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IMPROVED CONVERGENCE RESULTS FOR GENERALIZED EQUATIONS

Ioannis K. Argyros

Abstract. We revisit the study of finding solutions of equations containing a differentiable and a continuous term on a Banach space setting [1]-[5], [9]-[11]. Using more precise majorizing sequences than before [9]-[11], we provide a semilocal convergence analysis for the generalized Newton's method as well the generalized modified Newton's method. It turns out that under the same or even weaker hypotheses: finer error estimates on the distances involved, and an at least as precise information on the location of the solution can be obtained. The above benefits are obtained under the same computational cost.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the equation

$$F(x) + G(x) = 0, \tag{1}$$

where, F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in itself, and G is a continuous function, defined on some subset of X, with values in X.

We approximate vector x^* with a sequence generated implicitly by the generalized Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}\left[F(x_n) + G(x_{n+1})\right] \ , \quad (x_0 \in D), (\ n \geq 0) \tag{2}$$

or the generalized modified Newton's method

$$y_{n+1} = y_n - F'(y_0)^{-1}\left[F(y_n) + G(y_{n+1})\right] \ , \quad (y_0 = x_0 \in D), \ (n \geq 0). \tag{3}$$

If $G = 0$ these methods reduce, respectively to Newton's method, and the modified Newton's method [3], [6], [8]. A survey on local as well as semilocal convergence theorems concerning equation (1) in case $G = 0$ or not can be found in [1]-[5], [9]-[11], and the references there. Note that methods (2) and

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(3) can still be used to find solutions of $(F+G)$ when G is only continuous. In fact G can be taken to be multivalued [9], [11].

Let J denote the duality map given by $J(x) = \{y \in X^* : (x,y) = \|x\|^2 = \|y\|^2\}$. We recall that an operator $A : D(A) \subseteq X \to X$ is said to be accretive if for all $u, v \in D(A)$ there exists $l.c \in J(u-v)$ such that $(A(u) - A(v) \geq 0$. If in addition we have $R(I + \lambda A) = X$ for all $\lambda > 0$, then A is said to be l-accretive. In this case the Yosida approximation $(I+\lambda A)^{-1}$ exists as a single-valued non-expansive operator from X to X for every $\lambda > 0$. Finally, A is said to be closed if

$$(x_n \in D(A), \lim_{n\to\infty} x_n = x^*, \lim_{n\to\infty} A(x_n) = y)$$
$$\Longrightarrow (x^* \in D(A) \text{ and } A(x^*) = y).$$

The generalized Newton's method (2) makes sense if $(F'(x_n)^{-1} \in L(X,X)$ $(n \geq 0)$ the space of bounded linear operators from X into X, and $F'(x_n)^{-1}G$ is l-accretive, with nonempty domain $D(x_n)(n \geq 0)$. The generalized modified Newton's method is well defined provided that $F'(x_0)^{-1} \in L(X,X)$ and $F'(x_0)^{-1}G$ be l-accretive, with nonempty domain $D(x_0)$.

Here we provide a semilocal convergence analysis for methods (2) and (3) using some ideas of ours introduced in [1]-[3] and [9]-[11] for related works.

Our results are obtained in affine invariant form in contrast to the ones in [11]. The advantages of this approach have been explained in [3].

As in [11], assume that there exist non-negative constants ℓ, ℓ_0, $r > 0$ and $x_0 \in D$ with $F'(x_0)^{-1} \in L(X,X)$ such that

$$\left\|F'(x_0)^{-1}[F'(x) - F'(Y)]\right\| \leq \ell \left\|x - y\right\|, \tag{4}$$

and

$$\left\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\right\| \leq \ell_0 \left\|x - x_0\right\| \tag{5}$$

for all $x, y \in U(x_0, r) = \{x \in X : \|x - x_0\| \leq r\} \subseteq D$.

In view of (4) and (5)

$$\ell_0 \leq \ell \tag{6}$$

holds in general, and $\frac{\ell}{\ell_0}$ can be arbitrarily large [1]-[3].

Using more precise majorizing sequence than before [11], we provide under the same computational cost and the same or weaker hypotheses finer error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ $(n \geq 0)$, and on at least as precise information on the location of the solution x^*.

2. Semilocal convergence analysis of method (2)

We will need the following results on majorizing sequences:

Lemma 1. [1] *Assume there exist constants $\ell \geq 0$, $\ell_0 \geq 0$ with $\ell_0 \leq \ell$, $\eta \geq 0$ such that*

$$h_0 = (\ell_0 + \ell)\eta \leq 1 \tag{7}$$

Then, iteration $\{t_n\}$ $(n \geq 0)$ given by

$$t_0 = 0, t_1 = \eta, t_{n+1} = t_n + \frac{\ell(t_n - t_{n-1})^2}{2(1 - \ell_0 t_n)}, \quad (n \geq 1) \tag{8}$$

is non-decreasing, bounded above by $t^{**} = 2\eta$, and converges to some t^* such that

$$0 \leq t^* \leq t^{**} \tag{9}$$

Remark 2. *If $\ell_0 = \ell$ iteration (8) coincides with majorizing sequence $\{s_n\}$ essentially used in* [11] *and given by*

$$s_0 = 0, \ s_1 = n, \ s_{n+1} = s_n + \frac{\ell(s_n - s_{n-1})^2}{2(1 - \ell_0 s_{n-1})}\ , \ \ (n \geq 1). \tag{10}$$

We showed in [1] *that if the famous Newton-Kantorovich hypothesis* [6]

$$h = 2\ell\eta \leq 1 \tag{11}$$

holds, then

$$t_n \leq s_n, \quad (n \geq 2), \tag{12}$$

$$t_{n+1} - t_n \leq s_{n+1} - s_n, \quad (n \geq 1), \tag{13}$$

and

$$t^* \leq s^*, \tag{14}$$

$$t^* - t_n \leq s^* - s_n, \quad s^* = \lim_{n\to\infty} s_n, \quad (n \geq 0). \tag{15}$$

Moreover if strict inequality holds in (6), so does in (12) and (12).

Note also that

$$h \leq 1 \Longrightarrow h_0 \leq 1, \tag{16}$$

but not necessarily vice versa unless if $\ell_0 = \ell$.

We can show the following semilocal convergence theorem for method (2):

Theorem 3. *Let $F : U(x_0, r) \to X$ be a Fréchet-differentiable operator, and let G be a continuous operator defined on a subset of X with values in X. Suppose that $F'(x)^{-1}G$ is l-accretive and closed with domain $D(x)$, whenever $F'(x)^{-1}$ exists, and that there exist $\ell \geq 0$ and $\ell_0 \geq 0$ such that (4) and (5) hold.*

If $F'(x_0)^{-1}$ exists, set $\left\|F'(x_0)^{-1}[F(x_0) + G(x_0)]\right\| \leq \eta$. Moreover assume (7) holds, and $t^* \leq r$ or $t^{**} \leq r$. Then sequence $\{x_n\}$ $(n \geq 0)$ generated by generalized Newton's method (2) is well defined, remains in $U(x_0, t^*)$ for all $n \geq 0$, and converges to a solution x^* of equation $F(x) + G(x) = 0$ in $U(x_0, t^*)$.

Moreover the following error bounds hold for all $n \geq 1$:

$$\|x_{n+1} - x_n\| \leq \frac{\ell \|x_n - x_{n-1}\|^2}{2(1 - \ell_0 \|x_n - x_0\|)} \leq t_{n+1} - t_n, \tag{17}$$

and

$$\|x_n - x^*\| \leq t^* - t_n. \tag{18}$$

Proof. We shall show for all $k \geq 0$

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k \tag{19}$$

and

$$U(x_{n+1}, t^* - t_{k+1}) \leq U(x_k, t^* - t_k). \tag{20}$$

Since $F'(x_0)^{-1}G$ is l-accretive, it is easy to see that there exists a unique solution $x_1 \in D(x_0)$ satisfying (2). For every $z \in U(x_1, t^* - t_1)$,

$$\|z - x_0\| \leq \|z - x_1\| + \|x_1 - x_0\| \leq t^* - t_1 + t_1 = t^* - t_0,$$

implies $z \in U(x_0, t^* - t_0)$.

That is (19) and (20) holds for $k = 0$. Given they hold for $n = 0, 1, ..., k$, then since $F'(x_n)^{-1}G$ is l-accretive, it is easy to see that there exists a unique $x_{k+1} \in D(x_k)$ satisfying (2).

We also have

$$\|x_{k+1} - x_0\| \leq \sum_{i=1}^{k+1} \|x_i - x_{i-1}\| \leq \sum_{i=1}^{k+1} (t_i - t_{i-1}) = t_{k+1} - t_0 = t_{k+1},$$

and

$$\|x_k + \theta(x_{k+1} - x_k) - x_0\| \leq t_k + \theta(t_{k+1} - t_k) \leq t^*, \quad \theta \in [0, 1].$$

It also follows from (2), the accretivity of $F'(x_k)^{-1}$ that

$$\begin{aligned} (x_{k+1} - x_k, v) &\leq (x_{k+1} - x_k, v) + \left(F'(x_k)^{-1}G(x_{k+1}) - F'(x_k)G(x_k), v\right) \\ &= (F'(x_k)^{-1}[-F(x_k) - G(x_k)], v) \end{aligned} \tag{21}$$

for some $v \in J(x_{k+1} - x_k)$. In view of the definition of J, we obtain

$$\begin{aligned} \|x_{k+1} - x_k\| \leq & \left\|[F'(x_k)^{-1}F'(x_0)]\right\| \\ & \cdot \left\|F'(x_0)^{-1}(F(x_k) + G(x_k))\right\|. \end{aligned} \tag{22}$$

Using (5) and the induction hypotheses we obtain

$$\begin{aligned} \left\|F'(x_0)^{-1}[F'(x_k) - F'(x_0)]\right\| &\leq \ell_0 \|x_k - x_0\| \\ &\leq \ell_0 t^* \leq \ell_0 t^{**} = 2\ell_0 \eta < 1. \end{aligned} \tag{23}$$

It follows from (23), and the Banach Lemma on invertible operators that [3], [6] $F'(x_k)^{-1}$ exists, and

$$\left\|F'(x_k)^{-1}F'(x_0)\right\| \leq \frac{1}{1 - \ell_0 \|x_k - x_0\|}. \tag{24}$$

Therefore by (22) and (24) we obtain in turn:

(25)

$$\begin{aligned}
&\|x_{k+1}-x_k\| \leq \\
&\leq \frac{1}{1-\ell_0 t_k}\left\|F'(x_0)^{-1}[F(x_k)-F(x_{k-1})]-F'(x_{k-1})(x_k-x_{k-1})\right\| \\
&\leq \frac{1}{1-\ell_0 t_k}\left\|F'(x_0)\int_0^1 [F'(x_{k-1}+\theta(x_k-x_{k-1})-F'(x_{k-1})]\,(x_k-x_{k-1})d\theta\right\| \\
&\leq \frac{1}{1-\ell_0 t_k}\ell\,\|x_k-x_{k-1}\|^2 \leq \frac{\ell(t_k-t_{k-1})^2}{2(1-\ell_0 t_{k-1})}=t_{k+1}-t_k,
\end{aligned}$$

which shows (19) for all $k \geq 0$.Thus, for every $z \in U(x_{k+1}, t^*-t_{k+1})$, we have

$$\begin{aligned}
\|z-x_k\| &\leq \|z-x_{k+1}\|+\|x_{k+1}-x_k\| \leq t^*-t_{k+1}+t_{k+1}-t_k \\
&= t^*-t_k,
\end{aligned}$$

which implies $z \in U(x_k, t^*-t_k)$.That is (20) holds for all $k \geq 0$.

Lemma 1, (19) and (20) imply that sequence $\{x_n\}$ is Cauchy in a Banach space X, and as such is converges to some $x^* \in U(x_0, r)$ (since $U(x_0, r)$ is a closed set).

Let $A = F'(x_0)^{-1}G$. Then, clearly, we have:

$$\begin{aligned}
A(x_{k+1}) &= F'(x_0)^{-1}F'(x_k)(x_k-x_{k+1})-F'(x_0)^{-1}F(x_k) \\
&\to -F'(x_0)^{-1}F(x^*), \text{ as } k\to\infty.
\end{aligned}$$

Since $x_n \to x^*$ and A is closed, we deduce $A(x^*) = -F'(x_0)^{-1}F(x^*)$, from which it follows that x^* satisfies equation (1). □

Finally we provide a generalization of a semilocal converge theorem due to Mysovskikh [7], [8]:

Theorem 4. . *Let $F : U(x_0, r) \to X$ be a Fréchet-differentiable operator, and let G be a continuous operator from a subset of X into X. Suppose that $F'(x)^{-1}$ exists and that $F'(x)^{-1}G$ is l-accretive and closed, with domain $D(x)$, for all $x \in U(x_0, r)$. If there exists $\ell > 0$, $\eta > 0$, $\alpha > 0$ such that (4) holds, and*

$$\left\|F'(x)^{-1}F'(x_0)\right\| \leq \alpha, \text{ for all } x \in U(x_0, r)$$

$$p = \frac{\ell\eta^2}{2} < 1$$

$$\frac{2}{\ell\alpha}\sum_{k=0}^{\infty} p^{2^k} < r,$$

then there exists a solution x^* of equation (1) in $U(x_0, r)$.

Moreover, sequence $\{x_n\}$ generated by generalized Newton's method (2) is well defined, remains in $U(x_0, r)$ and converges to x^* so that

$$\|x^* - x_n\| \leq \frac{2}{\ell\eta} \sum_{k=n}^{\infty} p^{2^k}.$$

Proof. Replace F in Theorem 3 in [11] by $F'(x_0)^{-1}F$. □

3. Semilocal convergence analysis of method (3)

We can show the corresponding to 3 semilocal convergence results using method (3).

The proof is similar to Theorem 2 in [11, p.184] but we use the weaker (5) (which is actually needed) instead of (4) used in [11]. There are also some small differences in the proof. Moreover reference [11] cannot easily be found.

Theorem 5. *Let $F : U(x_0, r) \to X$ be a Fréchet-differentiable operator, and let G be a continuous operator defined on a subset of X with values in X. For some $x_0 \in U$ suppose $F(x_0)^{-1}$ exists, and $F'(x_0)^{-1}G$ is l_0-accretive and closed, with domain $D(x_0)$, and that there exists $l_0 \geq 0$ such that (5) holds. Moreover assume*

$$q = 2\ell_0\eta < 1 \tag{26}$$

and

$$r_0 = \frac{\eta}{1 + \sqrt{1-q}} \leq r \tag{27}$$

where η was defined in (10).

The sequence $\{y_n\}$ generated by the generalized modified Newton's method (3) remains in $U(x_0, r_0)$ for all $n \geq 0$ and converges to a unique solution $y^* \in U(x_0, r_0)$ of equation $F(x) + G(x) = 0$.Moreover the following estimated hold

$$\|y_{n+1} - y_n\| \leq \ell_0 r_0 \|y_n - y_{n-1}\| \leq [1 - \sqrt{1-q}]\, \|y_n - y_{n-1}\|, \quad (n \geq 1), \tag{28}$$

and

$$\|y_n - y^*\| \leq \frac{2\eta}{q}[1 - \sqrt{1-q}]^{n+1}. \tag{29}$$

Proof. Let $x \in U(x_0, r_0)$. Using the l_0-accretivity of $F'(x_0)^{-1}G$ we can define $P(x) \in D(x_0)$ by

$$P(x) + F'(x_0)^{-1}G(P(x)) = x - F'(x_0)^{-1}F(x) \tag{30}$$

By definition (30) and the accretivity of $F'(x_0)^{-1}G$ we get

$$(F'(x_0)^{-1}F(x) - x + P(x), v) \leq 0 \text{ for some } v \in J(P(x) - x_0). \tag{31}$$

Estimate (31) can be rewritten

$$(P(x) - x_0, v) \leq (x - x_0 - F'(x_0)^{-1}F(x), v)$$

or

$$
\begin{aligned}
(32) \qquad \|P(x) - x_0\| &\le \left\| x - x_0 - F'(x_0)^{-1}F(x) \right\| \\
&\le \left\| F'(x_0)^{-1}[F'(x_0)(x - x_0) + F(x_0) - F(x) - F(x_0)] \right\| \\
&\le \left\| F'(x_0)^{-1}[F(x) - F(x_0) - F'(x_0)(x - x_0)] \right\| \\
&+ \left\| F'(x_0)^{-1}F(x_0) \right\| \\
&\le \frac{\ell_0}{2} \|x - x_0\|^2 + \eta \le \frac{\ell_0}{2} r_0^2 + \eta \le r_0,
\end{aligned}
$$

by the definition of r_0.That is we showed P maps the ball $U(x_0, r_0)$ into itself. Moreover for any $x, y \in U(x_0, r_0)$, (3) and the accretivity of $F'(x_0)^{-1}G$ we get

$$
(33) \qquad (P(x) - P(y), v) + (F'(x_0)^{-1}[F(x) - f(y) - x + y, v) \le 0
$$

for some $v \in J(P(x) - P(y))$.

Estimate (33) can be written as

$$
(P(x) - P(y), v) \le (x - y) - F'(x_0)^{-1}[F(x) + F(y)], v)
$$

or

$$
\begin{aligned}
(34) \qquad \|P(x) - P(y)\| &\le \left\| F'(x_0)^{-1}[F(x) - F(y) - F'(x_0)(x - y)] \right\| \\
&\le \ell \|x - y\| \max\{\|x - x_0\|, \|y - x_0\|\} \\
&\le \ell r_0 \|x - y\| \le [1 - \sqrt{1 - q}] \|x - y\|,
\end{aligned}
$$

which implies that (28) holds and that P is a strict contraction mapping from the ball $U(x_0, r_0)$ to itself. Note that method (3) can be written as $y_{n+1} = P(y_n)$. It then follows by the Banach contraction mapping principle [6], [8] that there exists a $y^* \in U(x_0, r_0)$ such that $P(y^* = y^*$. We shall also show (29). Indeed we can have in turn:

$$
\begin{aligned}
\|y^* - y_n\| &= \|P(y^*) - P(y_{n-1})\| \\
&\le [1 - \sqrt{1 - q}]^n \|y^* - x_0\| \\
&\le \sqrt{1 - q}]^n r_0 = \frac{2\eta}{q}[1 - \sqrt{1 - q}]^{n+1}.
\end{aligned}
$$

The fact that y^* solves equation (1) follows from the same reasoning given in the proof of Theorem 3. Finally to show uniqueness, let us assume $y^{**} \in U(x_0, r_0)$ is a fixed point of P with $y^* \ne y^{**}$. Then we get

$$
\|y^* - y^{**}\| = \|P(y^*) - P(y^{**})\| < \|y^* - y^{**}\|
$$

which is a contradiction.

That completes the proof of 5. □

Remark 6. *If $\ell_0 = \ell$, then 5 reduces to in* [11]. *Otherwise it is an improvement. Indeed let*

$$
(35) \qquad \overline{q} = 2\ell\eta < 1,
$$

and

$$\overline{r} = \frac{\eta}{1+\sqrt{1-\overline{q}}} \leq r \tag{36}$$

Under conditions (35) and (36) the conclusions of Theorem 5 hold with $\overline{q}$, $\overline{r}$ replacing g and r_0 respectively.

Proof. However, we have

$$\overline{q} < 1 \Longrightarrow q < 1 \tag{37}$$

but not vice versa,
and

$$r_0 < \overline{r}. \tag{38}$$

Finally note that

$$1 - \sqrt{1-q} < 1 - \sqrt{1-\overline{q}},$$

which implies that the error bounds are also smaller with our approach. □

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Cameron University, Department of Mathematical Sciences, Lawton, OK 73505, U.S.A.

E-mail address: iargyros@cameron.edu