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A SYSTEM OF NONLINEAR VARIATIONAL INCLUSIONS WITH (A, η)-MONOTONE MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we introduce a system of nonlinear variational inclusions involving (A, η)-monotone mappings in the framework of Hilbert spaces. Based on the generalized resolvent operator technique associated with (A, η)-monotonicity, the approximation solvability of solutions using an iterative algorithm is investigated. Our results improve and extend the recent ones announced by many others.

1. Introduction

Variational inclusions problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics and transportation equilibrium and engineering sciences. Variational inclusions problems have been generalized and extended in different directions using the novel and innovative techniques. Various kinds of iterative algorithms to solve the variational inequalities and variational inclusions have been developed by many authors. For details, we can refer to [1-9]. Inspired and motivated by the recent research going on in this area, we introduce and analysis a new class of variational inclusions problems involving (A, η)-monotone mappings which was introduced by Verma [9] in the framework of Hilbert spaces. Since (A, η)-monotonicity generalizes A-monotonicity [7] and H-monotonicity [2, 3], our results improve and extend the recent ones announced by many others.

2. Preliminaries

In this section we explore some basic properties derived from the notion of (A, η)-monotonicity. Let H denote a real Hilbert space with the norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Let $\eta : H \times H :\to H$ be a single-valued mapping. The map η is called *τ-Lipschitz continuous* if there is a constant $\tau > 0$ such that

$$\|\eta(u, v)\| \le \tau \|y - v\|, \quad \forall u, v \in H.$$

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Definition 2.1. Let $\eta : H \times H \to H$ be a single-valued mapping and let $M : H \to 2^H$ be a multivalued mapping on H.

(1) The map M is said to be (r, η)-*strongly monotone* if

$$\langle u^* - v^*, \eta(u, v)\rangle \geq r\|u - v\|, \quad \forall (u, u^*), (v, v^*) \in \text{Graph}(M).$$

(2) η-*pseudomonotone* if $\langle v^*, \eta(u, v)\rangle \geq 0$ implies

$$\langle u^*, \eta(u, v)\rangle \geq 0, \quad \forall (u, u^*), (v, v^*) \in \text{Graph}(M).$$

(3) (m, η)-*relaxed monotone* if there exists a positive constant m such that

$$\langle u^* - v^*, \eta(u, v)\rangle \geq -m\|u - v\|^2, \quad \forall (u, u^*), (v, v^*) \in \text{Graph}(M).$$

Definition 2.2. ([2,3]) Let $H : X \to X$ be a nonlinear mapping on a Hilbert space X and let $M : X \to 2^X$ be a multivalued mapping on X. The map M is said to be H-*monotone* if $(H + \rho M)X = X$ for $\rho > 0$.

Definition 2.3. ([7]) Let $A : H \to H$ be a nonlinear mapping on a Hilbert space H and let $M : H \to 2^H$ be a multivalued mapping on H. The map M is said to be A-*monotone* if

(1) M is m-relaxed monotone.

(2) $A + \rho M$ is maximal monotone for $\rho > 0$.

Remark 2.4. A-monotonicity generalizes the notion of H-monotonicity introduced by Fang and Huang [2,3].

Definition 2.5. ([5]) A mapping $M : H \to 2^H$ is said to be *maximal* (m, η)-*relaxed monotone* if

(1) M is (m, η)-relaxed monotone,

(2) for $(u, u^*) \in H \times H$ and

$$\langle u^* - v^*, \eta(u, v)\rangle \geq -m\|u - v\|^2, \quad \forall (v, v^*) \in \text{graph}(M),$$

we have $u^* \in M(u)$.

Definition 2.6. ([5]) Let $A : H \to H$ and $\eta : H \times H \to H$ be two single-valued mappings. The map $M : H \to 2^H$ is said to be (A, η)-*monotone* if

(1) M is (m, η)-relaxed monotone,

(2) $R(A + \rho M) = H$ for $\rho > 0$.

Note that, alternatively, the map $M : H \to 2^H$ is said to be (A, η)-*monotone* if

(1) M is (m, η)-relaxed monotone,

(2) $A + \rho M$ is η-pseudomonotone for $\rho > 0$.

Remark 2.7. (A, η)-monotonicity generalizes the notion of A-monotonicity introduced by Verma [7].

Definition 2.8. Let $A : H \to H$ be an (r, η)-strong monotone mapping and let $M : H \to H$ be an (A, η)-monotone mapping. Then the *generalized resolvent* *operator* $J^{A,\eta}_{M,\rho} : H \to H$ is defined by

$$J^{A,\eta}_{M,\rho}(u) = (A + \rho M)^{-1}(u), \quad \forall u \in H,$$

where $\rho > 0$ is a constant.

Definition 2.9. The map $N : H \times H$ is said to be *relaxed* (β, γ)-*cocoercive* with respect to A if there exists two positive constants α, β such that

$$\langle Tx - Ty, Ax - Ay\rangle \geq (-\beta)\|Tx - Ty\|^2 + \gamma\|x - y\|^2, \quad \forall (x, y) \in H \times H.$$

Proposition 2.10. ([9]) *Let $\eta : H\times \to H$ be a single-valued mapping, $A : H \to H$ be (r, η)-strongly monotone mapping and $M : H \to 2^H$ be an (A, η)-monotone mapping. Then the mapping $(A + \rho M)^{-1}$ is single-valued.*

3. Results on algorithmic convergence analysis

Let $N_1, N_2 : H \to H$ and $\eta_1, \eta_2 : H \times H \to H$ be four nonlinear mappings. Let $M_1 : H \to 2^H$ be an (A, η)-monotone mapping and $M_2 : H \to 2^H$ be an (A_2, η_2)-monotone mapping, respectively. Then the nonlinear system of variational inclusion (NSVI) problem: determine elements $u, v \in H$ such that

$$0 \in A_1u - A_1v + \rho_1[N_1v + M_1u], \tag{3.1}$$

$$0 \in A_2v - A_2u + \rho_2[N_2u + M_2v]. \tag{3.2}$$

Next, we consider a special case of NSVI problem (3.1)-(3.2).

(I) If $M_1 = M_2 = M$, $N_1 = N_2 = N$, $u = v$, $\eta_1 = \eta_2 = \eta$ and $\rho_1 = \rho_2$ in NSVI (3.1)-(3.2), we have the following NVI problem:

Find an element $u \in H$ such that

$$0 \in Nu + Mu. \tag{3.3}$$

In order to prove our main results, we need the following lemmas.

Lemma 3.1. *Let H be a real Hilbert space and let $\eta : H \times H \to H$ be a τ-Lipschitz continuous nonlinear mapping. Let $A : H \to H$ be a (r, η)-strongly monotone and let $M : H \to 2^H$ be (A, η)-monotone. Then the generalized resolvent operator $J^{A,\eta}_{M,\rho} : H \to H$ is $\tau/(r - \rho m)$, that is,*

$$\|J^{A,\eta}_{M,\rho}(x) - J^{A,\eta}_{M,\rho}(y)\| \leq \frac{\tau}{r - \rho m}\|x - y\|, \quad \forall x, y \in H.$$

Lemma 3.2. *Let H be a real Hilbert space, let $A_i : H \to H$ be (r_i, η_i)-strongly monotone and let $M_i : H \to 2^H$ be (A_i, η_i)-monotone. Let $\eta_i : H \times H \to H$ be a τ_i-Lipschitz continuous nonlinear mapping for $i = 1, 2$. Then (u, v) is the solution of NSVI (3.1)-(3.2) if and only if it satisfies*

$$\begin{cases} u = J^{A_1,\eta_1}_{M_1,\rho_1}[A_1v - \rho_1N_1v], \\ v = J^{A_2,\eta_2}_{M_2,\rho_2}[A_2u - \rho_2N_2u]. \end{cases}$$

Next, we consider the following algorithms.

Algorithm 3.1. For any $u_0, v_0 \in H$, compute the sequences $\{u_n\}$ and $\{v_n\}$ by the iterative process:

$$\begin{cases} u_{n+1} = J_{M_1,\rho_1}^{A_1,\eta_1}[A_1 v_n - \rho_1 N_1 v_n], \\ v_n = J_{M_2,\rho_2}^{A_2,\eta_2}[A_2 u_n - \rho_2 N_2 u_n]. \end{cases}$$

If $M_1 = M_2 = M$, $N_1 = N_2 = N$, $u_n = v_n$, $\eta_1 = \eta_2 = \eta$ and $\rho_1 = \rho_2 = \rho$ in Algorithm 3.1, then we have the following algorithm:

Algorithm 3.2. For any $u_0 \in H$, compute the sequence $\{u_n\}$ by the iterative processes:

$$u_{n+1} = J_{M,\rho}^{A,\eta}[Au_n - \rho N u_n].$$

We remark that Algorithm 3.2 gives the approximate solution to the NVI (3.3)

Now, we are in the position to prove our main results.

Theorem 3.1. *Let H be a real Hilbert space, let $A_i : H \times H$ be (r_i, η_i)-strongly monotone and s_i-Lipschitz continuous and let $M_i : H \to 2^H$ be (A_i, η_i)-monotone. Let $\eta_i : H \times H \to H$ be a τ_i-Lipschitz continuous nonlinear mapping and let $N_i : H \to H$ be relaxed (β_i, γ_i)-cocoercive (with respect to A_i) and μ_i-Lipschitz continuous for $i = 1, 2$. Let (u^*, v^*) be the solution of NSVI problem (3.1)-(3.2), $\{u_n\}$ and $\{v_n\}$ be sequences generated by Algorithm 3.1. Suppose the following condition are satisfied: $\tau_1\tau_2\theta_1\theta_2 < (r_1 - \rho_1 m_1)(r_2 - \rho_2 m_2)$, where $\theta_1 = \sqrt{s_1^2 - 2\rho_1\gamma_1 + 2\rho_1\beta_1\mu_1^2 + \rho_1^2\mu_1^2}$ and $\theta_2 = \sqrt{s_2^2 - 2\rho_2\gamma_2 + 2\rho_2\beta_2\mu_2^2 + \rho_2^2\mu_2^2}$, then the sequences $\{u_n\}$ and $\{v_n\}$ converges strongly to u^*, v^*, respectively.*

Proof. Let $(u^*, v^*) \in H$ is the solution of NSVI problem (3.1)-(3.2), we have

$$\begin{cases} u^* = J_{M_1,\rho_1}^{A_1,\eta_1}[A_1 v^* - \rho_1 N_1 v^*], \\ v^* = J_{M_2,\rho_2}^{A_2,\eta_2}[A_2 u^* - \rho_2 N_2 u^*]. \end{cases}$$

It follows that

$$\begin{aligned} \|u_{n+1} - u^*\| &= \|J_{M_1,\rho_1}^{A_1,\eta_1}[A_1 v_n - \rho_1 N_1 v_n] - u^*\| \\ &= \|J_{M_1,\rho_1}^{A_1,\eta_1}[A_1 v_n - \rho_1 N_1 v_n] - J_{M_1,\rho_1}^{A_1,\eta_1}[A_1 v^* - \rho_1 N_1 v^*]\| \\ &\le \frac{\tau_1}{r_1 - \rho_1 m_1}\|A_1 v_n - A_1 v^* - \rho(N_1 v_n - N_1 v^*)\|. \end{aligned} \tag{3.4}$$

It follows from relaxed (β_1, γ_1)-cocoercive monotonicity and μ_1-Lipschitz continuity of N_1 that

$$\begin{aligned} &\|A_1 v_n - A_1 v^* - \rho_1(N_1 v_n - N_1 v^*)\|^2 \\ &= \|A_1 v_n - A_1 v^*\|^2 - 2\rho_1 \langle N_1 v_n - N_2 v^*, A_1 v_n - A_1 u^* \rangle \\ &\quad + \rho_1^2 \|N_1 v_n - N_1 v^*\|^2 \\ &\leq \theta_1^2 \|v_n - v^*\|^2, \end{aligned} \tag{3.5}$$

where $\theta_1 = \sqrt{s_1^2 - 2\rho_1\gamma_1 + 2\rho_1\beta_1\mu_1^2 + \rho_1^2\mu_1^2}$. On the other hand, one has

$$\begin{aligned} \|v_n - v^*\| &= \|J_{M_2,\rho_2}^{A_2,\eta_2}[A_2 u_n - \rho_2 N_2 u_n] - v^*\| \\ &= \|J_{M_2,\rho_2}^{A_2,\eta_2}[A_2 u_n - \rho_2 N_2 u_n] - J_{M_2,\rho_2}^{A_2,\eta_2}[A_2 u^* - \rho_2 N u^*]\| \\ &\leq \frac{\tau_2}{r_2 - \rho_2 m_2} \|A_2 u_n - A_2 u^* - \rho_2(N_2 u_n - N_2 u^*)\|. \end{aligned} \tag{3.6}$$

It follows from relaxed (β_2, γ_2)-cocoercive monotonicity and μ_2-Lipschitz continuity of N_2 that

$$\begin{aligned} &\|A_2 u_n - A_2 u^* - \rho_2(N_2 u_n - N_2 u^*)\|^2 \\ &= \|A_2 u_n - A_2 u^*\|^2 - 2\rho_2 \langle N_2 u_n - N_2 u^*, A_2 u_n - A_2 u^* \rangle \\ &\quad + \rho_2^2 \|N_2 u_n - N_2 u^*\|^2 \\ &\leq \theta_2^2 \|u_n - u^*\|^2, \end{aligned} \tag{3.7}$$

where $\theta_2 = \sqrt{s_2^2 - 2\rho_2\gamma_2 + 2\rho_2\beta_2\mu_2^2 + \rho_2^2\mu_2^2}$. Substituting (3.7) into (3.6) yields that

$$\|v_n - v^*\| \leq \frac{\tau_2\theta_2}{r_2 - \rho_2 m_2} \|u_n - u^*\|. \tag{3.8}$$

Substituting (3.8) into (3.5), we have

$$\|A_1 v_n - A_1 v^* - \rho_1(N_1 v_n - N_1 v^*)\| \leq \frac{\tau_2\theta_1\theta_2}{r_2 - \rho_2 m_2} \|u_n - u^*\|. \tag{3.9}$$

Again, Substituting (3.9) into (3.4), one has

$$\|u_{n+1} - u^*\| \leq \frac{\tau_1\tau_2\theta_1\theta_2}{(r_1 - \rho_1 m_1)(r_2 - \rho_2 m_2)} \|u_n - u^*\|. \tag{3.10}$$

Observing the assumption $\tau_1\tau_2\theta_1\theta_2 < (r_1 - \rho_1 m_1)(r_2 - \rho_2 m_2)$, we can obtain the desired conclusion. This completes the proof. □

From Theorem 3.1, we have the following result immediately.

Corollary 3.2. *Let H be a real Hilbert space, let $A : H \times H$ be (r, η)-strongly monotone and s-Lipschitz continuous and let $M : H \to 2^H$ be (A, η)-monotone. Let $\eta : H \times H \to H$ be a τ-Lipschitz continuous nonlinear mapping and let $N : H \to H$ be relaxed (β, γ)-cocoercive (with respect to A) and μ-Lipschitz continuous for $i = 1, 2$. Let u^* be the solution of NVI problem* (3.3) *and* $\{u_n\}$

be a sequence generated by Algorithm 3.2. Suppose the following condition are satisfied: $\tau\theta < (r - \rho m)$, *where* $\theta = \sqrt{s^2 - 2\rho\gamma + 2\rho\beta\mu^2 + \rho^2\mu^2}$, *then the sequence* $\{u_n\}$ *converges strongly to* u^*.

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