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CHARACTERIZATION OF OPERATORS TAKING P-SUMMABLE SEQUENCES INTO SEQUENCES IN THE RANGE OF A VECTOR MEASURE

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Abstract. We characterize operators between Banach spaces sending unconditionally weakly p-summable sequences into sequences that lie in the range of a vector measure of bounded variation. Further, we describe operators between Banach spaces taking unconditionally weakly p-summable sequences into sequences that lie in the range of a vector measure.

1. Introduction

The intriguing connection between the geometry of subsets of Banach spaces and vector measure theory is not confined to Radon-Nikodym considerations. Questions regarding the finer structure of the range of a vector measure have found interest since Liapounoff's discovery of his everintriguing convexity theorem which states that the range of a nonatomic vector measure with values in a finite dimensional space is compact and convex. The infinite dimensional version of Liapounoff's theorem remained resistant to analysis for a long time. It is an important fact, first established by Bartle, Dunford and Schwartz in the early fifties, that the range of a vector measure is always relatively weakly compact.

Among the relatively weakly compact subsets of Banach spaces, those that are the range of a vector measure occupy a special place ; a remarkable similarity to the relatively norm compact sets is evidenced. For instance, Diestel and Seifert [3] proved that any sequence in the range of a vector measure admits a subsequence with norm convergent arithmetic means, a phenomenon not shared by all weakly compact sets.

Any intuition gained by noting the similarities between relatively norm compact sets and sets arising as ranges of vector measures must be tempered by the fact that the closed unit ball of an infinite dimensional Banach space can be the range of a vector measure.

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Anantharaman and Garg [1] proved that the closed unit ball of a Banach space X is the range of a vector measure if and only if the dual of a Banach space X is isometrically isomorphic to a reflexive subspace of $L^1(\mu)$ for some probability measure μ.

Anantharaman and Diestel [2] found that every weakly compact subset of $BD1$ (the separable $\mathcal{L}_\infty$ space of Bourgain and Delbaen that has the weakly compact extension property) lies inside the range of a $BD1$-valued measure. They also proved that every weakly 2-summable sequence in a Banach space X lies inside the range of an X-valued measure.

Piñeiro and Rodriguez-Piazza [6] showed that the compact subset of a Banach space X lies inside the range of an X-valued measure if and only if the dual of a Banach space X can be embedded into an $L^1(\mu)$-space for a suitable measure μ.

It is an easy consequence of the celebrated Dvoretsky-Rogers theorem that given an infinite dimensional Banach space X, there is an X-valued measure that does not have finite variation [11]. Thus the question arose : Which Banach spaces X have the property that every compact subset of X lies inside the range of an X-valued measure of bounded variation ? This was settled by Piñeiro and Rodriguez-Piazza [6]: Only finite-dimensional Banach spaces have this property.

Piñeiro [8] characterized Banach spaces X having the property that every weakly p-summable sequence in X lies inside the range of an X^{**}-valued measure of bounded variation provided that $1 < p < \infty$.

Piñeiro [9] also gave descriptions of Banach spaces X for which every unconditionally weakly p-summable sequence in X lies inside the range of an X-valued measure when $p > 2$.

In this paper we deal with the above mentioned problems in the framework of operators acting between Banach spaces.

We introduce the space $\mathcal{R}(X_p^u, Y)$ of all operators from a Banach space X into a Banach space Y taking unconditionally weakly p-summable sequences in X into sequences that lie in the range of a Y-valued measure. In addition, we define $\mathcal{R}_{bv}(X_p^u, Y)$ as the set of all operators from a Banach space X into a Banach space Y sending unconditionally weakly p-summable sequences in X into sequences that lie in the range of a Y-valued measure with bounded variation.

We first provide a description of operators belonging to the space $\mathcal{R}_{bv}(X_p^u, Y)$ in terms of (1,p,1)-summing operators.

Next we give usable necessary and sufficient conditions for an operator to belong to the space $\mathcal{R}(X_p^u, Y)$.

Finally we turn to the consideration of sequences in the range of a vector measure and give usable necessary condition for a sequence in a Banach space X to lie in the range of an X-valued measure with relatively compact range under the hypothesis that every p-integral operator from X to ℓ_1 is 1-summing when $1 < p < \infty$.

2. Definitions and Notations

We present some of the definitions and notation to be used. Throughout this paper X and Y denote Banach spaces.

A function μ from a σ-field Σ of subsets of a set Ω to a Banach space X is called a countably additive vector measure if $\mu(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ in the norm topology of X for all sequences (E_n) of pairwise disjoint members of Σ such that $\cup_{n=1}^{\infty} E_n \in \Sigma$. The range of μ will be denoted by $\operatorname{rg}\mu$. The variation of μ is the extended nonnegative function $|\mu|$ whose value on a set $E \in \Sigma$ is given by $|\mu|(E) = \sup_{\pi} \sum_{A\in\pi} \|\mu(A)\|$, where the supremum is taken over all partitions π of E into a finite number of pairwise disjoint members of Σ. If $|\mu|(\Omega) = \operatorname{tv}(\mu) < \infty$ then μ will be called a measure of bounded variation. The semivariation of μ is the extended nonnegative function $\|\mu\|$ whose value on a set $E \in \Sigma$ is given by $\|\mu\|(E) = \sup\{|x^* \circ \mu|(E) : x^* \in X^*, \|x^*\| \leq 1\}$, where $|x^* \circ \mu|$ is the variation of the real-valued measure $x^* \circ \mu$. If $\|\mu\|(\Omega) = \operatorname{tsv}(\mu) < \infty$, then μ will be called a measure of bounded semivariation.

Notation. (1) The dual of a Banach space X is denoted by X^*.
(2) The closed unit ball of a Banach space X is denoted by B_X.
(3) The dual operator of an operator T is denoted by T^*.
(4) $\mathcal{B}(X,Y)$ denotes the set of all bounded linear operators from X into Y.
(5) For $1 < p < \infty$, the conjugate exponent of p is denoted by p', i.e. $1/p + 1/p' = 1$.

The space $\mathcal{R}(X)$ is defined to consist of all sequences (x_n) in X such that there exists an X-valued measure μ satisfying $\{x_n : n \in \mathbb{N}\} \subset \operatorname{rg}\mu$. For each $(x_n) \in \mathcal{R}(X)$, define $\|(x_n)\|_r = \inf \operatorname{tsv}(\mu)$, where the infimum is taken over all vector measures μ as above.

The space $\mathcal{R}_c(X)$ consists of all sequences in X that lie inside the range of an X-valued measure with relatively compact range. If (x_n) belongs to $\mathcal{R}_c(X)$ then proposition 1.4 of [6] ensures that there exists an unconditionally convergent series $\sum_{k=1}^{\infty} y_k$ in X for which $\{x_n : n \in \mathbb{N}\} \subset \{\sum_{k=1}^{\infty} \alpha_k y_k : (\alpha_k) \in \ell_\infty, \|(\alpha_k)\|_\infty \leq 1\}$. For each $(x_n) \in \mathcal{R}_c(X)$, define $\|(x_n)\|_{rc} = \inf \sup\{\sum_{k=1}^{\infty} |\langle x^*, y_k\rangle| : x^* \in B_{X^*}\}$, where the infimum is taken over all unconditionally convergent series $\sum_{k=1}^{\infty} y_k$ of the kind described above.

The space $\mathcal{R}_{bv}(X)$ is defined to consist of all sequences (x_n) in X such that there exists an X-valued measure μ with bounded variation satisfying $\{x_n : n \in \mathbb{N}\} \subset \operatorname{rg}\mu$. For each $(x_n) \in \mathcal{R}_{bv}(X)$, set $\|(x_n)\|_{bv} = \inf \operatorname{tv}(\mu)$, where the infimum is taken over all vector measures μ as above.

The space $\mathcal{R}_{bvc}(X)$ consists of all sequences (x_n) in X such that there exists an absolutely convergent series $\sum_{k=1}^{\infty} y_k$ in X satisfying $\{x_n : n \in \mathbb{N}\} \subset \{\sum_{k=1}^{\infty} \alpha_k y_k : (\alpha_k) \in \ell_\infty, \|(\alpha_k)\|_\infty \leq 1\}$. For each $(x_n) \in \mathcal{R}_{bvc}(X)$, let

$\|(x_n)\|_{bvc} = \inf \sum_{k=1}^{\infty} \|y_k\|$, where the infimum is extended over all such absolutely convergent series $\sum_{k=1}^{\infty} y_k$.

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. We say that the operator $T : X \to Y$ belongs to $\mathcal{A}^*(X, Y)$ provided there is a constant $C \geq 0$ such that regardless of the finite dimensional normed spaces E and F and operators $a \in \mathcal{B}(E, X), b \in \mathcal{B}(Y, F)$ and $U \in \mathcal{B}(F, E)$, the composition $E \xrightarrow{a} X \xrightarrow{T} Y \xrightarrow{b} F \xrightarrow{U} E$ satisfies $|\operatorname{tr}(UbTa)| \leq C \cdot \|a\| \cdot \|b\| \cdot \alpha(U)$. The collection of all such C has an infimum, which is denoted by $\alpha^*(T)$. The Banach operator ideal $[\mathcal{A}^*, \alpha^*]$ is called the adjoint operator ideal of $[\mathcal{A}, \alpha]$.

For $1 \leq p \leq \infty$, an operator $T \in \mathcal{B}(X, Y)$ is called p-integral if there are a probability measure μ and operators $A \in \mathcal{B}(L^p(\mu), Y^{**})$ and $B \in \mathcal{B}(X, L^{\infty}(\mu))$ such that $\kappa_Y \circ T = A \circ i_p \circ B$, where $i_p : L^{\infty}(\mu) \to L^p(\mu)$ is the formal identity. The p-integral norm of T is defined by $\imath_p(T) = \inf\{\|A\|\|B\|\}$, where the infimum is extended over all measures μ and operators A and B as above. The collection of all p-integral operators from X into Y is denoted by $\mathcal{I}_p(X, Y)$.

Let $1 \leq p < \infty$. The vector sequence (x_n) in X is weakly p-summable if the scalar sequences $(\langle x^*, x_n \rangle)$ are in ℓ_p for every $x^* \in X^*$. We denote by $\ell_p^{\mathrm{weak}}(X)$ the set of all such sequences in X. This is a Banach space under the norm

$$\|(x_n)\|_p^{\mathrm{weak}} = \sup\big\{(\sum_n |\langle x^*, x_n \rangle|^p)^{1/p} : x^* \in X^*,\ \|x^*\| \leq 1\big\}.$$

If $(x_n) \in \ell_p^{\mathrm{weak}}(X)$ and P is a finite subset of $\mathbb{N}$, $(x_n(P))$ is the sequence defined by

$$x_n(P) = \begin{cases} x_n & \text{if } n \in P, \\ 0 & \text{if } n \notin P \end{cases}$$

for all $n \in \mathbb{N}$. We denote by $\ell_p^u(X)$ the set of all sequences (x_n) such that the net $(x_n(P))_{P \in \mathcal{F}(\mathbb{N})}$ converges to (x_n) in $\ell_p^{\mathrm{weak}}(X)$, where $\mathcal{F}(\mathbb{N})$ is the set of all finite subsets of $\mathbb{N}$.

We write $\mathcal{R}(X_p^u, Y)$ (respectively $\mathcal{R}_c(X_p^u, Y)$) for the set of all operators T from X into Y such that for each sequence $(x_n) \in \ell_p^u(X)$, the sequence (Tx_n) belongs to $\mathcal{R}(Y)$ (respectively $\mathcal{R}_c(Y)$).

We denote by $\mathcal{R}_{bv}(X_p^u, Y)$ (respectively $\mathcal{R}_{bvc}(X_p^u, Y)$) the space of all operators T from X into Y such that for each sequences $(x_n) \in \ell_p^u(X)$, the sequence (Tx_n) belongs to $\mathcal{R}_{bv}(Y)$ (respectively $\mathcal{R}_{bvc}(Y)$).

For $1 \leq p \leq \infty$, an operator $T \in \mathcal{B}(X, Y)$ is said to be p-nuclear if it can be written in the form $T = \sum_{i=1}^{\infty} x_i^* \otimes y_i$, where (x_i^*) in X^* and (y_i) in Y satisfy $N_p((x_i^*), (y_i)) < \infty$. Here

$$N_p((x_i^*), (y_i)) = \begin{cases} \|(x_i^*)\|_1^{\mathrm{strong}} \cdot (\sup_i \|y_i\|) & \text{for } p = 1, \\ \|(x_i^*)\|_p^{\mathrm{strong}} \cdot \|(y_i)\|_{p'}^{\mathrm{weak}} & \text{for } 1 < p < \infty, \\ (\sup_i \|x_i\|) \cdot \|(y_i)\|_1^{\mathrm{weak}} & \text{for } p = \infty. \end{cases}$$

Each such representation of T is called a p-nuclear representation. The set of all p-nuclear operators from X into Y is denoted by $\mathcal{N}_p(X,Y)$. With each $T \in \mathcal{N}_p(X,Y)$ we associate its p-nuclear norm, $\nu_p(T) = \inf N_p((x_i^*),(y_i))$, where the infimum is taken over all p-nuclear representations of T.

For $1 \leq p < \infty$, an operator $T \in \mathcal{B}(X,Y)$ is called absolutely p-summing if there exists a constant $C \geq 0$ such that for any finite subset $\{x_i\}_{i=1}^n \subset X$, we have

$$\left(\sum_{i=1}^n \|Tx_i\|^p\right)^{1/p} \leq C \cdot \sup\big\{(\sum_{i=1}^n |\langle x^*, x_n\rangle|^p)^{1/p} : x^* \in X^*,\ \|x^*\| \leq 1\big\}.$$

The infimum of such C is the absolutely p-summing norm of T and denoted by $\pi_p(T)$. We write $\Pi_p(X,Y)$ for the set of all absolutely p-summing operators from X into Y.

Let $1 \leq q < \infty$, $1 \leq p, r \leq \infty$ and $1/q \leq 1/p + 1/r$. An operator $T \in \mathcal{B}(X,Y)$ is called absolutely (q,p,r)-summing if there exists a constant $C \geq 0$ such that for all finite subsets $\{x_i\}_{i=1}^n \subset X$ and $\{y_i^*\}_{i=1}^n \subset Y^*$, we have

$$\left(\sum_{i=1}^n |\langle Tx_i, y_i^*\rangle|^q\right)^{1/q} \leq C \cdot \|(x_i)\|_p^{\text{weak}} \cdot \|(y_i^*)\|_r^{\text{weak}}.$$

The infimum of such C is the absolutely (q,p,r)-summing norm of T and denoted by $\pi_{q,p,r}(T)$. We write $\Pi_{q,p,r}(X,Y)$ for the set of all absolutely (q,p,r)-summing operators from X into Y.

A Banach space X has the Radon-Nikodym property with respect to (Ω, Σ, μ) if for each μ-continuous vector measure $G : \Sigma \to X$ of bounded variation there exists $g \in L_1(\mu, X)$ such that $G(E) = \int_E g\,d\mu$ for all $E \in \Sigma$. A Banach space X has the Radon-Nikodym property if X has the Radon-Nikodym property with respect to every finite mesure space.

We will say that a Banach space X satisfies Grothendieck's theorem if every operator from X into a Hilbert space is absolutely 1-summing.

3. Results

Let us start with the problem which gives a description of operators belonging to the space $\mathcal{R}_{bv}(X_p^u, Y)$ in terms of (1,p,1)-summing operators.

Theorem 1. *Let $1 < p < \infty$. Then the following statements about an operator $T : X \to Y$ are equivalent :*

(i) $T \in \mathcal{R}_{bvc}(X_p^u, Y)$.
(ii) $T \in \mathcal{R}_{bv}(X_p^u, Y)$.
(iii) $T \in \Pi_{1,p,1}(X,Y)$.

Proof. (i)⇒(ii). This is an easy consequence of the fact that $\mathcal{R}_{bvc}(Y) \subset \mathcal{R}_{bv}(Y)$.

(ii)⇒(iii). Let us select any sequence $(x_n) \in \ell_p^u(X)$. The hypothesis (ii) leads us to have that given $\epsilon > 0$ there exists a vector measure $\mu : \sum \to Y$ with bounded variation for which $\{Tx_n : n \in \mathbb{N}\} \subset \operatorname{rg}\mu$ and $\operatorname{tv}(\mu) \leq \epsilon + \|(Tx_n)\|_{bv}$. Take any sequence (y_n^*) in $\ell_1^{\text{weak}}(Y^*)$ and associate with it the map $S : Y \to \ell_1$ given by $Sy = (\langle y_n^*, y\rangle)_n$. Consider the integration operator $I : L_\infty(|\mu|) \to Y : f \mapsto \int f d\mu$. Then $S \circ I \in \Pi_1(L_\infty(|\mu|), \ell_1)$. The Radon-Nikodym property of ℓ_1 ensures that $S \circ I$ is nuclear and so is $(S \circ I)^*$. Note that

$$(\cdot) \quad \|(S \circ I)^* e_n\| = \sup\{|\langle (S \circ I)^* e_n, f\rangle| : \|f\|_\infty \leq 1\}$$
$$= \sup\{|\langle \int f d\mu, y_n^*\rangle| : \|f\|_\infty \leq 1\} = \sup\{|\int f d(y_n^* \circ \mu)| : \|f\|_\infty \leq 1\}.$$

Choose $A_n \in \sum$ such that $\mu(A_n) = Tx_n$ for each $n \in \mathbb{N}$. Then it follows from $(\cdot)$ that

$$\sum_n |\int \chi_{A_n} d(y_n^* \circ \mu)| = \sum_n |y_n^* \circ \mu(A_n)| = \sum_n |\langle y_n^*, Tx_n\rangle|$$
$$\leq \sum_n \|(S \circ I)^* e_n\| = \nu_1((S \circ I)^*) \leq \nu_1(S \circ I) = \pi_1(S \circ I)$$
$$= \operatorname{tv}(S \circ \mu) \leq \|S\| \operatorname{tv}(\mu) \leq \|S\|(\epsilon + \|(Tx_n)\|_{bv}).$$

This permits us to create a continuous linear map $\phi : R_{bv}(TX) \to \mathbb{R}$ through $\phi(Tx_n) = \sum_{n=1}^\infty \langle Tx_n, y_n^*\rangle$ for all $(Tx_n) \in R_{bv}(TX)$. Another appeal to the hypothesis (ii) establishes that the natural map $J : \ell_p^u(TX) \to R_{bv}(TX)$ is continuous. Hence the composition $\phi \circ J : \ell_p^u(TX) \to \mathbb{R}$ is continuous. Then the operator $u : TX \to \ell_{p'}$ defined by $u(Tx) = (\langle Tx, y_n^*\rangle)_n$ is integral. The reflexivity of $\ell_{p'}$ assures us that u is nuclear. The upshot of all this is that the map $\Phi : \mathcal{B}(Y, \ell_1) \to \mathcal{N}_1(TX, \ell_{p'}) : S \mapsto i_{1p'} \circ S|_{TX}$ is continuous, where $i_{1p'} : \ell_1 \to \ell_{p'}$ is the formal inclusion map. Consequently for every $n \in \mathbb{N}$ there is a constant $C > 0$ such that

$$(\cdot\cdot) \qquad \nu_1(\sum_{k=1}^n y_k^* \otimes e_k : TX \to \ell_{p'}^n) \leq C \cdot \sup\{\sum_{k=1}^n |\langle y, y_k^*\rangle| : \|y\| \leq 1\}.$$

Now given $x_1, \cdots x_n$ in X and $y_1^*, \cdots y_n^*$ in Y^* we define operators $u : TX \to \ell_{p'}^n$ and $v : \ell_{p'}^n \to Y$ via $u(Tx) = (\langle Tx, y_i^*\rangle)_{i=1}^n$ and $v((a_i)_{i=1}^n) = \sum_{i=1}^n a_i Tx_i$, respectively. We call on condition $(\cdot\cdot)$ to obtain the following :

$$|\operatorname{tr}(u \circ v)| = |\sum_{i=1}^n \langle Tx_i, y_i^*\rangle| \leq \sum_{i=1}^n |\langle Tx_i, y_i^*\rangle|$$
$$\leq \imath_1(u \circ v) = \nu_1(u \circ v) \leq \|v\| \cdot \nu_1(u)$$
$$\leq C \cdot \sup\{(\sum_{i=1}^n |\langle y^*, Tx_i\rangle|^p)^{1/p} : \|y\| \leq 1\} \cdot \sup\{\sum_{i=1}^n |\langle y, y_i^*\rangle| : \|y\| \leq 1\}$$
$$\leq C' \cdot \|(x_i)_{i=1}^n\|_p^{\text{weak}} \cdot \|(y_i^*)_{i=1}^n\|_1^{\text{weak}}.$$

This signifies that $T \in \Pi_{1,p,1}(X, Y)$.

(iii)$\Rightarrow$(i). Take a finite sequence $(y_i^*)_{i=1}^n$ in Y^* so that the linear map $u : Y \to \ell_1^n : y \mapsto (\langle y_i^*, y\rangle)_{i=1}^n$ is bounded. For every $n \in \mathbb{N}$, we define a linear map $U_n : (X^n, \|\cdot\|_p^{\text{weak}}) \to \mathcal{N}_1(\ell_1^n, Y)$ by $U_n(x_i)_1^n = \sum_{i=1}^n e_i \otimes Tx_i = v_x$. We take account of hypothesis (iii) to deduce that

$$
\begin{aligned}
|\operatorname{tr}(u \circ v_x)| &= |\sum_{i=1}^n \langle Tx_i, y_i^*\rangle| \le \sum_{i=1}^n |\langle Tx_i, y_i^*\rangle| \\
&\le C \cdot \|(x_i)_{i=1}^n\|_p^{\text{weak}} \cdot \|(y_i^*)_{i=1}^n\|_1^{\text{weak}} = C \cdot \|(x_i)_{i=1}^n\|_p^{\text{weak}} \cdot \|u\|.
\end{aligned}
$$

This validates the following :

$$\nu_1(v_x) = \sup\{|\operatorname{tr}(u \circ v_x)| : u \in \mathcal{B}(Y, \ell_1^n), \|u\| \le 1\} \le C \cdot \|(x_i)_{i=1}^n\|_p^{\text{weak}}$$

and hence $\|U_n\| \le C$ for every $n \in \mathbb{N}$. This yields that a linear map $U : (X^{\mathbb{N}}, \|\cdot\|_p^{\text{weak}}) \to \mathcal{N}(\ell_1, Y)$ defined by $U(x_n) = \sum_n e_n \otimes Tx_n$, is continuous.

Now let us take any sequence $(x_n) \in \ell_p^u(X)$. Then the operator $S = \sum_n e_n \otimes Tx_n \in \mathcal{N}_1(\ell_1, Y)$. Fix $\epsilon > 0$ and choose a nuclear representation $S = \sum_n \beta_n \otimes z_n$ such that $\sum_n \|\beta_n\|_{\ell_\infty} \|z_n\| \le \nu_1(S) + \epsilon$. Writing $\alpha = (\alpha_n) \in \ell_1$ and $\beta_k = (\beta_{k,n})_n \in \ell_\infty$, we see that $S\alpha = \sum_k \langle \alpha, \beta_k\rangle z_k = \sum_n \alpha_n Tx_n = \sum_k (\sum_n \alpha_n \beta_{k,n}) z_k$, and so $Tx_n = \sum_k \frac{\beta_{k,n}}{\|\beta_k\|_\infty} \|\beta_k\|_\infty z_k$. Since $\sum_k \|\beta_k\|_\infty z_k$ is an absolutely convergent series in Y, it follows that $(Tx_n) \in \mathcal{R}_{bvc}(Y)$. This forces that $T \in \mathcal{R}_{bvc}(X_p^u, Y)$. □

Applying the above theorem we draw usable necessary condition which guarantees that every unconditionally weakly p-summable sequence in a Banach space X belongs to $\mathcal{R}_{bvc}(X)$.

Corollary 2. *Let $1 < p < \infty$. Suppose that $\ell_p^u(X) \subset \mathcal{R}_{bvc}(X)$. Then for every Banach space Y, we have $\Pi_1(X, Y) \subset \Pi_{1,p,1}(X, Y)$.*

Proof. Let us take any operator $T \in \Pi_1(X, Y)$. Then our hypothesis informs us that T takes each sequence (x_n) in $\ell_p^u(X)$ into a sequence (Tx_n) in $\mathcal{R}_{bv}(Y)$. Theorem 1 steps in to conclude that $T \in \Pi_{1,p,1}(X, Y)$. This gives us the desired inclusion. □

Theorem 1 enables us to find a special kind of Banach space Y with the property that any operator $T \in \mathcal{B}(X, Y)$ belongs to $\mathcal{R}_{bvc}(X_p^u, Y)$.

Corollary 3. *Let $1 < p < \infty$ and let $T \in \mathcal{B}(X, Y)$. If Y^* satisfies Grothendieck's theorem, then $T \in \mathcal{R}_{bvc}(X_p^u, Y)$.*

Proof. We select any operator $S \in \mathcal{B}(Y, \ell_1)$. As Y^* satisfies Grothendieck's theorem, we have that $S \in \Pi_2(Y, \ell_1)$ and hence $S|_{TX} \in \Pi_2(TX, \ell_1)$. Taking note of the fact that the formal inclusion map $i_{1p'} : \ell_1 \to \ell_{p'}$ is 1-summing we derive that $i_{1p'} \circ S|_{TX} \in \mathcal{N}_1(TX, \ell_{p'})$. Then we see from the proof of theorem 1 that $T \in \mathcal{R}_{bvc}(X_p^u, Y)$. □

In the next theorem we establish the following characterization of operators belonging to the space $\mathcal{R}(X_p^u, Y)$.

Theorem 4. *Let $1 < p < \infty$ and let $T \in \mathcal{B}(X, Y)$. Then the following statements are equivalent :*

(i) $T \in \mathcal{R}_c(X_p^u, Y)$.
(ii) $T \in \mathcal{R}(X_p^u, Y)$.
(iii) *There exists a constant $C > 0$ such that*

$$\sum_{i=1}^{n} |\langle Tx_i, y_i^* \rangle| \leq C \cdot \|(x_i)_{i=1}^n\|_p^{weak} \cdot \pi_1(\sum_{i=1}^{n} y_i^* \otimes e_i : Y \to \ell_1^n)$$

regardless of the choice of $n \in \mathbb{N}$, and the vectors $x_1, \cdots x_n$ in X and $y_1^*, \cdots y_n^*$ in Y^*.

Proof. (i)⇒(ii) is an immediate consequence of the fact that $\mathcal{R}_c(Y) \subset \mathcal{R}(Y)$.

(ii)⇒(iii). We select any sequence $(x_n) \in \ell_p^u(X)$. The hypothesis (ii) tells us that $(Tx_n) \in \mathcal{R}(Y)$ and so given $\epsilon > 0$ there exists a vector measure $\mu : \sum \to Y$ for which $\{Tx_n : n \in \mathbb{N}\} \subset \operatorname{rg} \mu$ and $\operatorname{tsv}(\mu) \leq \epsilon + \|(Tx_n)\|_r$. Let λ be a control measure for μ and let $I : L_\infty(\lambda) \to Y : f \mapsto \int f d\mu$ be the integration operator. Take a sequence (y_n^*) in Y^* so that the linear map $S : Y \to \ell_1 : y \mapsto (\langle y_n^*, y \rangle)_n$ is 1-summing. Then $S \circ I : L_\infty(\lambda) \to \ell_1$ is 1-summing. The Radon-Nikodym property of ℓ_1 indicates that $S \circ I$ is nuclear and so is $(S \circ I)^*$. Notice that

$$(\cdot) \quad \|(S \circ I)^* e_n\| = \sup\{|\langle (S \circ I)^* e_n, f \rangle| : \|f\|_\infty \leq 1\}$$
$$= \sup\{|\langle \int f d\mu, y_n^* \rangle| : \|f\|_\infty \leq 1\} = \sup\{|\int f d(y_n^* \circ \mu)| : \|f\|_\infty \leq 1\}.$$

Choose $A_n \in \sum$ so that $\mu(A_n) = Tx_n$ for each $n \in \mathbb{N}$. We deduce from $(\cdot)$ that

$$\sum_n |\int \chi_{A_n} d(y_n^* \circ \mu)| = \sum_n |y_n^* \circ \mu(A_n)| = \sum_n |\langle y_n^*, Tx_n \rangle|$$
$$\leq \sum_n \|(S \circ I)^* e_n\| = \nu_1((S \circ I)^*) \leq \nu_1(S \circ I) = \pi_1(S \circ I)$$
$$= \operatorname{tv}(S \circ \mu) \leq \pi_1(S) \operatorname{tsv}(\mu) \leq \pi_1(S)(\epsilon + \|(Tx_n)\|_r).$$

This allows us to define a continuous linear map $\phi : R(TX) \to \mathbb{R}$ via $\phi(Tx_n) = \sum_{n=1}^\infty \langle Tx_n, y_n^* \rangle$ for all $(Tx_n) \in R(TX)$. It takes another appeal to the hypothesis (ii) to reveal that the natural map $J : \ell_p^u(TX) \to R(TX)$ is continuous. Thus the composition $\phi \circ J : \ell_p^u(TX) \to \mathbb{R}$ is continuous. Then the operator $u : TX \to \ell_{p'}$ defined by $u(Tx) = (\langle Tx, y_n^* \rangle)_n$ is integral. The reflexivity of $\ell_{p'}$ guarantees that u is nuclear. The upshot of all this is that the map $\Phi : \Pi(Y, \ell_1) \to \mathcal{N}_1(TX, \ell_{p'}) : S \mapsto i_{1p'} \circ S|_{TX}$ is continuous, where $i_{1p'} : \ell_1 \to \ell_{p'}$ is the formal inclusion map. Hence for every $n \in \mathbb{N}$ there is a constant $C > 0$ such that

$$(\cdot\cdot) \qquad \nu_1(\sum_{k=1}^{n} y_k^* \otimes e_k : TX \to \ell_{p'}^n) \leq C \cdot \pi_1(\sum_{k=1}^{n} y_k^* \otimes e_k : Y \to \ell_1^n).$$

Now given $x_1, \cdots x_n$ in X and $y_1^*, \cdots y_n^*$ in Y^* we define operators $u : TX \to \ell_{p'}^n$ and $v : \ell_{p'}^n \to Y$ by $u(Tx) = (\langle Tx, y_i^* \rangle)_{i=1}^n$ and $v((a_i)_{i=1}^n) = \sum_{i=1}^n a_i Tx_i$, respectively. We make use of condition $(\cdot\cdot)$ to obtain the following :

$$
\begin{aligned}
|\operatorname{tr}(u \circ v)| = |\sum_{i=1}^n \langle Tx_i, y_i^* \rangle| &\le \sum_{i=1}^n |\langle Tx_i, y_i^* \rangle| \\
&\le \imath_1(u \circ v) = \nu_1(u \circ v) \le \|v\| \cdot \nu_1(u) \\
\le C \cdot \sup\{(\sum_{i=1}^n |\langle y^*, Tx_i \rangle|^p)^{1/p} : \|y^*\| \le 1\} &\cdot \pi_1(\sum_{i=1}^n y_i^* \otimes e_i : Y \to \ell_1^n) \\
&\le C' \cdot \|(x_i)_{i=1}^n\|_p^{\text{weak}} \cdot \pi_1(\sum_{i=1}^n y_i^* \otimes e_i : Y \to \ell_1^n).
\end{aligned}
$$

(iii)$\Rightarrow$(i). Take a finite sequence $(y_i^*)_{i=1}^n$ in Y^* so that the linear map $u : Y \to \ell_1^n : y \mapsto (\langle y_i^*, y \rangle)_{i=1}^n$ is 1-summing. For every $n \in \mathbb{N}$, we define a linear map $U_n : (X^n, \|\cdot\|_p^{\text{weak}}) \to \mathcal{N}_\infty(\ell_1^n, Y)$ by $U_n(x_i)_1^n = \sum_{i=1}^n e_i \otimes Tx_i = \upsilon_x$. As a consequence of hypothesis (iii) we have

$$
\begin{aligned}
&|\operatorname{tr}(u \circ \upsilon_x)| = |\sum_{i=1}^n \langle Tx_i, y_i^* \rangle| \le \sum_{i=1}^n |\langle Tx_i, y_i^* \rangle| \\
&\le C \cdot \|(x_i)_{i=1}^n\|_p^{\text{weak}} \cdot \pi_1(\sum_{i=1}^n y_i^* \otimes e_i : Y \to \ell_1^n) = C \cdot \|(x_i)_{i=1}^n\|_p^{\text{weak}} \cdot \pi_1(u).
\end{aligned}
$$

This validates the following :

$$\nu_\infty(\upsilon_x) = \sup\{|\operatorname{tr}(u \circ \upsilon_x)| : u \in \Pi_1(Y, \ell_1^n), \pi_1(u) \le 1\} \le C \cdot \|(x_i)_{i=1}^n\|_p^{\text{weak}}$$

and so $\|U_n\| \le C$ for every $n \in \mathbb{N}$. This gives us that a linear map $U : (X^{\mathbb{N}}, \|\cdot\|_p^{\text{weak}}) \to \mathcal{N}_\infty(\ell_1, Y)$ defined by $U(x_n) = \sum_n e_n \otimes Tx_n$, is continuous.

Now let us take any sequence $(x_n) \in \ell_p^u(X)$. Then the operator $S = \sum_n e_n \otimes Tx_n \in \mathcal{N}_\infty(\ell_1, Y)$. Given $\epsilon > 0$, we choose a ∞-nuclear representation $S = \sum_n \beta_n \otimes z_n$ such that $\sup_n \|\beta_n\| \cdot \|(z_n)\|_1^{\text{weak}} \le \nu_\infty(S) + \epsilon$. Writing $\alpha = (\alpha_n) \in \ell_1$ and $\beta_k = (\beta_{k,n})_n \in \ell_\infty$, we get that $S\alpha = \sum_k \langle \alpha, \beta_k \rangle z_k = \sum_n \alpha_n Tx_n = \sum_k (\sum_n \alpha_n \beta_{k,n}) z_k$, and thus $Tx_n = \sum_k \frac{\beta_{k,n}}{\|\beta_k\|_\infty} \|\beta_k\|_\infty z_k$. Since $\sum_k \|\beta_k\|_\infty z_k$ is an unconditionally convergent series in Y, we obtain that $(Tx_n) \in \mathcal{R}_c(Y)$. This implies that $T \in \mathcal{R}_c(X_p^u, Y)$. $\square$

In the following we find usable sufficient condition which implies that every unconditionally weakly p-summable sequence in a Banach space X lies inside the range of an X-valued measure with relatively compact range.

Corollary 5. *Let $1 < p < \infty$. If $\Pi_1(X, \ell_1) \subset \Pi_{1,p,1}(X, \ell_1)$, then $\ell_p^u(X) \subset \mathcal{R}_c(X)$.*

Proof. Let us take the operator $T = \sum_n x_n^* \otimes e_n \in \Pi_1(X, \ell_1)$. The hypothesis assures us that $T \in \Pi_{1,p,1}(X, \ell_1)$ and hence for any finite collection of vectors $x_1, \cdots x_n$ in X we have

$$\begin{aligned}\sum_{i=1}^n |\langle x_i, x_i^* \rangle| = \sum_{i=1}^n |\langle x_i, T^* e_i \rangle| = \sum_{i=1}^n |\langle T x_i, e_i \rangle| \\ \leq \pi_{1,p,1}(T) \cdot \|(x_i)_{i=1}^n\|_p^{\text{weak}} \cdot \sup\{(\sum_{i=1}^n |\langle e_i, y \rangle| : y \in B_{\ell_1}\} \\ \leq C' \cdot \|(x_i)_{i=1}^n\|_p^{\text{weak}} \cdot \|(x_i^*)_{i=1}^n\|_1^{\text{weak}}.\end{aligned}$$

We apply theorem 1 of [8] to produce that the operator $\sum_n x_n^* \otimes e_n \in \mathcal{N}_1(X, \ell_{p'})$. From the proof of theorem 4 we see that $\ell_p^u(X) \subset \mathcal{R}_c(X)$. □

Apply theorem 4 we find a special kind of Banach space Y with the property that any operator $T \in \mathcal{B}(X, Y)$ belongs to $\mathcal{R}_c(X_p^u, Y)$.

Corollary 6. *Let $1 < p < \infty$ and let $T \in \mathcal{B}(X, Y)$. Suppose that Y^* is isomorphic to a subspace of $L_1(\mu)$ for some measure μ. Then $T \in \mathcal{R}_c(X_p^u, Y)$.*

Proof. We select any operator $S \in \Pi_1(Y, \ell_1)$. The hypothesis enables us to invoke theorem 3.6 of [6] to infer that $S \in \mathcal{N}_1(Y, \ell_1)$ and so $i_{1p'} \circ S|_{TX} \in \mathcal{N}_1(TX, \ell_{p'})$, where $i_{1p'} : \ell_1 \to \ell_{p'}$ is the formal inclusion map. From the proof of theorem 4 we know that $T \in \mathcal{R}_c(X_p^u, Y)$. □

The next corollary shows that theorem 1 is equivalent to theorem 4 under some restrictions to the underlying Banach space.

Corollary 7. *Let $1 < p < \infty$. Suppose that Y and Y^* satisfy Grothendieck's theorem. Then the following statements about an operator $T : X \to Y$ are equivalent :*

(i) $T \in \mathcal{R}_{bvc}(X_p^u, Y)$.
(ii) $T \in \mathcal{R}_{bv}(X_p^u, Y)$.
(iii) $T \in \mathcal{R}(X_p^u, Y)$.
(iv) $T \in \mathcal{R}_c(X_p^u, Y)$.
(v) $T \in \Pi_{1,p,1}(X, Y)$.

Proof. The equivalence of (i),(ii) and (v) is covered by theorem 1. The implication (ii)⇒(iii) is trivial. In theorem 4 we showed that (iii)⇔(iv). To show that (iv) implies (v), we consider the operator $S = \sum_n y_n^* \otimes e_n \in \mathcal{B}(Y, \ell_1)$. As Y^* satisfies Grothendieck's theorem, we have that $S \in \Pi_2(Y, \ell_1)$ and hence there exists a factorization $S : Y \xrightarrow{v} \ell_2 \xrightarrow{u} \ell_1$. Since Y satisfies Grothendieck's theorem, it follows that $v \in \Pi_1(Y, \ell_2)$ and so $S \in \Pi_1(Y, \ell_1)$. This forces that there is a constant C such that

$$(*) \qquad \pi_1(S) \leq C\|S\|.$$

On account of hypothesis (iv), we take account of theorem 4 to deduce that there exists a constant $C > 0$ such that no matter how we select finitely many vectors $x_1, \cdots x_n$ from X and $y_1^*, \cdots y_n^*$ from Y^*, we have

$$\sum_{i=1}^{n} |\langle Tx_i, y_i^* \rangle| \leq C \cdot \|(x_i)_{i=1}^n\|_p^{\text{weak}} \cdot \pi_1(\sum_{i=1}^{n} y_i^* \otimes e_i : Y \to \ell_1^n).$$

Then it follows from condition $(*)$ that

$$\sum_{i=1}^{n} |\langle Tx_i, y_i^* \rangle| \leq C' \cdot \|(x_i)_{i=1}^n\|_p^{\text{weak}} \cdot \|(y_i^*)_{i=1}^n\|_1^{\text{weak}}.$$

This means that $T \in \Pi_{1,p,1}(X, Y)$. □

Now we pass on to the study of sequences lying in the range of a vector measure with relatively compact range.

Proposition 8. *Let $1 < p < \infty$. The following statements are equivalent :*

(i) $\mathcal{I}_p(X, \ell_1) \subset \Pi_1(X, \ell_1)$.

(ii) *If $(x_n) \in \mathcal{R}_c(X)$, then the operator $\sum_n e_n \otimes x_n \in \Pi_{p'}(\ell_1, X)$.*

Proof. (i)⇒(ii). Take any sequence $(x_n) \in \mathcal{R}_c(X)$. An appeal to proposition 1.4 of [6] yields that given $\epsilon > 0$ there exists an unconditionally convergent series $\sum_{k=1}^{\infty} y_k$ in X for which $x_n = \sum_{k=1}^{\infty} \alpha_k y_k$ and $\|(y_k)\|_1^{\text{weak}} < \epsilon + \|(x_n)\|_{rc}$, where $\|(\alpha_k)\|_\infty \leq 1$. The hypothesis (i) guarantees the existence of a constant $C > 0$ such that $\pi_1(T) \leq C \cdot \imath_p(T)$ for any operator $T = \sum_n x_n^* \otimes e_n \in \mathcal{I}_p(X, \ell_1)$. Then we have

$$\begin{aligned}\sum_n |\langle x_n, x_n^* \rangle| &= \sum_n |\langle \sum_k \alpha_k y_k, x_n^* \rangle| \leq \sum_k \sum_n |\langle y_k, x_n^* \rangle| \\ &= \sum_k \|Ty_k\| \leq \pi_1(T) \cdot \|(y_k)\|_1^{\text{weak}} \leq C \cdot \imath_p(T) \cdot (\epsilon + \|(x_n)\|_{rc}).\end{aligned}$$

The upshot of all this is that the linear map $\Phi : \mathcal{I}_p(X, \ell_1) \to \ell_1 : \sum_n x_n^* \otimes e_n \mapsto (\langle x_n, x_n^* \rangle)_n$ is continuous.

Given $T = \sum_n x_n^* \otimes e_n$ and $(\beta_n) \in \ell_\infty$, we use the trace duality to obtain the following :

$$\begin{aligned}\langle \Phi^*(\beta_n), T \rangle = \langle (\beta_n), \Phi(T) \rangle &= \sum_n \langle \beta_n x_n, x_n^* \rangle = \text{tr}(T \circ \Phi^*(\beta_n)) \\ \sum_k \langle T \circ \Phi^*(\beta_n) e_k,\, e_k \rangle &= \sum_k \langle \sum_n \langle x_n^*,\, \Phi^*(\beta_n) e_k \rangle e_n,\, e_k \rangle \\ &= \sum_n \langle x_n^*,\, \Phi^*(\beta_n) e_n \rangle.\end{aligned}$$

Therefore $\Phi^* : \ell_\infty \to \Pi_{p'}(\ell_1, X) : (\beta_n) \mapsto \sum_n e_n \otimes \beta_n x_n$. Then the operator $\sum_n e_n \otimes x_n \in \Pi_{p'}(\ell_1, X)$.

(ii)⇒(i). From the proof of theorem 4 we see that if the operator $S = \sum_n e_n \otimes x_n \in \mathcal{N}_\infty(\ell_1, X)$, then $(x_n) \in \mathcal{R}_c(X)$ and $\|(x_n)\|_{rc} < \nu_\infty(S)$.

Now let us select any sequence $(x_n) \in \mathcal{R}_c(X)$. We use this sequence to define an operator $T : \ell_1 \to X$ via $T(\alpha_n) = \sum_n \alpha_n x_n$. It takes another appeal to proposition 1.4 of [6] to establish that given $\epsilon > 0$ there exists an unconditionally convergent series $\sum_k y_k$ in X so that $x_n = \sum_k \delta_{k,n} y_k$ and $\|(y_k)\|_1^{\text{weak}} < \epsilon + \|(x_n)\|_{rc}$, where $\|(\delta_{k,n})_k\|_\infty \leq 1$. We exploit the fact that there exist a weakly summable sequence (z_k) in X and a sequence (λ_k) in B_{C_0} for which $y_k = \lambda_k z_k$ and $\|(z_k)\|_1^{\text{weak}} \leq \epsilon + \|(y_k)\|_1^{\text{weak}}$ to see that $T \in \mathcal{N}_\infty(\ell_1, X)$. In fact $T(\alpha_n) = \sum_n \alpha_n \sum_k \delta_{k,n} y_k = \sum_k \sum_n \alpha_n \delta_{k,n} \lambda_k z_k = \sum_k \langle \alpha, \lambda_k \delta_k \rangle z_k$, where $\delta_k = (\delta_{k,n})_n$. Thus $\nu_\infty(T) \leq \sup_k \|\lambda_k \delta_k\|_\infty \cdot \|(z_k)\|_1^{\text{weak}} \leq \epsilon + \|(y_k)\|_1^{\text{weak}} < 2\epsilon + \|(x_n)\|_{rc}$. Then the hypothesis (ii) leads us to have that $\mathcal{N}_\infty(\ell_1, X) \subset \Pi_{p'}(\ell_1, X)$. Using the trace duality we draw that $\mathcal{I}_p(X, \ell_1) \subset \Pi_1(X, \ell_1)$. $\square$

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