East Asian Math. J. **24** (2008), No. 2, pp. 169–176

FIXED POINT THEOREM FOR NONCOMPATIBLE DISCONTINUOUS MAPPINGS AND BEST APPROXIMATION

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Abstract. The aim of this paper is to prove a common fixed point theorem in normed linear spaces for noncompatible, discontinuous mappings without assuming completeness of the space. We also give an application of our theorem to best approximation theory.

1. Introduction and preliminaries

In 1976, Jungck [9] proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem. Banach fixed point theorem has many applications but suffers from a draw back- the theorem requires the continuity of the mapping throughout the space. Sessa [21] defined a generalization of commutativity, which is called weak commutativity. Further Jungck [10] introduced more generalized commutativity so called compatibility. Since then various fixed point theorems for compatible mappings satisfying contractive type condition and assuming continuity of at least one of the mapping, have been obtained by many authors.

It may be observed in this context that it is known since the paper of Kannan [14] in 1968 that there exist maps that have discontinuity in their domain but which have fixed points. However, the maps involved were continuous at the fixed points.

The study of common fixed points of noncompatible mappings is also very interesting. Work along these lines has recently been initiated by Pant [17], [18].

In 1998, Jungck and Rhoades [12] introduced the notion of weakly compatible maps and showed that compatible maps are weakly compatible but converse need not be true.

On the other hand, Aamri and Moutawakil [1] generalized the concept of noncompatible mappings by defining the property (E.A).

Received June 26, 2007.
2000 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. common fixed point, (E.A) property, best approximation.
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Definition 1 ([10])**.** Let X be a normed linear space and let $A, B : X \to X$ be two mappings. A and B are said to be compatible if whenever $\{x_n\}$ is a sequence in X such that Ax_n, $Bx_n \to t \in X, n \to \infty$ then

$$\|ABx_n - BAx_n\| \to 0 \text{ as } n \to \infty.$$

Definition 2 ([12])**.** Two self mappings A and B of a normed linear space X are said to be weakly compatible if they commute at their coincidence points.

It is easy to see that two compatible maps are weakly compatible but converse need not true.

Definition 3 ([1])**.** Let A and B be two self-mappings of a normed linear space X. A and B satisfy the property (E.A) if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t, \quad \text{for some } t \in X.$$

Remark 1 ([1]). It is clear from Jungck's definition [10] that two self-mappings A and B of a normed linear space X will be noncompatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$, for some $t \in X$ but $\lim_{n\to\infty} ||ABx_n - BAx_n||$ is either non zero or non-existent. Therefore two noncompatible self-mappings of a normed linear space X satisfy the property (E.A).

Let C be a subset of a normed linear space X and $A : X \to X$. The set of fixed points of A on X is denoted by $F(A)$. If $\bar{x}$ is a point of X, then for $0 < a \leq 1$, we define the set Da of best (C, a)-approximants to $\bar{x}$ consists of the point y in C such that

$$a||y - \bar{x}|| = \inf\{||z - \bar{x}|| : z \in C\}.$$

For $a = 1$ our definition reduces to the set D of best C-approximants to $\bar{x}$. A subset C of X is said to be starshaped with respect to a point $p \in C$ if, for all x in C and for all $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)p \in C$. The point p is called the star-centre of C. A convex set is star shaped with respect to each of its points, but not conversely. For an example the set $C = \{0\} \times [0, 1] \cup [1, 0] \times \{0\}$ is star shaped with respect to $(0, 0) \in C$ as the star-centre of C, but it is not convex.

Many authors have studied the applications of fixed point theorems to best approximation theory including [2]-[8], [13], [15], [16], [19], [20], [22]-[30].

In this paper, we prove a common fixed point theorem in normed linear spaces for noncompatible, discontinuous mappings, without assuming the completeness of the space. We improve and extend the result of Jungk [11] and many others. We also give an application of our main theorem to best approximation theory. Our application improves the results Pathak, Cho and Kang [19], Sharma and Deshpande [23], [24].

2. Main Results

Theorem 1. *Let A and B be two noncompatible weakly compatible mappings of a normed linear space X into itself satisfying*

$$(1)\quad ||Ax - Ay|| \leq a||Bx - By|| + b\max\{||Bx - Ax||, ||By - Ay||\} + c\{||Bx - Ay|| + ||By - Ax||\},$$

where $a, b, c > o, a + b + c = 1, a + 2c < b$. If $B(X)$ is closed subset of X,then A and B have a unique common fixed point.

Proof. Since A and B are two noncompatible mappings of a normed linear space X into itself. So A and B satisfy the property (E.A), Therefore there exists a sequence $\{x_n\}$ in X satisfying

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z, \quad \text{for some } z \in X.$$

Since BX is closed, we have $\lim_{n\to\infty} Bx_n = Bu$, for some $u \in X$. Thus $Bu = z$. We claim $Au = z$. If not then condition (1) implies

$$||Ax_n - Au|| \leq a||Bx_n - Bu|| + b\max\{||Bx_n - Ax_n||, ||Bu - Au||\} + c\{||Bx_n - Au|| + ||Bu - Ax_n||\}.$$

Taking the limit as $n \to \infty$ we obtain

$$||z - Au|| \leq (b + c)||z - Au||,$$

which is a contradiction, so we have $Au = z$. Thus $Au = Bu = z$, i. e. u is coincidence point of A, B and weak compatibility of A and B imply $ABu = BAu$ i. e. $Az = Bz$. We claim that $Az = z$. If not then condition (1) imply

$$\begin{aligned} ||Az - Au|| &= ||Az - z|| \\ &\leq a||Bz - Bu|| + b\max\{||Bz - Az||, ||Bu - Au||\} \\ &\quad + c\{||Bz - Au|| + ||Bu - Az||\}, \\ &\leq (a + 2c)||Az - z|| \\ &< b||Az - z||, \end{aligned}$$

which is a contradiction, so we have $Az = z$. Therefore $Az = Bz = z$ i. e. z is common fixed point of A and B.

For uniqueness suppose that $z_1$is another fixed point of A and B. Then using (1), we have

$$\begin{aligned} ||z - z_1|| &= ||Az - Az_1|| \\ &\leq a||Bz - Bz_1|| + b\max\{||Bz - Az||, ||Bz_1 - Az_1||\} \\ &\quad + c\{||Bz - Az_1|| + ||Bz_1 - Az||\} \\ &= (a + 2c)||z - z_1|| \\ &< b||z - z_1||, \end{aligned}$$

which is a contradiction. Therefore $z = z_1$. This completes the proof. ∎

Example 1. Let $X = [2, 20)$ with the usual norm. Define $A, B \colon X \to X$ by

$$Ax = \begin{cases} 2 & \text{if } x = 2 \text{ or } x > 5, \\ 3 & \text{if } 2 < x \leq 5, \end{cases}$$

$$Bx = \begin{cases} 2 & \text{if } x = 2, \\ 12 & \text{if } 2 < x \leq 5, \\ \frac{x+1}{3} & \text{if } x > 5. \end{cases}$$

If we take $a = \frac{1}{9,}$ $b = \frac{13}{18}, c = \frac{3}{18}$, we can see that A and B satisfy all the conditions of Theorem 1 and have a unique common fixed point $2 \in X$.

It may be noted in this example that the mapping A and B commute at coincidence point $2 \in X$. So A and B are weakly compatible maps. Also BX is closed subset of X.

Consider a sequence $\{x_n = 5 + \frac{1}{n}, n = 1, 2, 3, ...\}$, then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = 2.$$

Clearly A and B are noncompatible since

$$\lim_{n\to\infty} ||ABx_n - BAx_n|| = |3 - 2| = 1 \neq 0.$$

Also A and B are discontinuous mappings even at the common fixed point $x = 2$.

Since two non compatible self-mappings of a normed linear space X satisfy the property (E.A), we get the following result:

Corollary 1. *Let A and B be two weakly compatible mappings of a normed linear space X into itself satisfying:*

$$||Ax - Ay|| \leq a||Bx - By|| + b\max\{||Bx - Ax||, ||By - Ay||\} + c\{||Bx - Ay|| + ||By - Ax||\},$$

where $a, b, c > o, a + b + c = 1, a + 2c < b$. If $B(X)$ is closed subset of X, A and B satisfy the property (E.A), then A and B have a unique common fixed point.

Remark 2. The conclusion of Theorem 1 and Corollary 1 remains valid if we assume that $A(X)$ is closed instead of $B(X)$ provided that $A(X) \subset B(X)$.

Theorem 2. *Let A and B be two of mappings of a normed linear space X into itself and C be a nonempty, closed subset of X such that $A : \partial C \to C$ and $\bar{x} \in F(A) \cap F(B)$. Further, suppose that A and B satisfy* (1) *for all x, y in $D\prime a = Da \cup \{\bar{x}\} \cup E$, where*

$$E = \{q \in X : Ax_n, Bx_n \to q, \{x_n\} \subset Da\},$$

$a, b, c > o$, $a + b + c = 1, a + 2c < b$. If A and B are continuous on Da and A and B are noncompatible, weakly compatible in Da. If Da is nonempty, compact convex and $B(Da) = Da$ then $Da \cap F(A) \cap F(B) \neq \phi$.

Proof. Let $y \in Da$ and hence By is in Da since $B(Da) = Da$.

Further, if $y \in \partial C$ then Ay is in C. Since $A(\partial C) \subset C$, from (1), it follows that

$$\begin{aligned} ||Ay - \bar{x}|| = ||Ay - A\bar{x}|| &\\ &\leq a||By - B\bar{x}|| + b\max\{||By - Ay||, ||B\bar{x} - A\bar{x}||\} \\ &\quad + c\{||By - A\bar{x}|| + ||B\bar{x} - Ay||\} \\ &\leq a||By - \bar{x}|| + b\max\{||By - \bar{x}|| + ||\bar{x} - Ay||\} \\ &\quad + c\{||By - \bar{x}|| + ||\bar{x} - Ay||\}, \end{aligned}$$

which implies $a||Ay - \bar{x}|| \leq ||By - \bar{x}||$ and so Ay is in Da. Thus A maps Da into itself. Clearly $A(Da) \subset Da = B(Da)$.

Since A and B are noncompatible on Da, so A and B satisfy the property (E.A), therefore

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z, \quad \text{for some } z \in D(a). \tag{2}$$

Since $A(Da) \subset B(Da)$ there exists a point $v \in Da$ such that $z = Bv$. Then by (1), we have

$$\begin{aligned} ||Av - z|| &\leq ||Av - Ax_n|| + ||Ax_n - z|| \\ &\leq a||Bv - Bx_n|| + b\max\{||Bv - Av||, ||Bx_n - Ax_n|| \\ &\quad + c\{||Bv - Ax_n|| + ||Bx_n - Av||\} + ||Ax_n - z||. \end{aligned}$$

Taking the limit as $n \to \infty$, yields

$$||Av - z|| \leq (b + c)||z - Av|| = (1 - a)||z - Av||.$$

So we have $Av = z$. Therefore $Av = Bv = z$. Since A and B are weakly compatible then $ABv = BAv$ i.e. $Az = Bz$. By (1), we have

$$\begin{aligned} ||Az - \bar{x}|| = ||Az - A\bar{x}|| &\\ &\leq a||Bz - B\bar{x}|| + b\max\{||Bz - Az||, ||B\bar{x} - A\bar{x}||\} \\ &\quad + c\{||Bz - A\bar{x}|| + ||B\bar{x} - Az||\} \\ &= (a + 2c)||Az - \bar{x}|| \\ &< b||Az - \bar{x}||. \end{aligned}$$

Thus $Az = \bar{x}$. So $Az = Bz = \bar{x}$.

Next we consider

$$\begin{aligned} ||Az - Ax_n|| \leq a||Bz - Bx_n|| + b\max\{||Bz - Az||, ||Bx_n - Ax_n||\} &\\ + c\{||Bz - Ax_n|| + ||Bx_n - Az||\}. \end{aligned}$$

Taking the limit $n \to \infty$ yields

$$||\bar{x} - z|| \leq (a + 2c)||\bar{x} - z|| < b||\bar{x} - z||.$$

So $\bar{x} = z$ i.e. $z = Az = Bz$. By Theorem 1, z must be unique. Hence $E = \{z\}$, then $D\prime a = Da \cup \{z\}$.

Let $\{e_n\}$ be a monotonically nondecreasing sequence of real numbers such that $0 \leq e_n < 1$ and $\overline{\lim}_{n\to\infty} e_n = 1$. Let $\{x_j\}$ be a sequence in $D\prime a$ satisfying (2). For each $n \in N$, define a mapping $A_n : D\prime a \to D\prime a$ by

$$A_n x_j = e_n A x_j + (1 - e_n)p.$$

It is possible to define such a mapping A_n for each $n \in N$ since $D\prime a$ is starshaped with respect to $p \in F(B)$. We have

$$\lim_{j\to\infty} A_n x_j = e_n \lim_{j\to\infty} A x_j + (1 - e_n)z = e_n z + (1 - e_n)z = z.$$

Now, $A_n z = Bz = z$ and $A_n Bz = z = BA_n z$. Therefore, B and A_n commute at their coincidence point. Thus B and A_n weakly compatible on $D\prime a$ for each n and $A_n(D\prime a) \subset D\prime a = B(D\prime a)$.

On the other hand by (1), for all $x, y \in D\prime a$, we have for all $j \geq n$ and n fixed,

$$\begin{aligned}
||A_n x - A_n y|| &= e_n ||Ax - Ay|| \\
&\leq e_j ||Ax - Ay|| \\
&< ||Ax - Ay|| \\
&\leq a\|Bx - By\| + b \max\{||Bx - Ax||, ||By - Ay||\} \\
&\quad + c\{||Bx - Ay|| + ||By - Ax||\} \\
&\leq a||Bx - By|| + b \max\{||Ax - A_n x|| + ||A_n x - Bx||, \\
&\qquad ||Ay - A_n y| + ||A_n y - By||\} \\
&\quad + c\{||A_n y - Ay|| + ||A_n y - Bx|| + ||A_n x - Ax|| \\
&\qquad + ||A_n x - By||\} \\
&\leq a||Bx - By|| + b \max\{(1 - e_n)||Ax - p|| + ||A_n x - Bx||, \\
&\qquad (1 - e_n)||Ay - p|| + ||A_n y - By||\} \\
&\quad + c\{(1 - e_n)||Ay - p|| + ||A_n y - Bx|| + (1 - e_n)||Ax - p|| \\
&\qquad + ||A_n y - By||\}.
\end{aligned}$$

Hence for all $j \geq n$,we have

$$\begin{aligned}
||A_n x - A_n y|| \leq a||Bx - By|| + b \max\{(1 - e_j)||Ax - p| + ||A_n x - Bx||, \\
(1 - e_j)||Ay - p|| + ||A_n y - By||\} \\
+ c\{(1 - e_j)||Ay - p|| + ||A_n y - Bx|| \\
+ (1 - e_j)||Ax - p|| + ||A_n y - By||\}.
\end{aligned} \tag{3}$$

Thus, since $\overline{\lim}_{j\to\infty} e_j = 1$, from (3) for every $n \in N$, we have

$$\begin{aligned} &||A_n x - A_n y|| \\ &\quad \le \overline{\lim_{j\to\infty}}[a||Bx - By|| + b\max\{(1-e_j)||Ax - p|| + ||A_n x - Bx||, \\ &\qquad (1-e_j)||Ay - p|| + ||A_n y - By||\} \\ &\qquad + c\{(1-e_j)||Ay - p|| + ||A_n y - Bx|| \\ &\qquad\quad + (1-e_j)||Ax - p|| + ||A_n y - By||\}, \end{aligned}$$

which implies

$$\begin{aligned} ||A_n x - A_n y|| = a||Bx - By|| + b\max\{||A_n x - Bx||, ||A_n y - By|| \\ + c\{||A_n y - Bx|| + ||A_n y - By||\} \end{aligned}$$

for all $x, y \in D\prime a$. Therefore by Theorem 1, for every $n \in N$, A_n and B have a unique common fixed point x_n in $D\prime a$ i.e. for every $n \in N$, we have

$$F(A_n) \cap F(B) = \{x_n\}.$$

Now the compactness of Da ensures that $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ which converges to a point w in Da. Since

$$x_{n_i} = A_{n_i} x_{n_i} = e_{n_i} A_{n_i} + (1 - e_{n_i})p \tag{4}$$

and A is continuous, we have as $i \to \infty$ in (4) $w = Aw$ i.e. $w \in Da \cap F(A)$. Further, the continuity of B implies that

$$Bw = B(\lim_{i\to\infty} x_{n_i}) = \lim_{i\to\infty} Bx_{n_i} = \lim_{i\to\infty} x_{n_i} = w,$$

i.e. $w \in F(B)$. Therefore $w \in Da \cap F(A) \cap F(B)$ and so $Da \cap F(A) \cap F(B) \neq \phi$. This completes the proof. ∎

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