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CONSTRAINED JACOBI POLYNOMIAL AND CONSTRAINED CHEBYSHEV POLYNOMIAL

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Abstract. In this paper, we present the constrained Jacobi polynomial which is equal to the constrained Chebyshev polynomial up to constant multiplication. For degree $n = 4, 5$, we find the constrained Jacobi polynomial, and for $n \geq 6$, we present the normalized constrained Jacobi polynomial which is similar to the constrained Chebyshev polynomial.

1. Introduction

Degree reduction of Bézier curves is an old problem in CAGD (Computer Aided Geometric Design) or CAD/CAM. In general, degree reduction cannot be done exactly so that it invokes approximation problems. Thus many papers dealing with the problems have been published in the recent thirty years or so. Degree reduction has been developed in a variety of viewpoint, e.g., C^k constrained degree reduction [1, 2, 3, 8, 11, 12, 20], degree reduction using control points [6, 7, 9, 13, 15], degree reduction matrix [14, 17, 18], degree reduction on simplex domain [10, 16], etc.

The constrained Chebyshev polynomial is the error function of the best degree reduction with C^0 constraint at both end points. The author [1] presented the constrained Jacobi polynomial as the error function of the near best degree reduction with C^k constraint at both end points. So, A question follows: Does the constrained Jacobi polynomial which is equal to the constrained Chebyshev polynomial up to constant multiplication exist? In this paper we find the constrained Jacobi polynomial for degree $n = 4, 5$. But unfortunately, we cannot find the constrained Jacobi polynomial for $n \geq 6$. Thus we extend the constrained Jacobi polynomial of degree $n = 5$ to all degree $n \geq 6$. We also find an interpolation of normalization factor at even integer $2n$, so that we finally present the normalized constrained Jacobi polynomial for all degree. By plotting the graph of the polynomial, we can see that they are almost similar to the constrained Chebyshev polynomial.

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2. constrained Jacobi polynomial and constrained Chebyshev polynomial

The Jacobi polynomial defined [4, 5, 19] by

$$(2.1)\quad P_n^{(\alpha,\beta)}(x) = 2^{-n}\sum_{i=0}^{n}\binom{n+\alpha}{n-i}\binom{n+\beta}{i}(x-1)^i(x+1)^{n-i}, \qquad x\in[-1,1]$$

for $\alpha,\beta > -1$, is the well known orthogonal polynomial with respect to the weight $(1-x)^\alpha(1+x)^\beta$ such that

$$\begin{aligned}&\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)\\ &= \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}\delta_{m,n},\end{aligned}$$

where

$$\delta_{m,n} = \begin{cases}1 & (m=n)\\ 0 & (m\neq n)\end{cases}$$

and has the leading coefficient

$$2^{-n}\sum_{i=0}^{n}\binom{n+\alpha}{i}\binom{n+\beta}{n-i} = 2^{-n}\binom{2n+\alpha+\beta}{n}.$$

Note that

$$\begin{aligned}L_n(x) &= P_n^{(0,0)}(x)\\ T_n(x) &= 2^{2n}\binom{2n}{n}^{-1}P_n^{(-1/2,-1/2)}(x)\\ U_n(x) &= 2^{2n}\binom{2n+1}{n+1}^{-1}P_n^{(1/2,1/2)}(x)\end{aligned}$$

are called by *Legendre polynomial*, *Chebyshev polynomial of first kind*, and *Chebyshev polynomial of second kind*, which are the error functions of the best degree reduction with respect to the L_2-, L_∞-, and L_1-norm, respectively. The Chebyshev polynomial

$$T_n(x) = 2^{2n}\binom{2n}{n}^{-1}P_n^{(-1/2,-1/2)}(x) = \cos(n\cos^{-1}x)$$

has the leading coefficient 2^{n-1}. It has zeros at $x=\cos(\frac{2k+1}{2n}\pi)$, $k=0, 1, \ldots, n-1$, and its largest zero is $\cos(\pi/2n)$. The constrained Chebyshev polynomial is well known [11]

$$T_{n,1} = T_n(\cos(\pi/2n)x)$$

which is the error function of the best degree reduction with C^0-continuity at both end points. We define the C^0 *constrained Jacobi polynomial* $J_n^\alpha(x)$ by

$$(2.2)\qquad J_n^\alpha(x) = (x-1)(x+1)P_{n-2}^{(\alpha,\alpha)}(x), \qquad x\in[-1,1]$$

for $n \geq 2$.

Proposition 2.1. *$J_4^\alpha(x) \equiv T_{4,1}(x)$ up to constant multiplication if and only if $\alpha = \sqrt{2}$.*

Proof. Since $\cos(\pi/8) = (3 + 2\sqrt{2})/8$, we have

$$\begin{aligned} T_{4,1}(x) &= 2^3 \cos^4(\frac{\pi}{8})(x-1)(x+1)(x - \frac{\cos\frac{3}{8}\pi}{\cos\frac{\pi}{8}})(x + \frac{\cos\frac{3}{8}\pi}{\cos\frac{\pi}{8}}) \\ &= (3+2\sqrt{2})(x^2-1)(x^2 - (\sqrt{2}-1)^2) \end{aligned}$$

and by Equations (2.1)-(2.2),

$$J_4^\alpha(x) = \frac{(2\alpha+2)(2\alpha+3)}{2^2 \cdot 2!}(x^2-1)(x^2 - \frac{1}{2\alpha+3}).$$

Thus we have $1/(2\alpha+3) = (\sqrt{2}-1)^2$, and $\alpha = \sqrt{2}$ is the solution of $J_4^\alpha(x) \equiv T_{4,1}(x)$ up to constant multiplication. □

Proposition 2.2. *$J_5^\alpha(x) \equiv T_{5,1}(x)$ up to constant multiplication if and only if $\alpha = \frac{3\sqrt{5}-1}{4}$.*

Proof. Since $\cos(\pi/10) = (\sqrt{10+2\sqrt{5}})/4$, we have

$$\begin{aligned} T_{5,1}(x) &= 2^4 \cos^5(\frac{\pi}{10})x(x^2-1)(x^2 - (\frac{\cos\frac{3}{10}\pi}{\cos\frac{\pi}{10}})^2) \\ &= \frac{5\sqrt{50+22\sqrt{5}}}{4}x(x^2-1)(x^2 - \frac{3-\sqrt{5}}{2}) \end{aligned}$$

and

$$J_5^\alpha(x) = \frac{(2\alpha+4)(2\alpha+5)(2\alpha+6)}{2^3 \cdot 3!}x(x^2-1)(x^2 - \frac{3}{2\alpha+5}).$$

Thus we have $\frac{3-\sqrt{5}}{2} = \frac{3}{2\alpha+5}$, and $\alpha = (3\sqrt{5}-1)/4$ is the solution of $J_5^\alpha(x) \equiv T_{5,1}(x)$ up to constant multiplication. □

We call the polynomial $J_{2n}^\alpha(x)/|J_{2n}^\alpha(0)|$ by *normalized constrained Jacobi polynomial* (*NCJP*). But $J_{2n-1}^\alpha(0) = 0$. Thus it is required to find interpolation function of $|J_{2n+2}^\alpha(0)| = |P_{2n}^{(\alpha,\alpha)}(0)|$ at even integer $2n$.

Proposition 2.3. *$\delta_x^\alpha = \frac{\Gamma(x+\alpha+1)}{2^x\Gamma(x/2+1)\Gamma(x/2-\alpha+1)}$, $x > 0$, is an interpolation of $|P_{2n}^{(\alpha,\alpha)}(0)|$ at even integer $x = 2n$.*

Proof. Since

$$\begin{aligned} P_{2n}^{(\alpha,\alpha)}(0) &= 2^{-2n}\sum_{i=0}^{2n}(-1)^i\binom{2n+\alpha}{2n-i}\binom{2n+\alpha}{i} \\ &= \frac{(-1)^n}{2^{2n}}\frac{(n+1+\alpha)\cdots(2n+\alpha)}{n!}, \end{aligned}$$

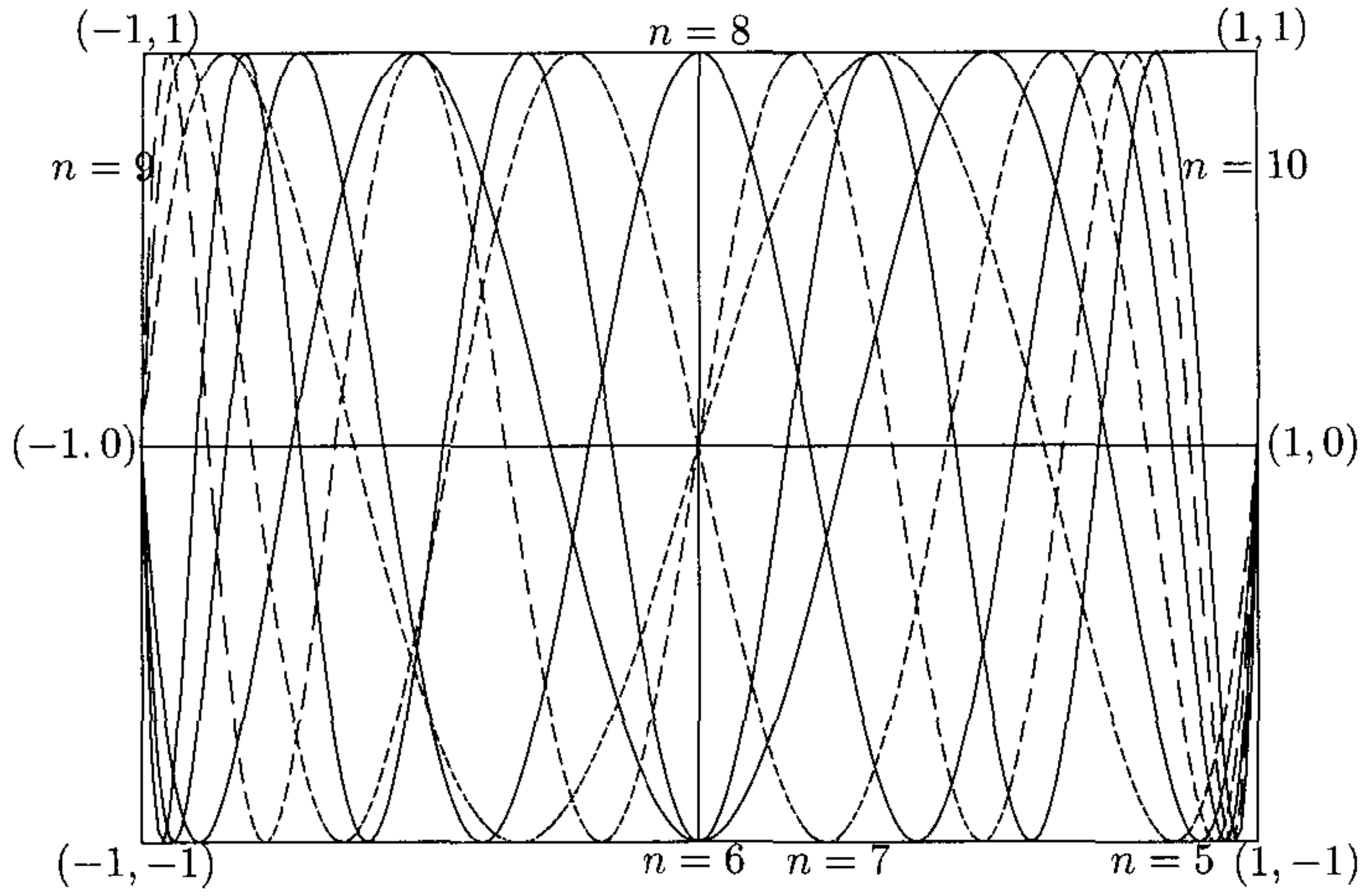

FIGURE 1. The constrained Chebyshev polynomials, $5 \le n \le 10$. They are plotted by dash lines for odd degree n and by solid lines for even n.

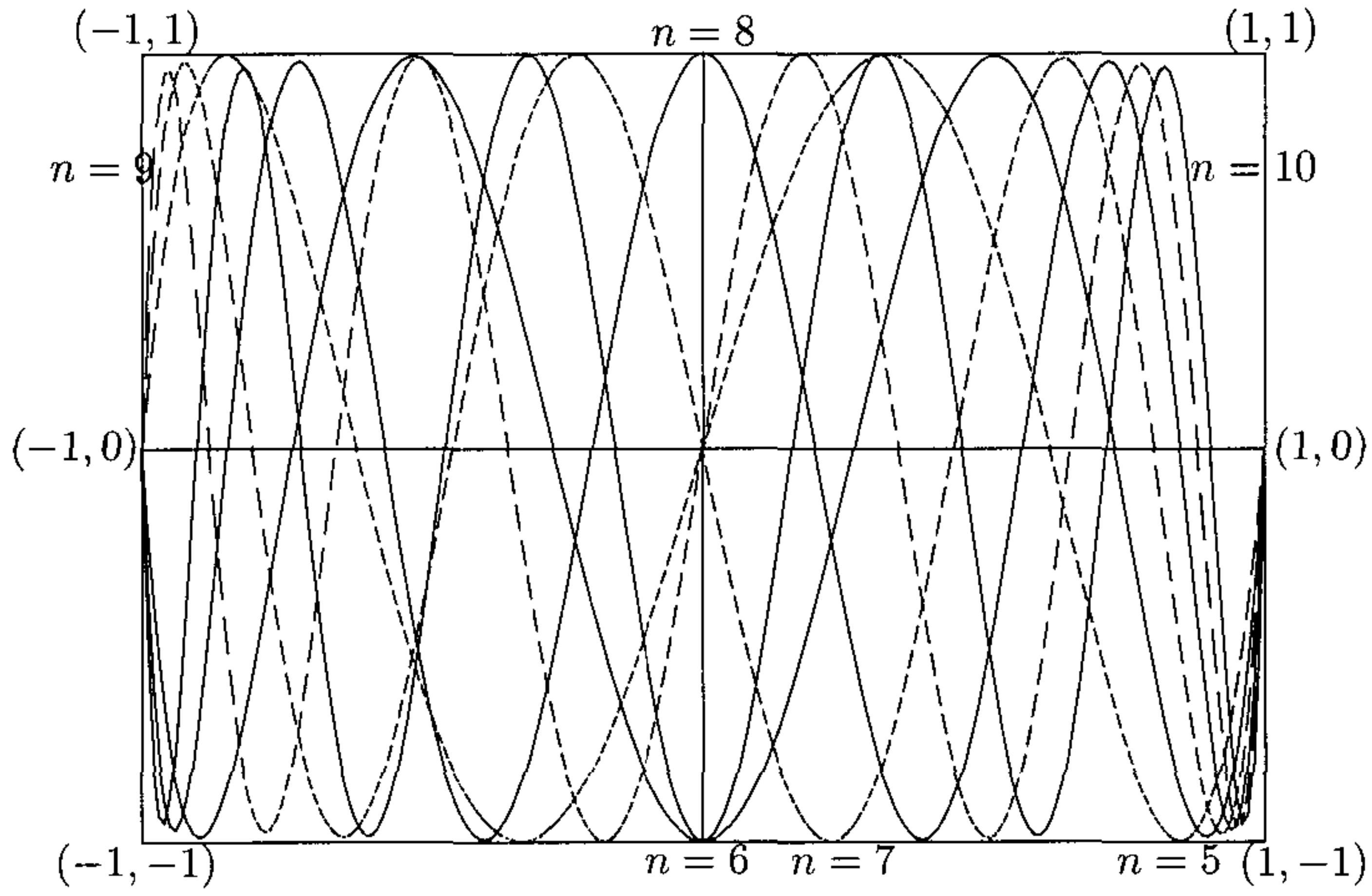

FIGURE 2. The normalized constrained Jacobi polynomials, $5 \le n \le 10$. They are plotted by dash lines for odd degree n and by solid lines for even n.

we have

$$\begin{aligned} \delta_{2n}^{\alpha} &= \frac{\Gamma(2n+\alpha+1)}{2^{2n}\Gamma(n+1)\Gamma(n+\alpha+1)} \\ &= \frac{1}{2^{2n}} \frac{(n+1+\alpha)\cdots(2n+\alpha)}{n!} = |P_{2n}^{(\alpha,\alpha)}(0)|. \end{aligned}$$

Thus the assertion follows. □

We present the polynomial NCJP $J_n^\alpha(x)/\delta_n^\alpha$ for all degree $n \geq 5$ as an error function of degree reduction with C^0 continuity at both end-points. For $\alpha = \frac{3\sqrt{5}-1}{4}$ and $5 \leq n \leq 10$, we can see that they are almost similar to the constrained Chebyshev polynomial, as shown in Figures 1-2.

References

[1] Y. J. Ahn, *Degree reduction of Bézier curves with C^k-continuity using Jacobi polynomials*, Comp. Aided Geom. Desi. **20** (2003), 423–434.

[2] ———, *Degree reduction of Bezier curves using constrained Chebyshev polynomials of second kind*, ANZIAM J. **45** (2003), 195–205.

[3] Y. J. Ahn, B. G. Lee, Y. B. Park, and J. C. Yoo, *Constrained polynomial degree reduction in the L_2-norm equals best weighted Euclidean approximation of Bézier coefficients*, Comp. Aided Geom. Desi. **21** (2004), 181–191.

[4] T. S. Chihara, *An introduction to orthogonal polynomials*, Gordon and Breach, New York, 1978.

[5] P. J. Davis, *Interpolation and approximation*, Dover publications, Ner York, 1975.

[6] M. Eck, *Degree reduction of Bézier curves*, Comp. Aided Geom. Desi. **10** (1993), 237–251.

[7] ———, *Least squares degree reduction of Bézier curves*, Computer-Aided Design **27** (1995), 845–851.

[8] H. J. Kim and Y. J. Ahn, *Good degree reduction of Bézier curves using Jacobi polynomials*, Comp. Math. Appl. **40** (2000), 1205–1215.

[9] H. O. Kim, J. H. Kim, and S. Y. Moon, *Degree reduction of Bézier curves and filter banks*, Comp. Math. Appl. **31** (1996), 23–30.

[10] H. S. Kim and Y. J. Ahn, *Constrained degree reduction of polynomials in Bernstein Bezier form over simplex domain*, Journal of Computational and Applied Mathematics, In Press, 2007.

[11] M. A. Lachance, *Chebyshev economization for parametric surfaces*, Comp. Aided Geom. Desi. **5** (1988), 195–208.

[12] ———, *Approximation by constrained parametric polynomials*, Rocky Mountain J. Math. **21** (1991), 473–488.

[13] B. G. Lee and Y. Park, *Distance for Bézier curves and degree reduction*, Bull. Australian Math. Soc. **56** (1997), 507–515.

[14] B. G. Lee, Y. Park, and J. Yoo, *Application of Legendre-Bernstein basis transformations to degree elevation and degree reduction*, Comp. Aided Geom. Desi. **19** (2002), 709–718.

[15] D. Lutterkort, J. Peters, and U. Reif, *Polynomial degree reduction in the L_2-norm equals best Euclidean approximation of Bézier coefficients*, Comp. Aided Geom. Desi. **16** (1999), 607–612.

[16] J. Peters and U. Reif, *Least squares approximation of Bézier coefficients provides best degree reduction in the L_2-norm*, J. Approximation Theory **104** (2000), 90–97.

[17] H. Sunwoo and N. Lee, *A unified matrix representation for degree reduction of Bezier curves*, Comp. Aided Geom. Desi. **21** (2004), 151–164.

[18] H. Sunwoo, *Matrix representation for multi-degree reduction of Bezier curves*, Comp. Aided Geom. Desi. **22** (2005), 261–273.

[19] G. Szego, *Orthogonal polynomials*, AMS Coll. Publ., 23, Providence.

[20] M. Watkins and A. Worsey, *Degree reduction for Bézier curves*, Computer-Aided Design **20** (1988), 398–405.

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