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COMPOSITE-EXPONENTIAL-FITTING INTERPOLATION RULES

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Abstract. This paper demonstrates how composite-exponential-fitting interpolation rules can be constructed to fit an oscillatory function using not only pointwise values of that function but also of that functions's derivative on a closed and bounded interval of interest. This is done in the framework of exponential-fitting techniques. These rules extend the classical composite cubic Hermite interpolating polynomials in the sense that they become the classical composite polynomials as a parameter tends to zero. Some examples are provided to compare the newly constructed rules with the classical composite cubic Hermite interpolating polynomials (or recently developed interpolation rules).

1. Introduction

Generally speaking, the Hermite interpolating polynomials using not only pointwise values of a function, but also of its derivative, can be constructed by extending Lagrange interpolating polynomials or by using the divided difference for more computable form [1]. Because of the derivative information, the Hermite interpolating polynomials provide a relatively more accurate approximation of the function over an interval of interest than the Lagrange interpolating polynomials. Admittedly however, there are certain categories of interpolation problems in which the Hermite interpolating polynomials cannot be appropriately applied. For example, neither Hermite interpolating polynomials nor Lagrange interpolating polynomials work effectively in approximating either oscillatory functions depending on high frequencies or data having highly oscillatory property. As an alternative method to manipulate the oscillatory property, the applicability of exponential-fitting interpolation rules was recently investigated, and the numerical examples resulted in appropriately interpolating highly oscillatory functions [9]. However, despite this breakthrough, difficulties still arise when the number of the selected nodes on the domain increases in the rules. That is, the condition numbers, which are involved with the linear system to generate the rules, increase as the number of the nodes

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increases in the rules. In a typical system, an important quantity that has a crucial influence in determining the numerical solution of a linear system is the condition number. This condition number depends on the norms of the associated matrix and its inverse [2]. As might be expected, the larger the condition number, the more ill-conditioned the linear system. In order to handle the difficulties presented by the condition number, this paper will first focus its attention toward constructing exponential-fitting rules with two nodes and then investigate the properties of the rules. This investigation will make it possible to construct composite interpolation rules in the framework of exponential-fitting techniques. In addition, this construction will show the extension of the classical interpolating polynomials which are known as composite (or piecewise) cubic Hermite interpolating polynomials in the sense that the new composite rules become the classical composite polynomials, which are more explained in Section 4, as the involved frequency tends to zero. It should be noted that Ixaru introduced the idea of using an exponential-fitting tool for numerical operations on oscillatory functions [4]. His work has been utilized and further developed into the areas of numerical integrations and ordinary differential equations [6, 7, 8].

Consider for a moment the mathematical demands made by modern computer graphics. Today's applications in computer graphics require the rapid generation of smooth curves that can be easily and quickly modified. For such purpose, it is considered that changing one portion of these curves should have little or no effect on other portions of the curves. In essence, this is why the composite cubic Hermite interpolating polynomials are preferred as curves for use in computer graphics. However, despite the proven merits of the classical polynomials, difficulties are encountered when fitting highly oscillatory functions. Based on the results demonstrated in this paper, the newly constructed rules are expected to be used more effectively in the area of computer graphics as well as interpolation than the classical composite polynomials.

In Section 2, approximate rules to interpolate an oscillatory function at two nodes are constructed using exponential-fitting techniques. The coefficients of the exponential-fitting rules are obtained by some functions in the Appendix. In Section 3, the distinct properties of the coefficients of the exponential-fitting rules are investigated. This allows for an analysis of the relationship between the rules constructed in Section 2 and classical cubic Hermite interpolating polynomials when the involved frequency tends to zero. In Section 4, Fig. 1 and 2 are provided to compare the composite-exponential-fitting interpolation rules with the composite cubic Hermite interpolating polynomials. Some discussions are also given between the composite-exponential-fitting rules and recently developed interpolation rules.

2. Basic exponential-fitting interpolation rules

From the ideas presented in Chapter 4 of [5] and [9], we consider an oscillatory function f with a frequency ω of the form

$$f(x) = f_1(x)\cos\omega x + f_2(x)\sin\omega x, \tag{1}$$

where f_1 and f_2 are assumed to be smooth enough to be approximated by polynomials, and its interpolation rule, denoted by R, which involves not only pointwise values of the function but also of its first derivative at two nodes, viz.:

$$f(c+th) \approx R(t) = \alpha_1 f(c-h) + \alpha_2 f(c+h) + h(\beta_1 f'(c-h) + \beta_2 f'(c+h)), \tag{2}$$

where c is the middle node on a closed interval, the length of the interval is $2h$ and $-1 \le t \le 1$. Thus R approximates f whose domain is $[c-h, c+h]$. From the definition of R given in (2), we construct a functional F:

$$\begin{aligned} F(f(x), h, \Lambda) &= f(x+th) - \alpha_1 f(x-h) - \alpha_2 f(x+h) \\ &\quad -h(\beta_1 f'(x-h) + \beta_2 f'(x+h)), \end{aligned} \tag{3}$$

where Λ is the vector of coefficients α_k and β_k which have to be expressed in terms of the variable t and the values of ω and h, $\Lambda = (\alpha_1, \alpha_2, \beta_1, \beta_2)$. When the values of the function f and its first derivative f' at two nodes are assumed to be known, our main concern is to determine the values of the coefficients α_k and β_k from the conditions

$$F(x^{n-1}\exp(\pm i\omega x), h, \Lambda) = 0 \quad (n = 1, 2) \tag{4}$$

because $\sin\omega x$ and $\cos\omega x$ are linear combinations of exponential functions $\exp(\pm i\omega x)$. By inserting $f(x) = \exp(i\omega x), \exp(-i\omega x)$ into (3), we have

$$\begin{gathered} F(\exp(\mu x), h, \Lambda) = \exp(\mu x)\phi(\mu h, \Lambda) \\ \text{and} \\ F(\exp(-\mu x), h, \Lambda) = \exp(-\mu x)\phi(-\mu h, \Lambda), \end{gathered} \tag{5}$$

where $\mu = i\omega$ and

$$\phi(u, \Lambda) = \exp(ut) - (\alpha_1\exp(-u) + \alpha_2\exp(u)) - u(\beta_1\exp(-u) + \beta_2\exp(u)).$$

With the following notations,

$$\begin{gathered} \Phi^+(Z, \Lambda) = \tfrac{1}{2}(\phi(u, \Lambda) + \phi(-u, \Lambda)) \\ \text{and} \\ \Phi^-(Z, \Lambda) = \tfrac{1}{2u}(\phi(u, \Lambda) - \phi(-u, \Lambda)), \end{gathered} \tag{6}$$

where $Z = u^2 = (\mu h)^2 = -\omega^2 h^2$, we get

$$\begin{aligned} \Phi^+(Z, \Lambda) &= \xi(Zt^2) - \alpha^+\xi(Z) + \beta^- Z\eta_0(Z), \\ \Phi^-(Z, \Lambda) &= t\eta_0(Zt^2) + \alpha^-\eta_0(Z) - \beta^+\xi(Z), \end{aligned} \tag{7}$$

where

$$\alpha^\pm = \alpha_1 \pm \alpha_2, \quad \beta^\pm = \beta_1 \pm \beta_2. \tag{8}$$

In (7) the functions ξ and η_0 which were introduced in Section 3.4 of [3] are given in the Appendix. It is now obtained that the four equations of (4) are equivalent to new forms of four equations,

$$\begin{aligned} &\Phi^+(Z,\Lambda)=0, \Phi^-(Z,\Lambda)=0, \\ &\tfrac{d}{dZ}\Phi^+(Z,\Lambda)=0, \tfrac{d}{dZ}\Phi^-(Z,\Lambda)=0, \end{aligned} \tag{9}$$

where the differentiation results like

$$\frac{\partial}{\partial\mu}F(\exp(\mu x),h,\Lambda)=F(x\exp(\mu x),h,\Lambda) \text{ and } \frac{dZ}{d\mu}=2\mu h^2$$

exploit to get the equivalent relation. Note that

$$\begin{gathered} \tfrac{d}{dZ}\Phi^+(Z,\Lambda)=\tfrac{1}{2}\left(t^2\eta_0(Zt^2)-\alpha^+\eta_0(Z)+\beta^-(\xi(Z)+\eta_0(Z))\right) \\ \text{and} \\ \tfrac{d}{dZ}\Phi^-(Z,\Lambda)=\tfrac{1}{2}\left(t^3\eta_1(Zt^2)+\alpha^-\eta_1(Z)-\beta^+\eta_0(Z)\right) \end{gathered} \tag{10}$$

which come from the differentiation properties given in the Appendix. The linear system of (9) governing $\alpha^+,\alpha^-,\beta^+,\beta^-$, provides the values of the coefficients in the rule (2) with the help of the algebraic relations in (8). That is, we have

$$\begin{aligned} \alpha_1 &= \tfrac{1}{2V_1V_2}(T_1V_2+T_2V_1), & \alpha_2 &= \tfrac{1}{2V_1V_2}(T_1V_2-T_2V_1), \\ \beta_1 &= \tfrac{1}{2V_1V_2}(W_1V_1+W_2V_2), & \beta_2 &= \tfrac{1}{2V_1V_2}(W_1V_1-W_2V_2), \end{aligned} \tag{11}$$

where

$$\begin{aligned} V_1 &= 1+\xi(Z)\eta_0(Z), \\ V_2 &= \eta_0^2(Z)-\xi(Z)\eta_1(Z), \\ T_1 &= \xi(Zt^2)(\xi(Z)+\eta_0(Z))-Zt^2\eta_0(Zt^2)\eta_0(Z), \\ T_2 &= t(-\eta_0(Zt^2)\eta_0(Z)+t^2\xi(Z)\eta_1(Zt^2)), \\ W_1 &= t(-\eta_0(Zt^2)\eta_1(Z)+t^2\eta_0(Z)\eta_1(Zt^2)), \\ W_2 &= \xi(Zt^2)\eta_0(Z)-t^2\xi(Z)\eta_0(Zt^2). \end{aligned} \tag{12}$$

From now on, we call the rule R with the coefficients of (11) to interpolate an oscillatory function at two nodes in the exponential-fitting techniques as a basic exponential-fitting interpolation rule while we do the classical cubic Hermite interpolating polynomial to interpolate the oscillatory function at the two nodes as a basic Hermite-interpolating polynomial which is given in Section 3. Next section it is considered that the rule R with the coefficients of (11) becomes the basic Hermite-interpolating polynomial as the frequency ω tends to zero.

3. Basic exponential-fitting interpolation rule in comparison with basic Hermite-interpolating polynomial

By applying general Hermite interpolation theories explained in Chapter 3 of [10] into the case at two nodes or using the results provided by Section 3.3 of [2], it is given the basic Hermite-interpolating polynomial H which agrees with f and f' at two nodes $c-h$ and $c+h$ as

$$H(x) = f(c-h)H_1(x)+f(c+h)H_2(x)+f'(c-h)\tilde{H}_1(x)+f'(c+h)\tilde{H}_2(x), \tag{13}$$

where

$$\begin{aligned} H_1(x) &= \frac{1}{4h^2}(1+\frac{1}{h}(x-(c-h)))(x-(c+h))^2, \\ H_2(x) &= \frac{1}{4h^2}(1-\frac{1}{h}(x-(c+h)))(x-(c-h))^2, \\ \tilde{H}_1(x) &= \frac{1}{4h^2}(x-(c-h))(x-(c+h))^2, \\ \tilde{H}_2(x) &= \frac{1}{4h^2}(x-(c+h))(x-(c-h))^2. \end{aligned} \tag{14}$$

By using the change of variables,

$$x = c + ht,$$

the basic Hermite-interpolating polynomial $H(x)$ of degree 3, defined on $[c-h, c+h]$, is transformed into a t-dependent function as follows:

$$\begin{aligned} H(x) &= H(c+ht) \\ &= \frac{1}{4}\left((t+2)(t-1)^2 f(c-h) + (2-t)(t+1)^2 f(c+h)\right) \\ &\quad + \frac{h}{4}\left((t+1)(t-1)^2 f'(c-h) + (t-1)(t+1)^2 f'(c+h)\right). \end{aligned} \tag{15}$$

It is now shown that, as the frequency approaches zero, the basic exponential-fitting interpolation rule R with the coefficients given in (11) tends to the basic Hermite-interpolating polynomial H given in (15) because of the properties of the functions ξ and η_s in the Appendix. In details, as ω tends to zero, equivalently $Z \to 0$, one of the four coefficients given in (11), α_1 becomes

$$\frac{3}{8}(\frac{2}{3}t^3 - 2t + \frac{4}{3}) = \frac{1}{4}(t+2)(t-1)^2,$$

which is exactly the same as the coefficient of $f(c-h)$ in the first term of $H(x)$ in (15). Likewise,

$$\begin{aligned} \alpha_2 &= \frac{1}{4}(2-t)(t+1)^2, \\ \beta_1 &= \frac{1}{4}(t+1)(t-1)^2, \\ \beta_2 &= \frac{1}{4}(t-1)(t+1)^2. \end{aligned}$$

Furthermore, in the rule R with the coefficients given in (11) we have

$$R(-1) = f(c-h) \tag{16}$$

irrespective of the values of ω because $\alpha_1 = 1$, $\alpha_2 = \beta_1 = \beta_2 = 0$ at $t = -1$. Also,

$$R(1) = f(c+h) \tag{17}$$

because $\alpha_2 = 1$, $\alpha_1 = \beta_1 = \beta_2 = 0$ at $t = 1$. By using algebraic relations given in (8) and the results of (11), we obtain

$$\begin{aligned}
\frac{d\alpha^+}{dt} &= \frac{Zt\eta_0(Zt^2)\xi(Z) - Zt\eta_0(Zt^2)\eta_0(Z) - Z^2t^3\eta_1(Zt^2)\eta_0(Z)}{1+\xi(Z)\eta_0(Z)}, \\
\frac{d\alpha^-}{dt} &= \frac{-\eta_0(Zt^2)\eta_0(Z) - Zt^2\eta_1(Zt^2)\eta_0(Z) + 3t^2\eta_1(Zt^2)\xi(Z) + Zt^4\eta_2(Zt^2)\xi(Z)}{\eta_0^2(Z) - \xi(Z)\eta_1(Z)}, \\
\frac{d\beta^+}{dt} &= \frac{-\eta_0(Zt^2)\eta_1(Z) - Zt^2\eta_1(Zt^2)\eta_1(Z) + 3t^2\eta_1(Zt^2)\eta_0(Z) + Zt^4\eta_2(Zt^2)\eta_0(Z)}{\eta_0^2(Z) - \xi(Z)\eta_1(Z)}, \\
\frac{d\beta^-}{dt} &= \frac{Zt\eta_0(Zt^2)\eta_0(Z) - 2t\eta_0(Zt^2)\xi(Z) - Zt^3\eta_1(Zt^2)\xi(Z)}{1+\xi(Z)\eta_0(Z)}
\end{aligned} \tag{18}$$

and then the relations in (8) again show that

$$\begin{aligned}
&\frac{d\alpha_1}{dt} = \frac{d\alpha_2}{dt} = \frac{d\beta_2}{dt} = 0, \quad \frac{d\beta_1}{dt} = 1 \quad \text{at t} = -1 \\
&\qquad\qquad\qquad\qquad \text{and} \\
&\frac{d\alpha_1}{dt} = \frac{d\alpha_2}{dt} = \frac{d\beta_1}{dt} = 0, \quad \frac{d\beta_2}{dt} = 1 \quad \text{at t} = 1
\end{aligned} \tag{19}$$

because we have

$$\begin{aligned}
&\frac{d\alpha^+}{dt} = \frac{d\alpha^-}{dt} = 0, \quad \frac{d\beta^+}{dt} = \frac{d\beta^-}{dt} = 1 \qquad \text{at} \quad t = -1 \\
&\qquad\qquad\qquad \text{and} \\
&\frac{d\alpha^+}{dt} = \frac{d\alpha^-}{dt} = 0, \quad \frac{d\beta^+}{dt} = 1, \quad \frac{d\beta^-}{dt} = -1 \quad \text{at} \quad t = 1.
\end{aligned} \tag{20}$$

Thus, the change of variables,

$$x = c + ht,$$

reveals that the rule R have the same slopes as f at two nodes $c-h$ and $c+h$. The slope properties of the basic exponential-fitting interpolation rule R as well as the results of (16) and (17) make it possible to consider the composite exponential-fitting interpolation rules explained in Section 4. In general, such properties play a crucial role in interpolating curves or data in relatively accurate approximations, as shown in Figure 1 and 2 which are explained in Section 4.

4. Composite interpolation rules and discussion

Let FTE be the first non-vanishing term of the error of the rule R with coefficients given in (11). Through applying some results about the first non-vanishing term of the error given by Eq. (23) of [9] (or Eq. (2.38) in [4] for more general form of the first non-vanishing term of the error) to the rule R, the following three statements are satisfied.

(i):

(21)
$$\mathrm{FTE} = \frac{1-\alpha_1-\alpha_2}{\omega^4}\left(\frac{d^2}{dx^2}+\omega^2\right)^2 f(x).$$

(ii): The value of $\alpha_1+\alpha_2$ is bounded by some constant.

(iii): The leading factor of ω in the FTE is $\frac{1}{\omega^2}$.

Therefore, as ω increases the FTE decreases as $\frac{1}{\omega^2}$. This property of the error makes exponential-base rules explained below more meaningful and valuable. For example, for the case of $f(x) = x^2\cos\omega x + x^2\sin\omega x$,

(22)
$$\mathrm{FTE} = \frac{8(\alpha_1+\alpha_2-1)(\cos\omega x+\sin\omega x)}{\omega^2}.$$

The term FTE is worth investigating because it gives the most powerful influence on the behavior of the whole error of the rule R. Readers may have any question about the whole error of the rule. But it does not seem easy to explain about it clearly. Thus much efforts are being made to answer to the question.

Let us consider a function

(23)
$$f(x) = x\sin x\cos\omega x + x\cos x\sin\omega x$$

which is an example of (1) and two types of its approximate values on the interval $[a,b]$. One is the composite-Hermite-interpolating polynomial, denoted by H_c, and the other is the composite-exponential-fitting interpolation rule, denoted by R_c. More detailed forms for both of them are followed. We select a set of distinct nodes $\{a = x_0, x_1, \ldots, x_n = b\}$ from the interval $[a,b]$, where $x_0 < x_1 < \cdots < x_n$. Then the interval $[a,b]$ is subdivided into n subintervals, that is

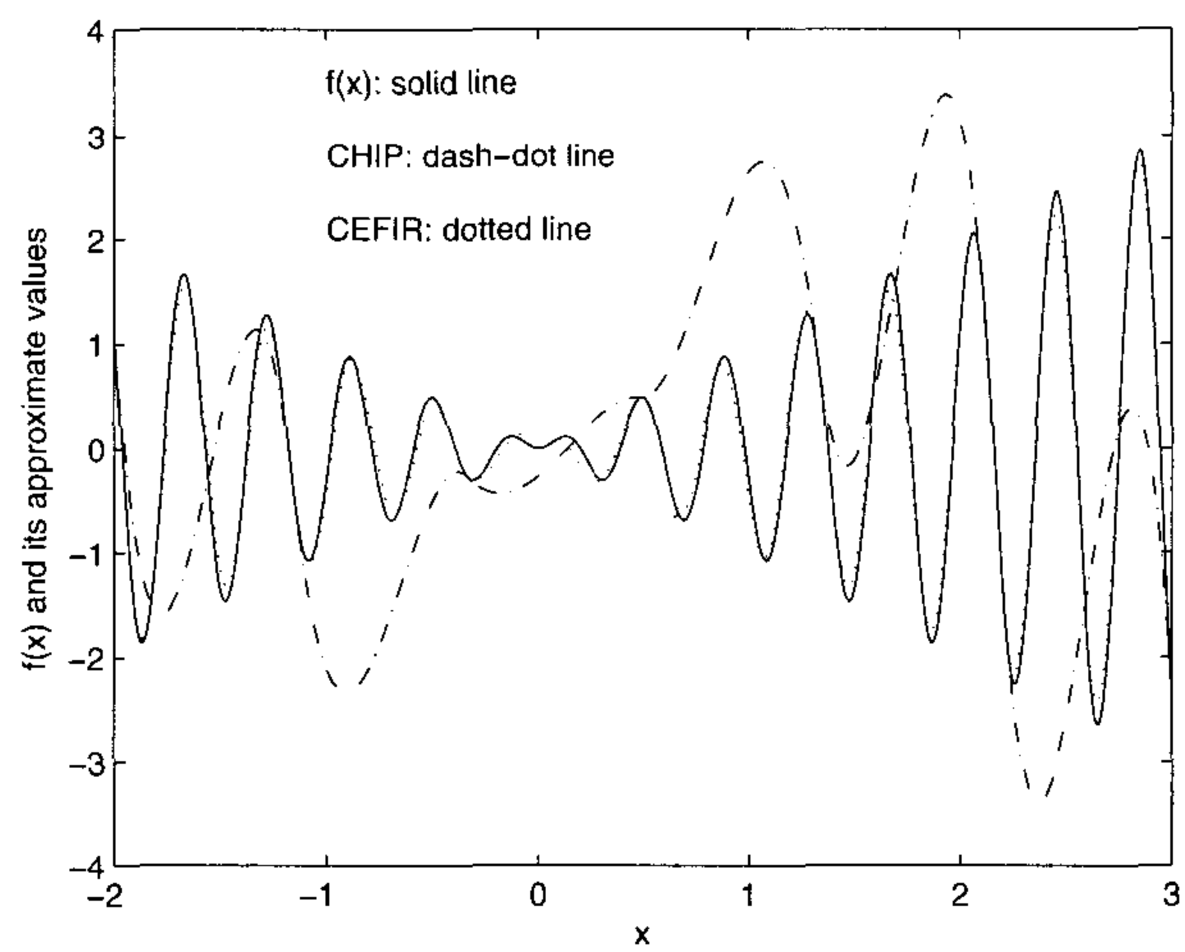
(24)
$$[a,b] = [x_0,x_1]\cup[x_1,x_2]\cup\cdots\cup[x_{n-1},x_n].$$

Through replacing c and h in (2) with

(25)
$$c_k = \frac{x_{k-1}+x_k}{2} \quad\text{and}\quad h_k = \frac{x_k-x_{k-1}}{2},$$

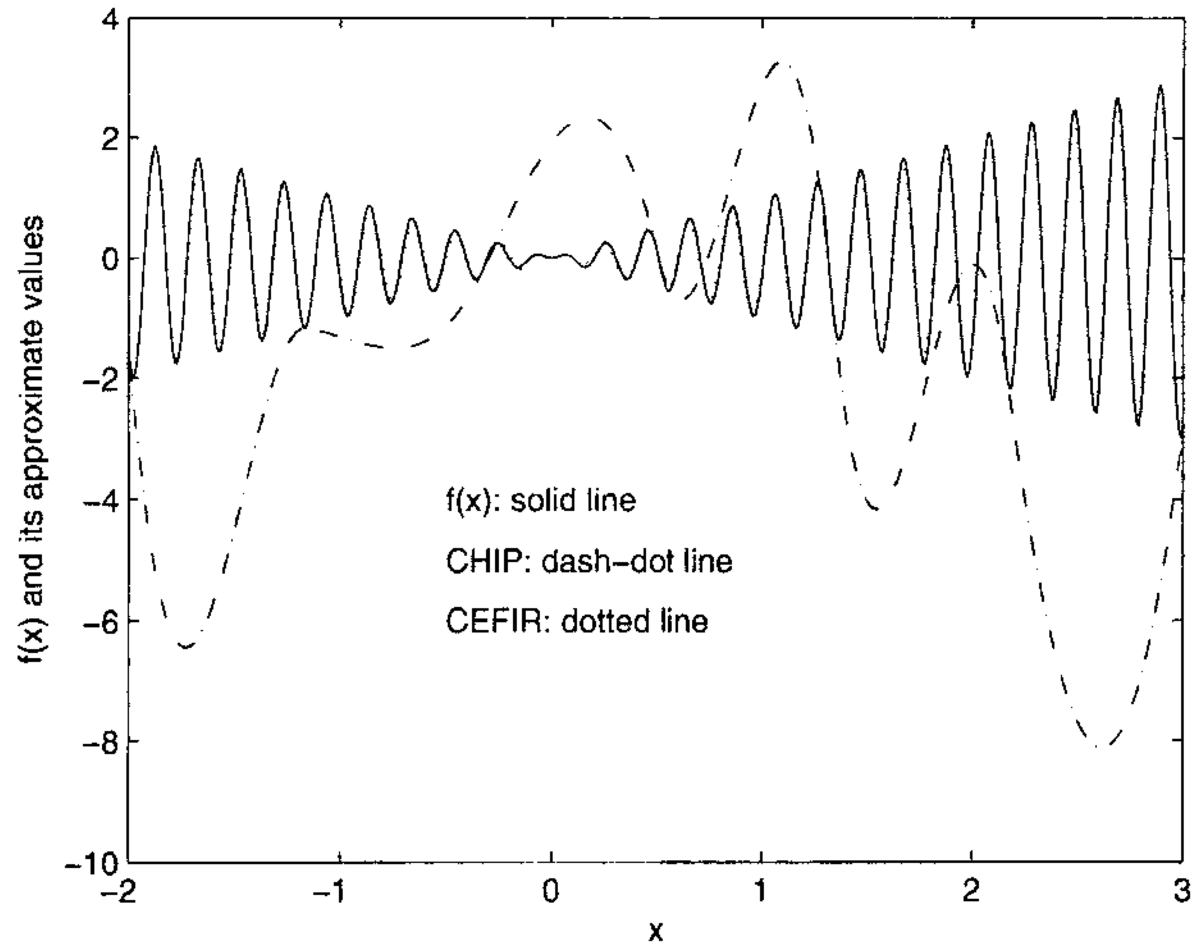
the composite interpolation rule R_c on $[a,b]$ is constructed by applying the rule R given in (2) to each subinterval $[x_{k-1},x_k]$ for $k = 1,2,\ldots,n$. Now the new rule R_c is governed by four coefficients of (11) depending on ω, h_k and t. Likewise, the composite-Hermite-interpolating polynomial H_c on $[a,b]$ is also obtained from repeatedly constructing the basic Hermite-interpolating polynomial H in (13) on each subinterval $[x_{k-1},x_k]$. The two approximate values, H_c and R_c, about the example function (23) on $[-2,3]$ are displayed in Fig. 1 and 2 where the interval $[-2,3]$ is equally subdivided into six subintervals, that is $n=6$ in (24) and $h_k = \frac{5}{12}$ for each k. In Fig. 1 and 2, "$f(x)$" represents the example function (23), "CHIP" means the results obtained from the composite-Hermite-interpolating polynomial H_c and "CEFIR" means the results obtained from the composite-exponential-fitting interpolation rule R_c. Fig. 1 and 2 are involved with the frequency $\omega = 15$ and $\omega = 30$, respectively. All computational results throughout the article are obtained by Matlab [11]. As visually detected in the figures, new rule R_c approximates the example function f more accurately and stably on the whole range of the domain than the classical version H_c. Note that the accuracy of H_c deteriorates severely when compared the behavior of H_c for $\omega = 15$ with the one for $\omega = 30$. As the frequency increases, such a gain in accuracy in the rule R_c is expected to be shown much more obviously because the oscillatory property of the function f becomes stronger than before. The expectation is justified by the fact that the rule R_c is constructed depending on the value of the frequency while the classical version H_c is done in a way irrespective of the frequency. The arguments in Section 3 give a theoretical background to extend the composite-Hermite-interpolating polynomial in the sense that the rule R_c becomes the classical version H_c as the frequency approaches zero. Numerical examples surely show that highly oscillatory functions can be more accurately approximated by the rule R_c than the composite interpolating polynomial H_c. This suggests that the rule R_c are worthy of investigation as practical methods for the area of interpolation as well as computer graphics.

FIGURE 1. $n = 6$ and $\omega = 15$



Let us consider another exponential-fitting interpolation rules using n nodes, denoted by I_n, investigated in [9] and compare them with the composite-exponential-fitting interpolation rule R_c about the example function given in (23) on the domain $[-1, 1]$ for simplicity. That is, for the function of $f(x) = x \sin x \cos \omega x + x \cos x \sin \omega x$ and $\omega = 30$, one is the rule I_7 using seven nodes whose values are $-1, -2/3, -1/3, 0, 1/3, 2/3, 1$ and the other is our new rule, denoted by $R_c^{n=6}$, when $a = -1, b = 1$ and $n = 6$ in (24) with equally spaced subintervals. As a result, I_7 uses the same nodes on the domain

FIGURE 2. $n = 6$ and $\omega = 30$



as $R_c^{n=6}$. Then we have

$$|f - I_7| < 6 \times 10^{-5} \quad \text{and} \quad |f - R_c^{n=6}| < 4 \times 10^{-2}. \tag{26}$$

As seen with the above results, the accuracy of I_7 is better than the $R_c^{n=6}$. But, each rule has its own merit. The condition numbers of the associated matrices involving the rule I_n will increase as the number of selected nodes increases [9]. For example, I_3, I_5 and I_7 are associated with the condition numbers $5 \times 10^4, 2 \times 10^6$ and 3×10^7, respectively. In particular, the error bound of the rule I_7, 6×10^{-5}, in the first inequality of (26) has been obtained after the associated systems are solved with the reduced condition number whose size is about half of 3×10^7. This reduction is done by multiplying the associated system by a diagonal matrix $D = (d_{ij})$ whose diagonal element d_{ii} is the reciprocal of the largest absolute value of elements in each i-row of the associated matrix. Thus, it is not recommended to use I_n with many nodes causing big condition numbers. But, the R_c can select a set of distinct nodes on the domain enough to get as error bound as desired. For example, the composite rule R_c with an error bound of about 6×10^{-5} instead of the error bound 4×10^{-2} in the second inequality of (26) can be obtained if the domain is subdivided into equally spaced subintervals whose number is more than or equal to 48. Furthermore, the condition numbers involving the R_c are less than 25.

On the other hand, it may be possible to extend the R_c with two nodes into the case with more nodes. Its extension is under investigation and requires complicated calculations to establish some properties as shown in (19) or (20).