(x^*)^{-1}(L - F'(x^*))\right\| \leq \ell_0 \|x^* - y^*\|^p < 1. \tag{17}$$

In view of (17) and the Banach lemma on invertible operators, we deduce operator L is invertible. By (3), we can write

$$L(y^* - x^*) = F(y^*) - F(x^*) \tag{18}$$

and we get $x^* = y^*$ by the invertibility of linear operator L.

The proof will end if we show estimate (10). Define

$$\gamma = \frac{1}{2}\left(\frac{\ell_0(1+p) + \ell 2^p}{1+p}\right)^{\frac{1}{p}} \tag{19}$$

and

$$\varepsilon_n = \gamma \|x_n - x^*\|. \tag{20}$$

With the above definitions the rest of the proof is identical to the corresponding one in [6, p. 288]. In order to avoid repetitions we refer the reader to [6] for the rest of the proof. That completes the proof. □

Remark 2. If $\ell_0 = \ell$, then our Theorem 1 reduces to Theorem 1 in [6]. Otherwise, it is an improvement. Indeed, for the radius r_{HQ} obtained in [6] simply, set $\ell_0 = \ell$ in (9). It then follows that, if $\ell_0 = \ell$

$$r_{HQ} < r \tag{21}$$

and

$$\gamma < \gamma_{HQ}, \tag{22}$$

where

$$\gamma_{HQ} = \frac{1}{2}\left[\frac{(1+p+2^p)\ell}{1+p}\right]^{\frac{1}{P}}. \tag{23}$$

Estimates (21) and (23) justify the claims made in the introduction of this study. The results obtained here are important in computational mathematics since as noted in [1], [4], [7], [8], the local convergence analysis can be used for projection methods and to develop the cheapest possible mesh refinement strategies. Finally note that these results can be used to solve equations when operator F satisfies autonomous differential equations [4] of the form where P is a continuous given operator. Since $F'(x^*) = P(F(x^*)) = P(0)$ we can use the results without actually knowing x^*.

We complete this study with a numerical example.

Example 3. Let $X = Y = \mathbb{R}$, $p = 1$, $U = D = U(0, 1)$, $x^* = 0$, and define functions f on D by

$$f(x) = e^x - 1. \tag{24}$$

Proof. Using (5), (6), and (24) for $[x, y; F] = \frac{f(x)-f(y)}{x-y}, (x \neq y)$, we obtain $\ell = \frac{e}{2} > \frac{e-1}{2} = \ell_0$. That is (7) holds as a strict inequality. Moreover, using (9), (22) and (23), we obtain

$$r_{HQ} = \frac{1}{4e} = .09196986 < r = \frac{1}{2(2e-1)} = .112699836$$

and

$$\gamma = \frac{2e-1}{2} = 2.218281828 < \gamma_{HQ} = e = 2.718281828.$$

Clearly, the numerical example justifies the theoretical results. Hence the advantages of our approach as stated above follow. □

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