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SENSITIVITY ANALYSIS OF SOLUTIONS FOR A SYSTEM OF PARAMETRIC GENERAL QUASIVARIATIONAL-LIKE INEQUALITIES

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Abstract. In this paper, we introduce and study a new class of system of parametric general quasivariational-like inequalities. Using η-subdifferential and η-proximal mappings of proper functionals in Hilbert spaces, we prove the equivalence between the system of parametric general quasivariational-like inequalities and a fixed point problem and construct two iterative algorithms. A few existence and uniqueness results as well as the sensitivity analysis of solutions are also established for the system of parametric general quasivariational-like inequalities, and some convergence results of iterative sequence generated by the algorithms are proved. Our results extend a few known results in the literature.

1. Introduction and preliminaries

Variational inequality theory has become a very powerful tool in pure and applied mathematics. In 1996, Zhu and Marcotte introduced and investigated a class of system of variational inequalities in $\mathbb{R}^n$. Afterwards, Liu, Hao, Lee and Kang [9], Nie, Liu, Kim and Kang [12], Verma [13], and Wu, Liu, Shim and Kang [15] studied the approximation and solvability of a few kinds of various systems of variational inequalities in Hilbert spaces. Recently, Dafermos [3] studied the sensitivity property of solutions of a parametric variational inequality in $\mathbb{R}^n$. Afterwards, using the ideas of Dafermos, many researchers including Agarwal, Cho and Huang [1], Liu, Debnath, Kang and Ume [8], Liu, Wang, Kang and Ume [11], Yen and Lee [16] and others have established the sensitivity analysis of solutions for various types of variational inequalities and quasi-variational inclusions in Hilbert spaces, respectively. At the same time, Ding and Luo [4] studied the quasivariational-like inequalities with η-subdifferential mapping in Hilbert spaces.

Inspired and motivated by the results [1, 4, 9-11], in this paper, we introduce and study a new class of system of parametric general quasivariational-like

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inequalities. We show its equivalence with a fixed point problem and establish the existence and sensitivity analysis of solutions for the system of parametric general quasivariational-like inequalities involving strongly monotone, Lipschitz continuous and η-subdifferential mappings and some convergence results of iterative sequence generated by the algorithm are proved. Our results extend and improve the corresponding results in [4, 7, 9-11, 14, 17].

2. Preliminaries

Let H be a real Hilbert space with a norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, respectively. Let P be a nonempty open subset of H in which the parameter λ takes values.

Definition 2.1. Let A, B, $C : H \to H$ and $N : H \times H \times H \times P \to H$ be mappings. The mapping N is said to be

(1) *Lipschitz continuous* in the first argument if there exists a constant $a > 0$ such that

$$\|N(x, u, v, \lambda) - N(y, u, v, \lambda)\| \le a\|x - y\|$$

for all $x, y, u, v \in H$ and $\lambda \in P$;

(2) *strongly monotone* with respect to A in the first argument if there exists a constant $r > 0$ such that

$$\langle N(Ax, u, v, \lambda) - N(Ay, u, v, \lambda), x - y\rangle \ge r\|x - y\|^2$$

for all $x, y, u, v \in H$ and $\lambda \in P$;

(3) *relaxed monotone* with respect to B in the second argument if there exists a constant $s > 0$ such that

$$\langle N(u, Bx, v, \lambda) - N(u, By, v, \lambda), x - y\rangle \ge -s\|x - y\|^2$$

for all $x, y, u, v \in H$ and $\lambda \in P$.

In the similar way, we can define the Lipschitz continuity of N in the second and third arguments, respectively.

Definition 2.2. ([4]) A functional $f : H \times H \to \mathbb{R} \cup \{+\infty\}$ is said to be 0-*diagonally quasi-concave* (in short, 0-DQCV) in x if for any finite set $\{x_1, \cdots, x_n\} \subset H$, and for any $y = \sum_{i=1}^n l_i x_i$ with $l_i \ge 0$ and $\sum_{i=1}^n l_i = 1, \min_{1\le i\le n} f(x_i, y) \le 0$.

Definition 2.3. ([4]) Let $\eta : H \times H \to H$ be a mapping. A proper functional $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is said to be η-*subdifferentiable* at a point $x \in H$ if there exists a point $f^* \in H$ such that

$$\phi(y) - \phi(x) \ge \langle f^*, \eta(y, x)\rangle, \quad \forall y \in H,$$

where f^* is called a η-subgradient of ϕ at x. The set of all η-subgradient of ϕ at x is denoted by $\Delta\phi(x)$. The mapping $\Delta\phi : H \to 2^H$ defined by

$$\Delta\phi(x) = \{f^* \in H : \phi(y) - \phi(x) \ge \langle f^*, \eta(y, x)\rangle, \forall y \in H\}, \quad x \in H \tag{2.1}$$

is said to be η-*subdifferential* of ϕ.

Definition 2.4. ([4]) Let $\phi : H \to \mathbb{R} \cup \{+\infty\}$ be a proper functional. For any given $x \in H$ and $\rho > 0$, if there exist a mapping $\eta : H \times H \to H$ and a unique point $u \in H$ such that

$$\langle u - x, \eta(y, u)\rangle \geq \rho\phi(u) - \rho\phi(y), \quad \forall y \in H,$$

then the mapping $x \mapsto u$, denoted by $J_\rho^{\Delta\varphi}(x)$, is said to be *$\eta$-proximal mapping* of ϕ.

By (2.1) and the definition of $J_\rho^{\Delta\varphi}(x)$, we have $x - u \in \rho\Delta\phi(u)$. It follows that

$$J_\rho^{\Delta\varphi}(x) = (I + \rho\Delta\phi)^{-1}(x),$$

where I is the identity mapping on H.

Let $A, B, C : H \to H$, $N : H \times H \times H \times P \to H$ and $\eta : H \times H \times P \to H$ be mappings and $\phi : H \times P \to \mathbb{R} \cup \{+\infty\}$ be a proper functional such that $\phi : H \times P \to H$ is lower semicontinuous and η-subdifferentiable on H. Let ρ and β be positive constants and f and g be arbitrary elements in H. For each $\lambda \in P$, we consider the following problem:

Find elements $x, y \in H$ such that

$$\text{(2.2)} \qquad \begin{cases} \langle \rho(N(Ay, By, Cy, \lambda) - f) + x - y, \eta(u, x, \lambda)\rangle \\ \quad \geq \rho\phi(x, \lambda) - \rho\phi(u, \lambda), \\ \langle \beta(N(Ax, Bx, Cx, \lambda) - g) + y - x, \eta(u, y, \lambda)\rangle \\ \quad \geq \beta\phi(y, \lambda) - \beta\phi(u, \lambda), \end{cases} \qquad \forall u \in H,$$

which is known as the system of parameter general quasivariational-like inequalities.

It is clear that the system of parameter general quasivariational-like inequalities (2.2) includes the systems of variational inequalities in [9, 12, 13] as special cases.

Lemma 2.1. *Suppose that $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, $\{c_n\}_{n\geq 0}$ and $\{t_n\}_{n\geq 0}$ are sequences of nonnegative numbers satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + t_n b_n + c_n, \quad \forall n \geq 0$$

with $\{t_n\}_{n\geq 0} \subseteq [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. ([4]) *Let $\eta : H \times H \to H$ be δ-strongly monotone and τ-Lipschitz continuous such that $\eta(x, y) = -\eta(y, x)$ for all $x, y \in H$ and for any given $x, u \in H$, the functional $h(y, u) = \langle x - u, \eta(y, u)\rangle$ is 0-DQCV in y. Let $\phi : H \to \mathbb{R}$ be a lower semicontinuous η-subdifferentiable proper functional and $\rho > 0$ be an arbitrary constant. Then the η-proximal mapping $J_\rho^{\Delta\phi}$ of ϕ is $\frac{\tau}{\delta}$-Lipschitz continuous.*

By virtue of Definition 2.6 and Theorem 2.8 in [4], we obtain the following

Lemma 2.3. *For a given $u \in H$, the element $z \in H$ satisfies the following inequality*

$$\langle u - z, \eta(v,u)\rangle \geq \rho\phi(u) - \rho\phi(v), \quad \forall v \in H,$$

if and only if $u = J_\rho^{\Delta\phi}(z)$, where $\rho > 0$ is a constant and $J_\rho^{\Delta\phi} = (I + \rho\Delta\phi)^{-1}$ is the η-proximal mapping of ϕ.

3. Iterative algorithm

It follows from Lemma 2.3 that

Lemma 3.1. *Let ρ and β be positive constants, and f and g be arbitrary elements in H and $\lambda \in P$. Then the following statements are equivalent to each other.*

(a) *the system of parameter general quasivariational-like inequalities* (2.2) *has a solution $(x,y) \in H \times H$;*

(b) *there exists $(x,y) \in H \times H$ satisfying*

$$\begin{aligned} x &= J_\rho^{\Delta\phi(\cdot,\lambda)}[y - \rho(N(Ay,By,Cy,\lambda) - f)],\\ y &= J_\beta^{\Delta\phi(\cdot,\lambda)}[x - \beta(N(Ax,Bx,Cx,\lambda) - g)]; \end{aligned} \tag{3.1}$$

(c) *the mapping $F(\cdot,\lambda) : H \to H$ defined by*

$$\begin{aligned} F(u,\lambda) = J_\rho^{\Delta\phi(\cdot,\lambda)}\{&J_\beta^{\Delta\phi(\cdot,\lambda)}(u - \beta(N(Au,Bu,Cu,\lambda) - g))\\ &- \rho[N(AJ_\beta^{\Delta\phi(\cdot,\lambda)}(u - \beta(N(Au,Bu,Cu,\lambda) - g)),\\ &\quad BJ_\beta^{\Delta\phi(\cdot,\lambda)}(u - \beta(N(Au,Bu,Cu,\lambda) - g)),\\ &\quad CJ_\beta^{\Delta\phi(\cdot,\lambda)}(u - \beta(N(Au,Bu,Cu,\lambda) - g))) - f]\}, \quad \forall u \in H \end{aligned} \tag{3.2}$$

has a fixed point $x \in H$ and $y = J_\beta^{\Delta\phi(\cdot,\lambda)}(x - \beta(N(Ax,Bx,Cx,\lambda) - g))$.

Remark 3.1. Lemma 2.1 in [9, 12] and Lemma 1.3 in [13] are special cases of Lemma 3.1.

Based on Lemma 3.1 we suggest the following perturbed iterative algorithms for the system of parameter general quasivariational-like inequalities (2.2).

Algorithm 3.1. *Let $A, B, C : H \to H$, $N : H \times H \times H \times P \to H$, $\eta : H \times H \times P \to H$ and $\phi : H \times P \to \mathbb{R}$ be mappings. For any given $x_0 \in H$, the iterative sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ are defined by*

$$\begin{aligned} z_n &= (1 - b_n)x_n + b_n F(x_n,\lambda) + p_n,\\ x_{n+1} &= (1 - a_n)x_n + a_n F_n(z_n,\lambda) + a_n u_n + v_n,\\ y_n &= J_\beta^{\Delta\phi(\cdot,\lambda)}(x_n - \beta(N(Ax_n,Bx_n,Cx_n,\lambda) - g)) + w_n, \quad \forall n \geq 0, \end{aligned} \tag{3.3}$$

where $F(\cdot,\lambda)$ is defined by (3.2) *and $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are any sequences in $[0,1]$ and $\{u_n\}_{n\geq 0}, \{v_n\}_{n\geq 0}, \{w_n\}_{n\geq 0}$, $\{p_n\}_{n\geq 0}$ are arbitrary sequences in H*

satisfying

$$\sum_{n=0}^{\infty} a_n = \infty, \quad \sum_{n=0}^{\infty} \|v_n\| < \infty, \qquad (3.4)$$
$$\lim_{n\to\infty} \|u_n\| = \lim_{n\to\infty} \|w_n\| = \lim_{n\to\infty} \|p_n\| = 0.$$

Algorithm 3.2. *Let A, B, C, N, η and ϕ be same as in Algorithm 3.1. For each $x_0 \in H$, compute the sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ by the iterative procedure*

$$\begin{aligned} y_n &= J_\beta^{\Delta\phi(\cdot,\lambda)}(x_n - \beta(N(Ax_n, Bx_n, Cx_n, \lambda) - g)) + w_n, \\ x_{n+1} &= (1 - a_n)x_n + a_n J_\rho^{\Delta\phi(\cdot,\lambda)}(y_n - \rho(N(Ay_n, By_n, Cy_n, \lambda) - f)) \\ &\quad + a_n u_n + v_n, \quad \forall n \geq 0, \end{aligned} \qquad (3.5)$$

where $\{a_n\}_{n\geq 0}$ is any sequence in $[0, 1]$ and $\{u_n\}_{n\geq 0}, \{v_n\}_{n\geq 0}$ and $\{w_n\}_{n\geq 0}$ are arbitrary sequences in H satisfying

$$\sum_{n=0}^{\infty} a_n = \infty, \quad \sum_{n=0}^{\infty} \|v_n\| < \infty, \quad \lim_{n\to\infty} \|u_n\| = \lim_{n\to\infty} \|w_n\| = 0. \qquad (3.6)$$

4. Existence and convergence

Theorem 4.1. *Let $A, B, C : H \to H$ be Lipschitz continuous with contants m, n and l, respectively. Let $N : H \times H \times H \times P \to H$ be Lipschitz continuous with constants a, b, c in the first, second and third arguments, respectively, and N be strongly monotone with constants r with respect to A in the first argument, relaxed monotone with constant s with respect to B in the second argument. Let $\eta : H \times H \times P \to H$ be δ-strongly monotone and τ-Lipschitz continuous with $\eta(x, y, \lambda) = -\eta(y, x, \lambda), \forall x, y \in H, \lambda \in P$ and for each $x, u \in H, \lambda \in P$ the function $h(y, u, \lambda) = \langle x - u, \eta(y, u, \lambda)\rangle$ is 0-DQCV in y. Let $\phi : H \times P \to H$ be a lower semicontinuous η-subdifferentiable proper functional. Let ρ and β be positive constants. If there exists a constant θ satisfying*

$$\begin{aligned} \theta =&\left(\frac{\tau}{\delta}\right)^2 (\sqrt{1 - 2\rho(r - s) + \rho^2(am + bn)^2} + \rho cl) \\ &\times (\sqrt{1 - 2\beta(r - s) + \beta^2(am + bn)^2} + \beta cl) < 1, \end{aligned} \qquad (4.1)$$

then for any given $f, g \in H$, $\lambda \in P$, the system of parameter general quasivariational-like inequalities (2.2) *has a unique solution $(x, y) \in H \times H$.*

Proof. For each given $\lambda \in P$, we assert that $F(\cdot, \lambda) : H \to H$ defined by (3.2) is a contraction mapping. Since N is both a-Lipschitz continuous and r-strongly monotone in the first argument, b-Lipschitz continuous and s-relaxed monotone in the second argument and c-Lipschitz continuous in the third argument, A, B, C are Lipschitz continuous with constants m, n and l, respectively, it follows from Lemma 2.2 that

$$
\begin{aligned}
&\|F(u,\lambda)-F(v,\lambda)\| \\
&\le \frac{\tau}{\delta}\|J_\beta^{\Delta\phi(\cdot,\lambda)}(u-\beta(N(Au,Bu,Cu,\lambda)-g)) \\
&\qquad - J_\beta^{\Delta\phi(\cdot,\lambda)}(v-\beta(N(Av,Bv,Cv,\lambda)-g)) \\
&\qquad - \rho[N(AJ_\beta^{\Delta\phi(\cdot,\lambda)}(u-\beta(N(Au,Bu,Cu,\lambda)-g)), \\
&\qquad\quad BJ_\beta^{\Delta\phi(\cdot,\lambda)}(u-\beta(N(Au,Bu,Cu,\lambda)-g)), \\
&\qquad\quad CJ_\beta^{\Delta\phi(\cdot,\lambda)}(u-\beta(N(Au,Bu,Cu,\lambda)))) \\
&\qquad - N(AJ_\beta^{\Delta\phi(\cdot,\lambda)}(v-\beta(N(Av,Bv,Cv,\lambda)-g)), \\
&\qquad\quad BJ_\beta^{\Delta\phi(\cdot,\lambda)}(v-\beta(N(Av,Bv,Cv,\lambda)-g)), \\
&\qquad\quad CJ_\beta^{\Delta\phi(\cdot,\lambda)}(v-\beta(N(Av,Bv,Cv,\lambda))))]\| \\
&\le \theta\|u-v\|
\end{aligned}
\tag{4.2}
$$

for all $u,v\in H$. (4.1) and (4.2) mean that $F(\cdot,\lambda)$ is a contraction mapping and hence it has a unique fixed point $x\in H$. Set $y=J_\beta^{\Delta\phi(\cdot,\lambda)}(x-\beta(N(Ax,Bx,Cx,\lambda)-g))$. It follows from Lemma 3.1 that the system of parameter general quasivariational-like inequalities (2.2) has a solution $(x,y)\in H\times H$. Now we claim that (x,y) is the unique solution of the system of parameter general quasivariational-like inequalities (2.2). In fact, if $(u,v)\in H\times H$ is also a solution of the the system of parameter general quasivariational-like inequalities (2.2), by Lemma 3.1 we know that $u=F(u,\lambda)$ and $v=J_\beta^{\Delta\phi(\cdot,\lambda)}(u-\beta(N(Au,Bu,Cu,\lambda)-g))$. It follows from the uniqueness of fixed point of $F(\cdot,\lambda)$ that $u=x$ and hence $v=J_\beta^{\Delta\phi(\cdot,\lambda)}(u-\beta(N(Au,Bu,Cu,\lambda)-g))=y$. This completes the proof. □

Theorem 4.2. *Let the conditions of Theorem 4.1 hold. If there exists a positive constant θ satisfying* (4.1)*, then for any given $f,g\in H$, $\lambda\in P$, the system of parameter general quasivariational-like inequalities* (2.2) *has a unique solution $(x,y)\in H\times H$ and $\lim_{n\to\infty}x_n=x$ and $\lim_{n\to\infty}y_n=y$, where $\{x_n\}_{n\ge 0}$ and $\{y_n\}_{n\ge 0}$ are the sequences generated by Algorithm 3.1.*

Proof. It follows Theorem 4.1 that the system of parameter general quasivariational-like inequalities (2.2) has a unique solution $(x,y)\in H\times H$. Now we claim the sequences $\{x_n\}_{n\ge 0}$ and $\{y_n\}_{n\ge 0}$ generated by Algorithm 3.1 converge strongly to x and y, respectively. As in the proof of Theorem 4.1, we know that (4.2) holds. In view of (3.3), (4.1) and (4.2), we conclude that

$$\|x_{n+1}-x\|\le[1-(1-\theta)a_n]\|x_n-x\|+a_n\theta\|p_n\|+a_n\|u_n\|+\|v_n\| \tag{4.3}$$

and

$$\|y_n-y\|\le\frac{\tau}{\delta}\big(\sqrt{1-2\beta(r-s)+\beta^2(a+b)^2}+\beta c\big)\|x_n-x\|+\|w_n\| \tag{4.4}$$

for all $n \geq 0$, where $F(\cdot, \lambda)$ and θ are defined by (3.2) and (4.1), respectively. It follows from Lemma 2.2, (3.4) and (4.3) that $\lim_{n\to\infty} x_n = x$. Letting $n \to \infty$ in (4.4), by (3.4) we infer that $\lim_{n\to\infty} y_n = y$. This completes the proof. □

Theorem 4.3. *Let A, B, C, N, η and ϕ be as in Theorem 4.1. If there exists a positive constant θ satisfying* (4.1), *then for any given $f, g \in H, \lambda \in P$, the system of parameter general quasivariational-like inequalities* (2.2) *has a unique solution $(x, y) \in H \times H$ and $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, where $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ are the sequences generated by Algorithm 3.2.*

Proof. Theorem 4.1 ensures that the system of parameter general quasivariational-like inequalities (2.2) has a unique solution $(x, y) \in H \times H$. As in the proof of Theorems 4.1 and 4.2, we conclude that (4.4) holds and

$$\begin{aligned} \|x_{n+1} - x\| \leq {} & [1 - (1-\theta)a_n]\|x_n - x\| \\ & + a_n(\|u_n\| + \|w_n\|) + \|v_n\|, \quad \forall n \geq 0. \end{aligned} \tag{4.5}$$

It follows from Lemma 2.2, (3.6) and (4.5) that $x_n \to x$ as $n \to \infty$. Thus (4.4) and (3.6) yield that $y_n \to y$ as $n \to \infty$. This completes the proof. □

Remark 4.1. Theorems 4.1∼4.3 extend, improve and unify Theorem 3.6 in [4] and Theorems 2.1∼2.3 in [9, 12, 13].

5. Sensitivity analysis

Now we analyze the sensitivity of solutions for the system of parameter general quasivariational-like inequalities (2.2).

Theorem 5.1. *Let the conditions of Theorem 4.1 be satisfied. Assume that N is continuous* (*resp. uniformly continuous or Lipschitz continuous*) *with respect to the fourth argument, η is continuous* (*resp. uniformly continuous or Lipschitz continuous*) *with respect to the third argument and ϕ is continuous* (*resp. uniformly continuous or Lipschitz continuous*) *with respect to the second argument. Suppose that there exists ζ satisfying*

$$\|J_\rho^{\Delta\phi(\cdot,\lambda)}(z) - J_\rho^{\Delta\phi(\cdot,\overline{\lambda})}(z)\| \leq \zeta\|\lambda - \overline{\lambda}\|, \quad \forall z \in H,\ \lambda, \overline{\lambda} \in P. \tag{5.1}$$

Then the solutions of the system of parameter general quasivariational-like inequalities (2.2) *are continuous* (*resp. uniformly continuous or Lipschitz continuous*).

Proof. Let $F(\cdot, \lambda)$ be defined by (3.2). It follows from Theorem 4.1 that for any $\lambda \in P$ there exists a unique $(x, y) \in H \times H$ denoted by $x(\lambda)$ and $y(\lambda)$ such that they are the solution of the system of parameter general quasivariational-like inequalities (2.2). Hence for each $\lambda, \overline{\lambda} \in P$, we get that

$$x(\lambda) = F(x(\lambda), \lambda), \quad x(\overline{\lambda}) = F(x(\overline{\lambda}), \overline{\lambda}),$$

$$\begin{aligned} y(\lambda) = {} & J_\beta^{\Delta\phi(\cdot,\lambda)}(x(\lambda) - \beta(N(A(x(\lambda)), B(x(\lambda)), C(x(\lambda)), \lambda) - g)), \\ y(\overline{\lambda}) = {} & J_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(x(\overline{\lambda}) - \beta(N(A(x(\overline{\lambda})), B(x(\overline{\lambda})), C(x(\overline{\lambda})), \overline{\lambda}) - g)), \end{aligned} \tag{5.2}$$

$$\|x(\lambda)-x(\overline{\lambda})\| \leq \|F(x(\lambda),\lambda)-F(x(\lambda),\overline{\lambda})\| + \|F(x(\lambda),\overline{\lambda})-F(x(\overline{\lambda}),\overline{\lambda})\|, \tag{5.3}$$

$$\begin{aligned}
&\|y(\lambda)-y(\overline{\lambda})\| \\
&= \|J_\beta^{\Delta\phi(\cdot,\lambda)}(x(\lambda)-\beta(N(A(x(\lambda)),B(x(\lambda)),C(x(\lambda)),\lambda)-g)) \\
&\quad - J_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(x(\overline{\lambda})-\beta(N(A(x(\overline{\lambda})),B(x(\overline{\lambda})),C(x(\overline{\lambda})),\overline{\lambda})-g))\|.
\end{aligned} \tag{5.4}$$

Set

$$X(\lambda,\lambda) = x(\lambda)-\beta(N(A(x(\lambda)),B(x(\lambda)),C(x(\lambda)),\lambda)-g),$$
$$X(\lambda,\overline{\lambda}) = x(\lambda)-\beta(N(A(x(\lambda)),B(x(\lambda)),C(x(\lambda)),\overline{\lambda})-g).$$

It follows from Lemma 2.2 and (5.1) that

$$\begin{aligned}
&\|F(x(\lambda),\lambda)-F(x(\lambda),\overline{\lambda})\| \\
&\leq \zeta\|\lambda-\overline{\lambda}\| + \frac{\tau}{\delta}\{(\|J_\beta^{\Delta\phi(\cdot,\lambda)}(X(\lambda,\lambda)) - J_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda))\| \\
&\quad + \|J_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)) - J_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\overline{\lambda}))\|) \\
&\quad + \rho[\|N(AJ_\beta^{\Delta\phi(\cdot,\lambda)}(X(\lambda,\lambda)),BJ_\beta^{\Delta\phi(\cdot,\lambda)}(X(\lambda,\lambda)),CJ_\beta^{\Delta\phi(\cdot,\lambda)}(X(\lambda,\lambda)),\lambda) \\
&\qquad - N(AJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),BJ_\beta^{\Delta\phi(\cdot,\lambda)}(X(\lambda,\lambda)),CJ_\beta^{\Delta\phi(\cdot,\lambda)}(X(\lambda,\lambda)),\lambda)\| \\
&\quad + \|N(AJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),BJ_\beta^{\Delta\phi(\cdot,\lambda)}(X(\lambda,\lambda)),CJ_\beta^{\Delta\phi(\cdot,\lambda)}(X(\lambda,\lambda)),\lambda) \\
&\qquad - N(AJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),BJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),CJ_\beta^{\Delta\phi(\cdot,\lambda)}(X(\lambda,\lambda)),\lambda)\| \\
&\quad + \|N(AJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),BJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),CJ_\beta^{\Delta\phi(\cdot,\lambda)}(X(\lambda,\lambda)),\lambda) \\
&\qquad - N(AJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),BJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),CJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),\lambda)\| \\
&\quad + \|N(AJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),BJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),CJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),\lambda) \\
&\qquad - N(AJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\overline{\lambda})),BJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),CJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),\lambda)\| \\
&\quad + \|N(AJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\overline{\lambda})),BJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),CJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),\lambda) \\
&\qquad - N(AJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\overline{\lambda})),BJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\overline{\lambda})),CJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),\lambda)\| \\
&\quad + \|N(AJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\overline{\lambda})),BJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\overline{\lambda})),CJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\lambda)),\lambda) \\
&\qquad - N(AJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\overline{\lambda})),BJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\overline{\lambda})),CJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\overline{\lambda})),\lambda)\| \\
&\quad + \|N(AJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\overline{\lambda})),BJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\overline{\lambda})),CJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\overline{\lambda})),\lambda) \\
&\qquad - N(AJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\overline{\lambda})),BJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\overline{\lambda})),CJ_\beta^{\Delta\phi(\cdot,\overline{\lambda})}(X(\lambda,\overline{\lambda})),\overline{\lambda})\|]\} \\
&\leq \zeta\Big[1+\frac{\tau}{\delta}(1+3\rho(am+bn+cl))\Big]\|\lambda-\overline{\lambda}\|
\end{aligned} \tag{5.5}$$

$$+\left(\frac{\tau}{\delta}\right)^2\beta[1+\rho(am+bn+cl)]\|N(A(x(\lambda)),B(x(\lambda)),C(x(\lambda)),\lambda) - N(A(x(\lambda)),B(x(\lambda)),C(x(\lambda)),\overline{\lambda})\| + \frac{\rho\tau}{\delta}\|N(A(v),B(v),C(v),\lambda)-N(A(v),B(v),C(v),\overline{\lambda})\|,$$

where $v = J_\beta^{\Delta\phi(\cdot,\lambda)}(x(\lambda)-\beta(N(A(x(\lambda)),B(x(\lambda)),C(x(\lambda)),\overline{\lambda})-g))$. It follows from (4.2) that

$$\|F(x(\lambda),\overline{\lambda})-F(x(\overline{\lambda}),\overline{\lambda})\| \le \theta\|x(\lambda)-x(\overline{\lambda})\|. \tag{5.6}$$

Combining (5.3), (5.5) and (5.6), we infer that

$$\begin{aligned}
&\|x(\lambda)-x(\overline{\lambda})\| \\
&\le (1-\theta)^{-1}\zeta[1+\frac{\tau}{\delta}(1+3\rho(am+bn+cl))]\|\lambda-\overline{\lambda}\| \\
&\quad+\left(\frac{\tau}{\delta}\right)^2\beta[1+\rho(am+bn+cl)]\|N(A(x(\lambda)),B(x(\lambda)),C(x(\lambda)),\lambda) \\
&\quad- N(A(x(\lambda)),B(x(\lambda)),C(x(\lambda)),\overline{\lambda})\| \\
&\quad+\frac{\rho\tau}{\delta}\|N(A(v),B(v),C(v),\lambda)-N(A(v),B(v),C(v),\overline{\lambda})\|.
\end{aligned} \tag{5.7}$$

From (5.4), we get that

$$\begin{aligned}
&\|y(\lambda)-y(\overline{\lambda})\| \\
&\le \zeta\|\lambda-\overline{\lambda}\|+\frac{\tau}{\delta}(1+\beta(am+bn+cl))\|x(\lambda)-x(\overline{\lambda})\| \\
&\quad+\frac{\beta\tau}{\delta}\|N(A(x(\lambda)),B(x(\lambda)),C(x(\lambda)),\lambda) \\
&\quad- N(A(x(\lambda)),B(x(\lambda)),C(x(\lambda)),\overline{\lambda})\|.
\end{aligned} \tag{5.8}$$

It follows from (5.7), (5.8) and the continuity of N (resp. uniform continuity or Lipschitz continuity) with respect to the fourth argument that the solutions of the system of parameter general quasivariational-like inequalities (2.2) are continuous (resp. uniformly continuous or Lipschitz continuous). This completes the proof. □

Remark 5.1. Theorem 5.1 extends and improves Theorem 3.4 in [10], Theorem 2.1 in [11], Theorem 3.3 in [14] and Theorem 3.1 in [17].

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