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LOCALIZATION OF AZUKAWA PSEUDOMETRIC

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Abstract. We prove a localization theorem of Azukawa pseudometric at a local plurisubharmonic peak point of a domain in the complex Euclidean space.

1. Introduction

The localization of invariant metrics is one of the important issues in the study of their boundary behavior.

Graham [5] first studied the localization of the Kobayashi metric of a smooth strongly pseudoconvex domain. In order to get the result, he explored a global peak function at each boundary point of a domain. In 1987, Forstneric and Rosay [4] obtained a quantitive estimate about the localiztion of this metric at a local peak point of a bounded domain under some growth condition for the peak function.

Yu [9] studied the localization of the higher order Kobayashi metrics at a boundary point of a taut domain whose boundary does not contain nontrivial varieties through this point. In order to study the localization of the singular Kobayashi metric, Nikolov [6] proved the uniform localization of the higher order Kobayashi metrics at a local peak point of a bounded domain. Nikolov [7] extended this result for an arbitrary domain in $\mathbb{C}^n$. Coman [3] studied the boundary behavior of the Green function of a bounded domain, which also gives the localization of the Azukawa metric.

In this paper, we first prove a plurisubhamonic version of Schwarz lemma, and show a localization theorem of Azukawa pseudometric at a local plurisubhamonic peak point of a domain in $\mathbb{C}^n$.

2. Localization

We begin with a subharmonic vesion of the Schwarz lemma. Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane.

Theorem 1. *Let* $f : \Delta \to [0, 1)$ *be a function such that*

(a) $\log f(z)$ *is subharmonic, and*

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(b) $\limsup_{z\to 0} \frac{f(z)}{|z|} < \infty$.

Then we have $f(z) \le |z|$. *Moreover if* $f(z) = |z|$ *for some* $z \ne 0$, *then* $f(z) = |z|$.

Proof. Since $\log f(z)/|z| = \log f(z) - \log|z|$ is subharmonic, so is $f(z)/|z|$. Since $f(z)/|z|$ is subharmonic and since $\limsup_{|z|\to 1} f(z)/|z| \le 1$, we have $f(z)/|z| \le 1$ by Maximum principle. In other words, we have $f(z) \le |z|$ for all $z \in \Delta$. If $f(z) = |z|$ for some $z \ne 0$, then $f(z) = |z|$ by the maximal principal for $f(z)/|z|$. ☐

We can see another version of the Schwarz lemma in [8].

Let G be a domain in $\mathbb{C}^n$. For $a \in G$, let $K_G(a)$ be the family of functions $u\colon G \to [0,1)$ such that

(1) $\log u$ is plurisubharmonic, and
(2) there exist $M, r > 0$ such that $u(z) \le M \cdot |z-a|$ for $z \in B(a;r) \subset G$ and $u(z)/|z-a| \le O(1)$ as $z \to a$.

Then the Azukawa pseudometric F_G [1] on G is defined by

$$F_G(a;X) = \sup\left\{ \limsup_{\lambda\to 0} \frac{u(a+\lambda X)}{|\lambda|} \ : \ u(z) \in K_G(a) \right\}.$$

We know that F is a contracting family of pseudometrics, in other words,

(1) for the unit disc Δ, F_Δ is the Pioncaré metric, and
(2) for every holomorphic function $f\colon G \to \Omega$, $F_G(a;X) \ge F_\Omega(f(a); df|_a(X))$.

We now consider the maximal function

$$g_G(z,a) := \sup\{u(z) \ : \ u(z) \in K_G(a)\}.$$

This function is closely related to the Azukawa pseudometric.

Lemma 1. ([1]) *Let G be a domain in $\mathbb{C}^n$. Let $a \in G$, $X \in \mathbb{C}^n$. Then we have*

$$F_G(a;X) = \limsup_{\lambda\to 0} \frac{g_G(a+\lambda X, a)}{|\lambda|}.$$

Proof. Let $g_G(\cdot,a)^*$ be the upper semi-continuous regulation of $g_G(\cdot,a)$. Since, for $u \in K_G(a)$,

$$\begin{aligned} u(z;a) &\le \frac{1}{A(\partial B(z;r))} \int_{\partial B(z;r)} u(z+w)\,dw \\ &\le \frac{1}{A(\partial B(z;r))} \int_{\partial B(z;r)} g(z+w,a)\,dw, \end{aligned}$$

we have

$$g(z,a) \le \frac{1}{A(\partial B(z;r))} \int_{\partial B(z;r)} g(z+w,a)\,dw,$$

where $A(\partial B(z;r))$ is the area of $\partial B(z;r)$. This implies that $\log g(\cdot,a)^*$ is plurisubhamonic.

For $z \in B(a;r) \subset G$, since $g_G(z,a) \le \frac{1}{r}\|z-a\|$, we have $g_G(z,a)^* \le \frac{1}{r}\|z-a\|$. This implies that there exists a postive number r such that $g(z,a)^* \le \frac{1}{r}\|z-a\|$ for all $z \in B(a,r) \subset G$. Then since $\log g(\cdot,a)^*$ is pluri-subhamonic, we obtain that $g_G(z,a)^* \in K_G(a)$. From the definition of $g_G(z,a)$ we can see that

$$g_G(z,a)^* = g_G(z,a) \in K_G(a).$$

Therefore, we have the result. □

Berteloot [2] showed localization for Sibony pseudometric [8]. Here we prefer Azukawa pseudometric in the family of plurisubhamornic functions, and present the localization theorem of this pseudometric.

Theorem 2. *Let Ω be a domain in $\mathbb{R}^n$ and $p \in \partial\Omega$. Let $\sigma\colon V_p \to \mathbb{R}$ be a local continuous plurisubharmonic peak function at p. Then for any neiborhood U of p, there exists a neiborhood W of p such that $f(\Delta_r) \subset U$ for every holomorphic fuction $f\colon \Delta \to \Omega$ with $f(0) \in W$.*

Proof. Choose constants r and R with $0 < r < R$ and $B(p;R) \subset\subset V_p$. By choosing a constant $k > 0$ such that $\sigma(z) \le -k$ for all $z \in \overline{\Omega} \cap \{|z-p| = r\}$, we have $\sigma(z) \cdot 2R^2/k \le -2R^2$. Thus we can assume that $\sigma(z) \le -2R^2$ for $z \in \overline{\Omega} \cap \{|z-p| = r\}$. Define $\tilde{\sigma}\colon \Omega \to \mathbb{R}$ by

$$\tilde{\sigma}(z) = \begin{cases} \max(\sigma(z) + |z-p|^2 - R^2, -2R^2), & \text{on } \overline{\Omega} \cap \{|z-p| < r\}, \\ -2R^2, & \text{on } \overline{\Omega} \cap \{|z-p| \ge r\}. \end{cases}$$

Then $\tilde{\sigma}$ is a negative continuous p.s.h. function. Near p, $\tilde{\sigma}$ is strictly p.s.h. Choose $a > 0$ such that $\tilde{\sigma}(z) = \sigma(z) + |z-p|^2 - R^2$ for all $z \in \overline{\Omega} \cap \{|z-p| < 2a\}$.

Now let $\theta\colon [0,\infty) \to \mathbb{R}$ be a smooth, nondecreasing function such that

$$\theta(x) = \begin{cases} x, & \text{if } x \le \frac{1}{2}, \\ 1, & \text{if } x \ge \frac{3}{4}. \end{cases}$$

Let $z_0 \in \Omega \cap \{|z-p| < a\}$. Let $s > 0$ with $B(z_0;s) \subset \{|z-z_0| < a\}$. For any $\lambda > 0$, define

$$\Psi_\lambda(z) = \begin{cases} \theta\left(\frac{|z-z_0|}{s}\right) \exp(\lambda\tilde{\sigma}(z)), & \text{for } z \in \Omega \cap \{|z-z_0| < s\}, \\ \exp(\lambda\tilde{\sigma}(z)), & \text{for } z \in \Omega \backslash \{|z-z_0| < s\}. \end{cases}$$

In a neighborhood of $\Omega \backslash B(z_0;s)$, $\log \Psi_\lambda(z) = \lambda\tilde{\sigma}(z)$ is plurisubharmonic. Also in a neighborhood of $\overline{B(z_0;s/2)}$, $\log \Psi_\lambda(z) = \log \frac{|z-z_0|}{s} + \lambda\tilde{\sigma}(z)$ is plurisubharmonic. On $B(z_0;s)$, we have

$$\Psi_\lambda(z) = \theta\left(\frac{|z-z_0|}{s}\right) \exp(\lambda\tilde{\sigma}(z)).$$

Let $h(x) = \log\theta(x)$. Then, by considering the eigenvalues of the Levi form of $h(|z|^2)$, we know that there exists $A > 0$ such that

$$\langle \mathcal{L}h(|z|^2)X, X\rangle \ge -A|X|^2$$

for all $z \in B(0;1)\backslash\{0\}$ and $X \in \mathbb{C}^n$. Then we have

$$\langle \mathcal{L} \log \Psi_\lambda(z) X, X\rangle \geq \left(-\frac{A}{s^2} + c\lambda\right)|X|^2$$

Now by choosing $\lambda = A/(cs^2)$, $\log \Psi_\lambda(z)$ is plurisubharmonic in a neighborhood of

$$\overline{B(z_0; s)\backslash B(z; s/2)}.$$

Therefore there exists $\lambda > 0$ such that $\Psi_\lambda \in K_\Omega(z_0)$. Since

$$\lim_{\lambda\to 0} \frac{\frac{|\lambda X|}{s} \cdot \exp\left(\frac{A}{cs^2}\tilde{\sigma}(z_0 + \lambda X)\right)}{|\lambda|} = \frac{1}{s} \cdot \exp\left(\frac{A}{cs^2}\tilde{\sigma}(z_0)\right)|X|$$

we have

$$F_\Omega(z_0; X) \geq \frac{1}{s} \cdot \exp\left(\frac{A}{cs^2}\tilde{\sigma}(z_0)\right)|X|.$$

The proof is done. □

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