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# DEFINABLE $C^r$ FIBER BUNDLES AND DEFINABLE $C^rG$ VECTOR BUNDLES

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ABSTRACT. Let $G$ and $K$ be compact subgroups of orthogonal groups and $0 \leq r < \infty$. We prove that every topological fiber bundle over a definable $C^r$ manifold whose structure group is $K$ admits a unique strongly definable $C^r$ fiber bundle structure up to definable $C^r$ fiber bundle isomorphism.

We prove that every $G$ vector bundle over an affine definable $C^rG$ manifold admits a unique strongly definable $C^rG$ vector bundle structure up to definable $C^rG$ vector bundle isomorphism.

## 1. Introduction

J. Bochnak, M. Coste and M. F. Roy proved that every topological vector bundle over a semialgebraic set admits a unique semialgebraic vector bundle structure up to semialgebraic vector bundle isomorphism (12.7.8. [1]). They also proved that any topological vector bundle over an affine Nash manifold admits a unique strongly Nash vector bundle structure up to Nash vector bundle isomorphism (12.7.14. [1]). The proof of 12.7.14. [1] also showed that every semialgebraic vector bundle over an affine Nash manifold admits a unique strongly Nash vector bundle structure up to Nash vector bundle isomorphism. An equivariant version of 12.7.8 [1] is studied in [2] and an equivariant $C^\infty$ version of 12.7.14 [1] is studied in [8].

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ be an $o$-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. The term "definable" means "definable with parameters in $\mathcal{M}$". Many results in semialgebraic geometry over $\mathcal{R}$ hold true in the more general setting of $o$-minimal expansions of $\mathcal{M}$. This theory of $o$-minimal structures has presented a strong interest since A. Wilkie [25] proved that $\mathbf{R}_{\exp} = (\mathbb{R}, <, +, \cdot, \exp)$ is $o$-minimal. See also [5], [7], [19] for other examples and constructions of $o$-minimal expansions of the field of reals. General references on $o$-minimal structures are [4], [6], see also

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[22]. It is known in [21] that there exist uncountably many $o$-minimal expansions of $\mathcal{R}$. Any definable category is a generalization of the semialgebraic category and the definable category on $\mathcal{R}$ coincides with the semialgebraic one,

Let $G$ and $K$ be compact subgroups of orthogonal groups. In this paper $G$ and $K$ denote such groups unless otherwise stated. Then they are compact algebraic groups (p.133 [20]). Hence it is definable. Everything is considered in $\mathcal{M}$, all definable maps are assumed to be continuous and $0 \leq r < \infty$ unless otherwise stated. In this paper we prove that every topological fiber bundle over a definable $C^r$ manifold whose structure group is $K$ admits a unique strongly definable $C^r$ fiber bundle structure up to definable $C^r$ fiber bundle isomorphism. Moreover we prove that every $G$ vector bundle over an affine definable $C^rG$ manifold admits a unique strongly definable $C^rG$ vector bundle structure up to definable $C^rG$ vector bundle isomorphism.

Definable fiber bundles, principal definable fiber bundles, definable $C^r$ manifolds, definable $C^r$ fiber bundles and principal definable $C^r$ fiber bundles are studied in [10], [12], [14], [15].

By the construction (19.6 [24]) of the $n$-universal principal bundle $\mathcal{B}_K = (B_K, p_K, X_K)$ relative to $K$, it is a Nash fiber bundle and $E_K$ and $X_K$ are compact affine Nash manifolds. Let $F$ be an affine definable $C^r$ manifold with an effective definable $C^rK$ action. Then by Proposition 2.7, the associated fiber bundle $\mathcal{B}_K[F] := (E, p, X_K, F, K)$ is a definable $C^r$ fiber bundle. A definable $C^r$ fiber bundle $\eta = (E, p, X, F, K)$ is *strongly definable* if there exist some $\mathcal{B}_K = (B_K, p_K, X_K)$ and a definable $C^r$ map $f : X \to X_K$ such that $\eta$ is definably $C^r$ fiber bundle isomorphic to $f^*(\mathcal{B}_K[F])$. *Strongly definable fiber bundles* are defined similarly.

**Theorem 1.1.** *Let $\eta = (E, p, X, F, K)$ be a strongly definable fiber bundle over a compact definable $C^r$ manifold.*

(1) *There exists a strongly definable $C^r$ fiber bundle $\zeta$ over $X$ such that $\zeta$ is definably fiber bundle isomorphic to $\eta$.*
(2) *If $\zeta'$ is another strongly definable $C^r$ fiber bundle over $X$ such that $\zeta'$ is definably fiber bundle isomorphic to $\eta$, then $\zeta'$ and $\zeta$ are definably $C^r$ fiber bundle isomorphic.*

*In particular,* (1) *and* (2) *say that $\eta$ admits a unique definable $C^r$ fiber bundle structure up to definable $C^r$ fiber bundle isomorphism.*

**Theorem 1.2.** *Let $\eta = (E, p, X, F, K)$ be a fiber bundle over a definable set.*

(1) *There exists a strongly definable fiber bundle $\zeta$ over $X$ such that $\zeta$ is fiber bundle isomorphic to $\eta$.*
(2) *If $\zeta'$ is another strongly definable fiber bundle over $X$ such that $\zeta'$ is fiber bundle isomorphic to $\eta$, then $\zeta'$ and $\zeta$ are definably fiber bundle isomorphic.*

*In particular,* (1) *and* (2) *say that $\eta$ admits a unique definable fiber bundle structure up to definable fiber bundle isomorphism.*

**Corollary 1.3.** *Let $\eta = (E, p, X, F, K)$ be a fiber bundle over a definable $C^r$ manifold.*

(1) *There exists a strongly definable $C^r$ fiber bundle $\zeta$ over $X$ such that $\zeta$ is fiber bundle isomorphic to $\eta$.*
(2) *If $\zeta'$ is another strongly definable $C^r$ fiber bundle over $X$ such that $\zeta'$ is fiber bundle isomorphic to $\eta$, then $\zeta'$ and $\zeta$ are definably $C^r$ fiber bundle isomorphic.*

*In particular, (2) and (3) say that $\eta$ admits a unique definable $C^r$ fiber bundle structure up to definable $C^r$ fiber bundle isomorphism.*

We next consider definable $C^rG$ vector bundle versions of the above results. *Strongly definable $C^rG$ vector bundles* and *strongly definable $G$ vector bundles* are defined similarly (see Definition 3.2).

**Theorem 1.4.** *Let $\eta = (E, p, X)$ be a definable $G$ vector bundle over an affine definable $C^rG$ manifold $X$.*

(1) [1.2 [10]] *$\eta$ is strongly definable.*
(2) *There exists a strongly definable $C^rG$ vector bundle $\zeta$ over $X$ such that $\zeta$ is definably $G$ vector bundle isomorphic to $\eta$.*
(3) *If $\zeta'$ is another strongly definable $C^rG$ vector bundle over $X$ such that $\zeta'$ is definably $G$ vector bundle isomorphic to $\eta$, then $\zeta'$ and $\zeta$ are definably $C^rG$ vector bundle isomorphic.*

*In particular, (2) and (3) say that $\eta$ admits a unique definable $C^rG$ vector bundle structure up to definable $C^rG$ vector bundle isomorphism.*

**Theorem 1.5** (1.3 [10])**.** *Let $X$ be a definable $G$ set. Then every $G$ vector bundle over $X$ admits a unique definable $G$ vector bundle structure up to definable $G$ vector bundle isomorphism.*

We have the following result as a corollary of Theorem 1.4 and 1.5.

**Corollary 1.6.** *Let $\eta = (E, p, X)$ be a $G$ vector bundle over an affine definable $C^rG$ manifold.*

(1) *There exists a strongly definable $C^rG$ vector bundle $\zeta$ over $X$ such that $\zeta$ is $G$ vector bundle isomorphic to $\eta$.*
(2) *If $\zeta'$ is another strongly definable $C^rG$ vector bundle over $X$ such that $\zeta'$ is $G$ vector bundle isomorphic to $\eta$, then $\zeta'$ and $\zeta$ are definably $C^rG$ vector bundle isomorphic.*

*In particular, (1) and (2) say that $\eta$ admits a unique strongly definable $C^rG$ vector bundle structure up to definable $C^r$ vector bundle isomorphism.*

## 2. Definable fiber bundles and definable $C^r$ fiber bundles

A *definable set* means a definable subset of some $\mathbb{R}^n$. A group $G$ is a *definable group* if $G$ is a definable set such that the group operations $G \times G \to G$ and

$G \to G$ are definable. Let $G$ be a definable group. A *definable $G$ action* on a definable set $X$ is a group action $G \times X \to X$ such that it is definable.

A *definable space* is an object obtained by pasting finitely many definable sets together along definable open subsets, and definable maps between definable spaces are defined similarly (see Chapter 10 [4]). Definable spaces are generalizations of semialgebraic spaces in the sense of [3].

Recall the definition of definable fiber bundles [14].

**Definition 2.1.** (1) A topological fiber bundle $\eta = (E, p, X, F, L)$ is called a *definable fiber bundle* over $X$ with fiber $F$ and structure group $L$ if the following two conditions are satisfied:

(a) The total space $E$ is a definable space, the base space $X$ is a definable set, the structure group $L$ is a definable group, the fiber $F$ is a definable set with an effective definable $L$ action, and the projection $p : E \to X$ is a definable map.

(b) There exists a finite family of local trivializations $\{U_i, \phi_i : p^{-1}(U_i) \to U_i \times F\}_i$ of $\eta$ such that each $U_i$ is a definable open subset of $X$, $\{U_i\}_i$ is a finite open covering of $X$. For any $x \in U_i$, let

$$\phi_{i,x} : p^{-1}(x) \to F, \phi_{i,x}(z) = \pi_i \circ \phi_i(z),$$

where $\pi_i$ stands for the projection $U_i \times F \to F$. For any $i$ and $j$ with $U_i \cap U_j \neq \emptyset$, the transition function $\theta_{ij} := \phi_{j,x} \circ \phi_{i,x}^{-1} : U_i \cap U_j \to L$ is a definable map. We call these trivializations *definable*.

Definable fiber bundles with compatible definable local trivializations are identified.

(2) Let $\eta = (E, p, X, F, L)$ and $\zeta = (E', p', X', F, L)$ be definable fiber bundles whose definable local trivializations are $\{U_i, \phi_i\}_i$ and $\{V_j, \psi_j\}_j$, respectively. A definable map $\overline{f} : E \to E'$ is said to be a *definable fiber bundle morphism* if the following two conditions are satisfied:

(a) The map $\overline{f}$ covers a definable map, namely there exists a definable map $f : X \to X'$ such that $f \circ p = p' \circ \overline{f}$.

(b) For any $i, j$ such that $U_i \cap f^{-1}(V_j) \neq \emptyset$ and for any $x \in U_i \cap f^{-1}(V_j)$, the map $f_{ij}(x) := \psi_{j,f(x)} \circ \overline{f} \circ \phi_{i,x}^{-1} : F \to F$ lies in $L$, and $f_{ij} : U_i \cap f^{-1}(V_j) \to L$ is a definable map.

We say that a bijective definable fiber bundle morphism $\overline{f} : E \to E'$ is a *definable fiber bundle equivalence* if it covers a definable homeomorphism $f : X \to X'$ and $(\overline{f})^{-1} : E' \to E$ is a definable fiber bundle morphism covering $f^{-1} : X' \to X$. A definable fiber bundle equivalence $\overline{f} : E \to E'$ is called a *definable fiber bundle isomorphism* if $X = X'$ and $f = id_X$.

(3) A continuous section $s : X \to E$ of a definable fiber bundle $\eta = (E, p, X, F, L)$ is a *definable section* if for any $i$, the map $\phi_i \circ s|U_i : U_i \to U_i \times F$ is a definable map.

(4) We say that a definable fiber bundle $\eta = (E, p, X, F, L)$ is a *principal definable fiber bundle* if $F = L$ and the $L$ action on $F$ is defined by the multiplication of $L$. We write $(E, p, X, L)$ for $(E, p, X, F, L)$.

A *definable $C^r$ manifold* is a $C^r$ manifold with a finite system of charts whose transition functions are definable, and definable $C^r$ maps, definable $C^r$ diffeomorphisms and definable $C^r$ imbeddings are defined similarly ([12], [15]). A definable $C^r$ manifold is *affine* if it is definably $C^r$ imbeddable into some $\mathbb{R}^n$. If $\mathcal{M} = \mathcal{R}$, a definable $C^\omega$ manifold (resp. affine definable $C^\omega$ manifold) is called a *Nash manifold* (resp. an *affine Nash manifold*). By [13], every definable $C^r$ manifold is affine. The definable $C^\omega$ case is complicated. Even if $\mathcal{M} = \mathcal{R}$, it is known that for every compact or compactifiable $C^\omega$ manifold of positive dimension admits a continuum number of distinct nonaffine Nash manifold structures (IV.1.3 [23]), and its equivariant version is proved in [16].

A *definable $C^rG$ action* on a definable $C^r$ manifold $X$ is a group action $G \times X \to X$ such that it is a definable $C^r$ map.

Recall the definition of definable $C^r$ fiber bundles [12].

**Definition 2.2** ([12]). (1) A definable fiber bundle $\eta = (E, p, X, F, L)$ is a *definable $C^r$ fiber bundle* if the total space $E$ and the base space $X$ are definable $C^r$ manifolds, the structure group $L$ is a definable $C^r$ group, the fiber $F$ is a definable $C^rL$ manifold with an effective action, the projection $p$ is a definable $C^r$ map and all transition functions of $\eta$ are definable $C^r$ maps. A *principal definable $C^r$ fiber bundle* is defined similarly.

(2) *Definable $C^r$ fiber bundle morphisms*, *definable $C^r$ fiber bundle equivalences*, *definable $C^r$ fiber bundle isomorphisms* between definable $C^r$ fiber bundles and *definable $C^r$ sections* of a definable $C^r$ fiber bundle are defined similarly.

Let $f : X \to Y$ be a definable map between definable sets. We say that $f$ is *proper* if for any compact subset $C$ of $Y$, $f^{-1}(C)$ is compact.

Let $E$ be an equivalence relation on a definable set $X$. We call $E$ *proper* if $E$ is a definable subset of $X \times X$ and the projection $E \to X$ defined by $(x, y) \mapsto x$ is proper.

**Theorem 2.3** (Definable quotients (e.g. 10.2.15 [4])). *Let $E$ be a proper equivalence relation on a definable set $X$. Then $X/E$ exists as a proper quotient, namely $X/E$ is a definable subset of some $\mathbb{R}^n$ and the projection $X \to X/E$ is a surjective proper definable map.*

The following is a corollary of Theorem 2.3.

**Corollary 2.4** (e.g. 10.2.18 [4]). *Let $X$ be a definable set with a definable $G$ action. Then $X/G$ is a definable subset of some $\mathbb{R}^n$ and the orbit map $p : X \to X/G$ is a surjective proper definable map.*

By a similar proof of 2.10 [17] and Corollary 2.4, we have the following.

**Proposition 2.5.** *Let $(E, p, X, K)$ be a principal definable fiber bundle and $F$ a definable set with an effective definable $K$ action. Then $(E \times_K F, p', X, F, K)$ is a definable fiber bundle, where $p' : E \times_K F \to X$ denotes the projection defined by $p'([z, f]) = p(z)$.*

We have the definable $C^r$ version of Proposition 2.5 similarly.

**Proposition 2.6.** *Let $(E, p, X, K)$ be a principal definable $C^r$ fiber bundle over a definable $C^r$ manifold $X$, $F$ an affine definable $C^r$ manifold with an effective definable $C^r K$ action. Then $(E \times_K F, p', X, F, K)$ is a definable $C^r$ fiber bundle, where $p' : E \times_K F \to X$ denotes the projection defined by $p'([z, f]) = p(z)$.*

As a corollary of Proposition 2.6, we have the following proposition.

**Proposition 2.7.** *Let $\mathcal{B}_K = (B_K, p_K, X_K)$ be the $n$-universal principal bundle relative to $K$, $F$ an affine definable $C^r$ manifold with an effective definable $C^r K$ action. Then the associated fiber bundle $\mathcal{B}_K[F] := (E, p, X_K, F, K)$ is a definable $C^r$ fiber bundle.*

**Definition 2.8.** A definable $C^r$ fiber bundle $(E, p, X, F, K)$ over a definable $C^r$ manifold $X$ is *strongly definable* if there exists a definable $C^r$ map $f : X \to X_K$ such that $\eta$ is definably $C^r$ fiber bundle isomorphic to $f^*(\mathcal{B}_K[F])$. *Strongly definable fiber bundles* are defined similarly.

Strongly definable $C^r$ fiber bundles have the following extension property.

**Proposition 2.9.** *Let $\eta = (E, p, X, F, K)$ be a definable $C^r$ fiber bundle over a definable $C^r$ manifold $X$ in $\mathbb{R}^n$. Then $\eta$ is strongly definable if and only if there exists a strongly definable $C^r$ fiber bundle $\eta'$ over a definable $C^r$ manifold $Y$ such that $X \subset Y$ and $\eta$ is definably $C^r$ fiber bundle isomorphic to $\eta'|X$.*

To prove the above proposition, we need existence of a Nash tubular neighborhood of an affine Nash manifold in $\mathbb{R}^l$. For later use, we state a definable $C^r$ version of it.

**Proposition 2.10** ([9], [11], [15]). *Let $0 \le r \le \omega$. Then every definable $C^r$ submanifold $X$ of $\mathbb{R}^l$ has a definable $C^r$ tubular neighborhood $(U, \theta)$ of $X$ in $\mathbb{R}^l$, namely $U$ is a definable open neighborhood of $X$ in $\mathbb{R}^l$ and $\theta : U \to X$ is a definable $C^r$ map with $\theta|X = id_X$.*

*Proof of Proposition 2.9.* Assume that $\eta$ is strongly definable. By the hypothesis, there exists a definable $C^r$ map $f : X \to X_K$ such that $\eta$ is definably $C^r$ fiber bundle isomorphic to $f^*(\mathcal{B}_K[F])$. We may assume that $X_K$ is a definable $C^r$ submanifold of $\mathbb{R}^p$ and $j = (i, f) : X \to \mathbb{R}^{n+p}$ is a definable $C^r$ imbedding, where $i : X \to \mathbb{R}^n$ denotes the inclusion. Identifying $X$ with $j(X)$, we may suppose that $f$ is the restriction to $X$ of the canonical projection $\pi : \mathbb{R}^{n+p} \to \mathbb{R}^p$. By Proposition 2.10, we have a definable open neighborhood $A$ of $X_K$ in $\mathbb{R}^p$ with a definable $C^r$ retraction $\theta : A \to X_K$. Thus we obtain a definable $C^r$ manifold $U := \pi^{-1}(A)$ in $\mathbb{R}^{n+p}$ and a definable $C^r$ map

$F := \theta \circ (\pi|U) : U \to X_K$ such that $F|X = f$ and $F^*(\mathcal{B}_K[F])$ is a strongly definable $C^r$ fiber bundle over $U$. □

**Proposition 2.11.** *Let $X_1, X_2$ be definable $C^r$ manifolds, $\eta_1, \eta_2$ principal definable $C^r$ fiber bundles over $X_1, X_2$ whose structure groups are $K$, respectively. Let $f_1 : X_1 \to X_2$ be a definable $C^r$ map.*

(1) *There exists a definable $C^r$ fiber bundle $(\eta_1, \eta_2, f)$ over $X_1$ such that its definable $C^r$ sections are in bijective correspondence with the definable $C^r$ fiber bundle morphisms $\eta_1 \to \eta_2$ covering $f$.*

(2) *If $\eta_1, \eta_2$ are strongly definable, then $(\eta_1, \eta_2, f)$ is strongly definable.*

*Proof.* (1) Let $\{U_i\}$ and $\{V_h\}$ be coordinate neighborhoods of $\eta_1$ and $\eta_2$, respectively, $\{g^1_{ij} : U_i \cap U_j \to K\}$ and $\{g^2_{hk} : V_h \cap V_k \to K\}$ cocycles of $\eta_1$ and $\eta_2$, respectively. Then its cocycle of $f^*(\eta_2)$ is $\{g^*_{hk} := g^2_{hk} \circ f : f^{-1}(V_h) \cap f^{-1}(V_k) \to K\}$. The definable open sets $\{W_a := U_i \cap f^{-1}(V_h)\}$ $(a = (i, h))$ is a refinement of $\{U_i\} \cup \{f^{-1}(V_h)\}$.

By construction, $(\eta_1, \eta_2, f)$ has $K$ as fiber and $K \times K$ as structure group, acting on $K$, $(k_1, k_2)k = k_1 k k_2^{-1}$. A family of coordinate neighborhoods such a bundle is given by $\{W_a\}$ and a cocycle by $\{g^1_{ab} := g^1_{ij}|W_a \cap W_b \times g^*_{ab} := g^*_{hk}|W_a \cap W_b \to K \times K\}, a = (i, h), b = (j, k)$.

Suppose that there exists a definable $C^r$ section $s$ of $(\eta_1, \eta_2, f)$. Then there exists a family $\{c_a : W_a \to K\}$ of definable $C^r$ maps such that $c_a(x) = (g_{ab}(x), g^*_{ab}(x))c_b(x), x \in W_a \cap W_b$. Namely, $g^*_{ab}(x) = c_a^{-1}(x) g_{ab}(x) c_b(x)$. This means that $\eta_1$ and $f^*(\eta_2)$ are definable $C^r$ fiber bundle equivalent. Then composing this equivalence with the canonical bundle map $f^*(\eta_2) \to \eta_2$, we have a definable $C^r$ fiber bundle morphism $F : \eta_1 \to \eta_2$ covering $f$.

Conversely let $F$ be a definable $C^r$ fiber bundle morphism covering $f$. There exists a family $\{c_a : W_a \to K\}$ of definable $C^r$ maps such that $g^*_{ab}(x) = c_a^{-1}(x) g_{ab}(x) c_b(x)$. Hence $c_a(x) = (g_{ab}(x), g^*_{ab}(x))c_b(x), x \in W_a \cap W_b$ holds. This implies that $(\eta_1, \eta_2, f)$ has a definable $C^r$ section $s$. Since the functions $s \mapsto F$ and $F \mapsto s$ are inverse to each other, (1) is proved.

(2) By assumption, there exists a definable $C^r$ map $h_l : X_l \to X_K, (l = 1, 2)$ such that $\eta_l$ is definably $C^r$ fiber bundle isomorphic to $h_l^*(\mathcal{B}_K)$. Let $\{A_m\}$ be a family of coordinate neighborhoods of $X_K$ and $\{g'_{mn}\}$ the relative cocycle. Then a cocycle of $\eta_l$ is $\{g^l_{mn} := g'_{mn} \circ h_l\}$ on the definable open covering $\{h_l^{-1}(A_m)\}$ of $X_l$. Consider the definable $C^r$ fiber bundle $\mathcal{B}_K \times \mathcal{B}_K = (B_K \times B_K, p_K \times p_K, X_K \times X_K)$. It is universal and a cocycle on $\{A_m, \times A_m\}$ is given by $\{(g_{mn}, g_{m'n'}) : (A_m \times A_{m'}) \cap (A_n \times A_{n'}) \to K \times K\}, (g_{mn}, g_{m'n'})(y, y') = (g_{mn}(y), g_{m'n'}(y'))$. If we consider the fiber bundle $(\mathcal{B}_K \times \mathcal{B}_K)[K]$ associated with $\mathcal{B}_K \times \mathcal{B}_K$ and with fiber $K$ (the group $K \times K$ acting on $K$ in the previous manner), then $(\eta_1, \eta_2, f)$ is induced from it under the map $H = (h_1, h_2 \circ f) : X_1 \to X_K \times X_K$. □

**Proposition 2.12.** *Let $\eta = (E, p, X, F, K)$ be a strongly definable $C^r$ fiber bundle over a definable $C^r$ manifold $X$ in $\mathbb{R}^m$ whose fiber $F$ is a definable*

*$C^r$ manifold in $\mathbb{R}^n$. If $\eta$ has a continuous section, then it has a definable $C^r$ section.*

To prove Proposition 2.12, we need the following definable approximation theorem.

**Theorem 2.13** ([11]). *If $0 \leq s < r < \infty$, then every definable $C^s$ map between definable $C^r$ manifolds is approximated in the definable $C^s$ topology by definable $C^r$ maps.*

*Proof of Proposition 2.12.* By the hypothesis, there exists a definable $C^r$ map $f : X \to X_K$ such that $\eta$ is definably $C^r$ fiber bundle isomorphic to $f^*(\mathcal{B}_K[F])$. Moreover we can suppose that $X_K$ is imbedded into $\mathbb{R}^p$.

Consider the definable $C^r$ imbedding $\psi : X \to \mathbb{R}^{m+p}$ defined by $\psi = (id_X, f)$. Identifying $X$ with $\psi(X)$, $f$ is identified with the restriction to $X$ of the canonical projection $\pi : \mathbb{R}^{m+p} \to \mathbb{R}^p$. Let $A'$ be a definable open tubular neighborhood of $X_K$ in $\mathbb{R}^p$ and $\theta : A' \to X_K$ a definable $C^r$ retraction. Thus $A := \pi^{-1}(A')$ is a definable open neighborhood of $X$ in $\mathbb{R}^{m+p}$ and we have a definable $C^r$ map $H := \theta \circ (\pi|A)$. Consider the definable $C^r$ fiber bundle $H^*(\mathcal{B}_K[F])$ over $A$. Then its restriction to $X$ is definably $C^r$ fiber bundle isomorphic to $\eta$ and hence we can identify these two bundles. A section $s$ can be identified with a section of $H^*(\mathcal{B}_K([F])|X$.

The next step is to extend $s$ to a section defined on a definable open neighborhood $U$ with $X \subset U \subset A$. Let $\{U_j\}_{j=1}^l$ be a finite definable open cover of $X$ given by definable open sets of $A$ such that the closure $\overline{U_j}$ of each $U_j$ is contained in some coordinate neighborhood $V_h$ of $H^*(\mathcal{B}_K[F])$. Assume that $\overline{U_1} \subset V_h$. Let $\phi_h : V_h \times F \to p^{-1}(V_h)$ be a coordinate function and $p_h : p^{-1}(V_h) \to F, p_h(b) = q \circ \phi_h^{-1}(b)$, where $q : V_h \times F \to F$ denotes the projection. Consider the function $p_h \circ s|\overline{U_1} \cap X : \overline{U_1} \cap X \to F \subset \mathbb{R}^n$. Let $W$ be a definable $C^r$ tubular neighborhood $F$ of $\mathbb{R}^n$ and $\theta' : W \to F$ a definable $C^r$ retraction.

Extend $p_h \circ s$ to a map $s_1' : \overline{U_1} \to \mathbb{R}^n$. Shrinking $U_1$, if necessary, we may assume that $s_1'(\overline{U_1}) \subset W$. Thus we can define a local section of $H^*(\mathcal{B}_K[F])$ on $\overline{U_1} \cup X$ by setting

$$s_1(x) = \begin{cases} s(x), & x \in X, \\ \phi_h(x, \theta' \circ s_1'(x)), & x \in \overline{U_1}. \end{cases}$$

Continuing this process, we have a sequence $\{s_j\}$ of local sections such that $X \subset U := \cup_{j=1}^l U_j$, $s_i$ is defined on $X \cup (\cup_{j=1}^i U_j)$ and $s_i|X \cup (\cup_{j=1}^{i-1} U_j) = s_{i-1}$, where $s_0 = s$. Then $\overline{s} := s_l$ is a section defined on $U$ extending $s$.

Consider now the strongly definable $C^r$ fiber bundle $H^*(\mathcal{B}_K[F])|U$, which has a continuous section $\overline{s}$. Using Theorem 2.13, a similar proof of III.2.3 [23] provides a definable $C^r$ section. □

*Proof of Theorem 1.1.* (1) Since $\eta$ is strongly definable, there exist the $n$-universal bundle $\mathcal{B}_K$ and a definable map $f : X \to X_K$ such that $f^*(\mathcal{B}_K[F])$ is definably fiber bundle isomorphic to $\eta$. By Theorem 2.13, we have a definable $C^r$ map $h : X \to X_K$ as an approximation of $f$. In particular $h$ is definably homotopic to $f$. Since $X$ is compact and by [14], $\zeta := h^*(\mathcal{B}_K[F])$ is definably fiber bundle isomorphic to $f^*(\mathcal{B}_K[F])$ and $\zeta$ is a strongly definable $C^r$ fiber bundle.

(2) Let $\zeta'$ be another strongly definable $C^r$ fiber bundle over $X$ such that $\zeta'$ is definably fiber bundle isomorphic to $\eta$. Consider the strongly definable $C^r$ fiber bundle $(\zeta, \zeta', id_X)$ whose sections represent the fiber bundle isomorphisms between $\zeta$ and $\zeta'$. It has a continuous section and by Proposition 2.12, it admits a definable $C^r$ section. This section gives a definable $C^r$ fiber bundle isomorphism between $\zeta$ and $\zeta'$. □

**Theorem 2.14** (3.3 [9]). *Let $X$ be a definable set. Then there exist a compact definable subset $Y$ of $X$ and a definable map $\rho : X \to Y$ with $\rho|Y = id_Y$.*

*Proof of Theorem 1.2.* (1) By Theorem 2.14, there exist a compact definable subset $Y$ of $X$ and a definable retraction $\rho : X \to Y$. By this fact and by [18], the sets of fiber bundle isomorphism classes of fiber bundles over $X$ and $Y$ coincide. Thus we may assume that $X$ is compact at the beginning.

By 19.3 [24], there exist the $n$-universal bundle $\mathcal{B}_K$ and a continuous map $f : X \to X_K$ such that $n > \dim X$ and $\eta$ is fiber bundle isomorphic to $f^*(\mathcal{B}_K[F])$.

Since $X$ is compact, applying the polynomial approximation theorem, we have a polynomial map $h' : X \to \mathbb{R}^n$, where $X_K \subset \mathbb{R}^n$. If this approximation is sufficiently close, $h(X)$ is contained in a some definable open neighborhood $(U, \phi)$ of $X_K$ in $\mathbb{R}^n$. Thus we obtain a definable map $h : X \to X_K$ approximating $f$. In particular $h$ is homotopic to $f$. Thus $\eta$ is fiber bundle isomorphic to a strongly definable fiber bundle $h^*(\mathcal{B}_K[F])$.

(2) A similar proof of Theorem 1.1 proves (2). □

*Proof of Corollary 1.3.* (1) By Theorem 1.2 (1), there exists a strongly definable fiber bundle $\zeta$ which is fiber bundle isomorphic to $\eta$. Namely there exists a definable map $f : X \to X_K$ such that $\eta$ is fiber bundle isomorphic to $f^*(\mathcal{B}_K[F])$. By Theorem 2.13, we have a definable $C^r$ map $h : X \to X_K$ as an approximation of $f$. In particular, $h$ is homotopic to $f$. By [18], $h^*(\mathcal{B}_K[F])$ is fiber bundle isomorphic to $f^*(\mathcal{B}_K[F])$. Thus $\eta$ is fiber bundle isomorphic to a strongly definable fiber bundle $\zeta := h^*(\mathcal{B}_K[F])$.

(2) A similar proof of Theorem 1.1 proves (2). □

## 3. Definable $G$ vector bundles and definable $C^rG$ vector bundles

A *representation* of $G$ means a group homomorphism from $G$ to some $O_n(\mathbb{R})$ which is a definable $C^r$ map and the *representation space* of this representation is $\mathbb{R}^n$ with the linear action induced from the representation. In this paper, we always assume that every representation is orthogonal. A *definable $G$ set* is a $G$ invariant definable subset of a representation space of $G$. A *definable $C^rG$ submanifold* of a representation space $\Omega$ of $G$ is a $G$ invariant definable

$C^r$ submanifold of $\Omega$. We say that a definable $C^rG$ manifold is *affine* if it is definably $C^rG$ diffeomorphic (definably $G$ homeomorphic if $r = 0$) to a definable $C^rG$ submanifold of some representation space of $G$.

Recall universal $G$ vector bundles.

**Definition 3.1** ([12]). Let $\Omega$ be an $n$-dimensional representation space of $G$ induced by a definable $C^r$ group homomorphism $B : G \to O_n(\mathbb{R})$. Suppose that $M(\Omega)$ denotes the vector space of $n \times n$-matrices with the action $(g, A) \in G \times M(\Omega) \to B(g)AB(g)^{-1} \in M(\Omega)$. For any positive integer $k$, we define the vector bundle $\gamma(\Omega, k) = (E(\Omega, k), u, G(\Omega, k))$ as follows:

$$G(\Omega, k) = \{A \in M(\Omega) | A^2 = A, A = A', TrA = k\},$$

$$E(\Omega, k) = \{(A, v) \in G(\Omega, k) \times \Omega | Av = v\},$$

$$u : E(\Omega, k) \to G(\Omega, k), u((A, v)) = A,$$

where $A'$ denotes the transposed matrix of $A$ and $Tr\ A$ stands for the trace of $A$. Then $\gamma(\Omega, k)$ is an algebraic vector bundle. Since the action on $\gamma(\Omega, k)$ is algebraic, it is an algebraic $G$ vector bundle. We call it *the universal $G$ vector bundle associated with $\Omega$ and $k$*. Remark that $G(\Omega, k) \subset M(\Omega)$ and $E(\Omega, k) \subset M(\Omega) \times \Omega$ are nonsingular algebraic $G$ sets.

**Definition 3.2.** A definable $C^rG$ vector bundle $\eta$ over an affine definable $C^rG$ manifold is *strongly definable* if there exist a representation space $\Omega$ of $G$ and a definable $C^rG$ map $f : X \to G(\Omega, k)$ such that $\eta$ is definably $C^rG$ vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where $k$ denotes the rank of $\eta$. Similarly a *strongly definable $G$ vector bundle* over a definable $G$ set is defined.

To prove Theorem 1.4, we need the following equivariant version of Theorem 2.13.

**Theorem 3.3** ([11]). *If $0 \leq s < r < \infty$, then every definable $C^sG$ map between affine definable $C^rG$ manifolds is approximated in the definable $C^s$ topology by definable $C^rG$ maps.*

*Proof of Theorem 1.4.* (2) By (1), there exist a representation space $\Omega$ of $G$ and a definable $G$ map $f : X \to G(\Omega, k)$ such that $\eta$ is definably $G$ vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where $k$ denotes the rank of $\eta$. By Theorem 3.3, we have a definable $C^rG$ map $h : X \to G(\Omega, k)$ as an approximation of $f$. In particular, $f$ and $h$ are $G$ homotopic. By [18], $f^*(\gamma(\Omega, k))$ is $G$ vector bundle isomorphic to $h^*(\gamma(\Omega, k))$. Thus by Theorem 1.5, they are definably $G$ vector bundle isomorphic. Therefore $\eta$ is definably $G$ vector bundle isomorphic to a strongly definable $C^rG$ vector bundle $\zeta := h^*(\gamma(\Omega, k))$.

(3) By a way similar to the proof of 3.1 [8], $\mathrm{Hom}(\zeta, \zeta')$ is a strongly definable $C^rG$ vector bundle. Since $\zeta$ and $\zeta'$ are definably $G$ vector bundle isomorphic, it gives a definable $G$ section $s$ of $\mathrm{Hom}(\zeta, \zeta')$. Using Theorem 3.3, by a similar proof of 3.3 [8] proves that $s$ is approximated by a definable $C^rG$ section $s'$. On the other hand, $Iso(\zeta, \zeta')$ is open in $\mathrm{Hom}(\zeta, \zeta')$. If this approximation

is sufficiently close, then $s'$ gives a definable $C^rG$ vector bundle isomorphism between $\zeta$ and $\zeta'$. □

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