

STABLE STRICTLY CONVEX BODIES

KYEONGHEE JO

ABSTRACT. In this article, we find a counter example which shows that the well-known necessary condition about the boundary regularity of a stable strictly convex body is not a sufficient condition.

1. Introduction

A convex body W is called *stable* if $\{W\}$ is closed in the quotient space of the space of convex bodies under the action of projective transformations, i.e., if W is projectively equivalent to any convex body W' which is a limit of a sequence $\{g_n W\}$ for some $\{g_n\} \subset \text{PGL}(n+1, \mathbb{R})$.

Based on Benzécri's results in [2], we can prove that every stable strictly convex body has a special boundary (Theorem 8.1): (i) it is at least C^1 , (ii) either it is conic or it fails to be twice differentiable on a dense subset, (iii) if it is not conic, then it is twice differentiable a.e. and furthermore the Hessian is degenerate everywhere it exists.

So it is natural to ask if or not this boundary condition is also a sufficient condition for a strictly convex domain to be stable. Actually it is not true and we'll see a counter example in the last section.

2. Spaces of convex bodies in $P(\mathbb{R}^{n+1})$

Throughout this paper we'll denote the n -dimensional real projective space by $P(\mathbb{R}^{n+1})$ and the projectivization of the group of all $(n+1)$ by $(n+1)$ matrices by $\text{PM}(n+1, \mathbb{R})$. We say that $W \subset P(\mathbb{R}^{n+1})$ is a *convex body* if there is an $(n-1)$ -dimensional projective subspace P_∞ such that W is a compact convex subset with non-empty interior in the affine space $P(\mathbb{R}^{n+1}) \setminus P_\infty$.

Let $\mathcal{C}(n)$ denote the set of all convex bodies in $P(\mathbb{R}^{n+1})$, with the topology induced from the Hausdorff metric on the set of all closed subsets of $P(\mathbb{R}^{n+1})$. Let

$$\mathcal{C}_*(n) = \{(W, x) \in \mathcal{C}(n) \times P(\mathbb{R}^{n+1}) \mid x \in \text{int}W\}$$

be the corresponding set of pointed convex bodies, with a topology induced from the product topology on $\mathcal{C}(n) \times P(\mathbb{R}^{n+1})$. $\text{PGL}(n+1, \mathbb{R})$ acts continuously

Received November 14, 2007.

2000 *Mathematics Subject Classification.* 51N15, 52A05.

Key words and phrases. stable, strictly convex, convex body.

on $\mathcal{C}(n)$ and $\mathcal{C}_*(n)$. Recall that an action of a group Γ on a space X is *syndetic* if there exists a compact subset $K \subset X$ such that $\Gamma K = X$.

Theorem 2.1 ([2]). $\text{PGL}(n+1, \mathbb{R})$ acts properly and syndetically on $\mathcal{C}_*(n)$. In particular, the quotient $\mathcal{L}_*(n) = \mathcal{C}_*(n)/\text{PGL}(n+1, \mathbb{R})$ is a compact Hausdorff space.

While the quotient $\mathcal{C}_*(n)/\text{PGL}(n+1, \mathbb{R})$ is Hausdorff, the space of equivalence classes of convex bodies $\mathcal{L}(n) = \mathcal{C}(n)/\text{PGL}(n+1, \mathbb{R})$ is not Hausdorff.^a Actually, $\mathcal{L}(n)$ is not a T_1 -space. The following example shows that.

Example 2.2. Consider the convex body

$$W = \{(x, y) \in \mathbb{R}^2 \mid -5 \leq x \leq 5, x^2 \leq y \leq 25\}$$

and a diagonal matrix $g_i = \text{diag}(2^i, 4^i)$ for each positive integer i . Then $g_i W$ converges to the ellipse W_0 in $\text{P}(\mathbb{R}^3)$, which is bounded by the equation $x^2 = yz$. But W and W_0 are not projectively equivalent, so the equivalence class $[W]$ of W is not a closed point in $\mathcal{L}(2)$.

Definition 2.3. Let W be a convex body in $\text{P}(\mathbb{R}^{n+1})$.

- (1) W is *stable* if $\overline{\{[W]\}} = \{[W]\}$, that is, $\{[W]\}$ is closed in $\mathcal{L}(n)$: if $g_n W$ converges to a convex body $W' \subset \text{P}(\mathbb{R}^{n+1})$ for some $\{g_n\} \subset \text{PGL}(n+1, \mathbb{R})$, then W is projectively equivalent to W' .
- (2) W is *minimal* if $\overline{\{[W']\}} = \overline{\{[W]\}}$ for any $[W'] \in \overline{\{[W]\}}$,
- (3) A proper face F of W is a *maximal face* if there is no proper face of W whose closure contains F .

Definition 2.4.

- (1) f is a q -dimensional section of $a \in \mathcal{L}(n)$ if $W_f \in \mathcal{C}(q)$ is a q -dimensional section of $W_a \in \mathcal{C}(n)$ for $f = \overline{[W_f]}$ and $a = \overline{[W_a]}$.
- (2) For $A \subset \mathcal{L}(n)$, we define

$$S_q(A) = \{f \in \mathcal{L}(p) \mid f \text{ is a } q\text{-dimensional section of } a \in A\}.$$

Proposition 2.5 ([2]). $S_q(\overline{A}) = \overline{S_q(A)}$ for any $A \subset \mathcal{L}(n)$. So if W is a stable convex body, then $S_q([W]) = \overline{S_q([W])}$.

3. The action of projective transformations on convex bodies

Since $\text{PM}(n+1, \mathbb{R})$ is a compactification of $\text{PGL}(n+1, \mathbb{R})$, any infinite sequence of projective transformations contains a convergent subsequence. For a singular projective transformation g we will denote the projectivization of the kernel space and the range space of g by $\text{Ker}(g)$ and $\text{Im}(g)$. Then g maps $\text{P}(\mathbb{R}^{n+1}) \setminus \text{Ker}(g)$ onto $\text{Im}(g)$. We will need later the following well-known basic facts.

^aSome notations and statements related to the space of convex bodies are quoted from Goldman's Lecture notes [4]. More precisely, 15 lines including Theorem 2.1 right before the mark are almost the same as in C.15 and C.16 of [4].

Proposition 3.1 ([2]). *Let W be a convex body in $P(\mathbb{R}^{n+1})$ and $h_i \in PGL(n+1, \mathbb{R})$ a sequence of projective transformations converging to a singular projective transformation h . Then for any compact neighborhood U of $W \cap \text{Ker}(h) \neq \emptyset$,*

- (1) $h_i(W)$ converges if and only if $h_i(W \cap U)$ converges,
- (2) $\lim_{i \rightarrow \infty} h_i(W \cap U) = \lim_{i \rightarrow \infty} h_i(W)$ if exists.

Lemma 3.2 ([2]). *Let $\{W_{1i}\}$ (respectively, $\{W_{2i}\}$) be a sequence of convex bodies in $P(\mathbb{R}^{n+1})$ which converges to a convex body W_1^0 (respectively, W_2^0); Let $\{g_i\} \in PGL(n+1, \mathbb{R})$ be a sequence of projective transformations converging to a singular projective transformation g such that $g_i(W_{1i}) = W_{2i}$. Then*

- (1) *there is a face F of W_1^0 such that $\text{Ker}(g)$ contains the support of F and $\text{Ker}(g) \cap \bar{\Omega} = \text{Ker}(g) \cap \partial\Omega = \bar{F}$,*
- (2) $\text{Im}(g) \cap W_2^0 = \text{Im}(g) \cap \partial W_2^0$,
- (3) $\text{Im}(g)$ is the support of $\text{Im}(g) \cap W_2^0$.

Here the support of a subset A means the projective space generated by A .

Using the above two facts we can easily prove the following useful proposition, which was proved first by Benzécri:

Proposition 3.3 ([2]). *Let W and W' be convex bodies in $P(\mathbb{R}^{n+1})$ such that $[W'] \neq [W]$ and $[W'] \in \overline{\{[W]\}}$. Then we can find a face F of W such that $[W'] \in \overline{\{[W \cap U]\}}$ for any closed convex neighborhood U of \bar{F} .*

Proof. Since $[W'] \in \overline{\{[W]\}}$, there is a sequence of projective transformations $\{g_i\}$ such that $g_i(W)$ converges to W' . We may assume that $\{g_i\}$ converges to a projective transformation g by taking a subsequence if necessary. The projective transformation g cannot be non-singular because $[W'] \neq [W]$. By Lemma 3.2, there is a face F of W such that $\text{Ker}(g) \cap \partial W = \bar{F}$. Now we can apply Proposition 3.1 and conclude that $\lim_{i \rightarrow \infty} g_i(W \cap U) = \lim_{i \rightarrow \infty} g_i(W) = W'$ for any closed convex neighborhood U of \bar{F} . □

The converse of Proposition 3.3 holds for a maximal face:

Proposition 3.4. *Let W be a convex body in $P(\mathbb{R}^{n+1})$ and F a maximal face of W . Then we can find a convex body W' such that $[W'] \in \overline{\{[W]\}}$ and $[W'] \in \overline{\{[W \cap U]\}}$ for every closed convex neighborhood U of the face F .*

Proof. Choose a point $z \in F$ and a sequence $\{z_n\}$ of points in $\text{int}W$ converging to z . Then $[W, z_n]$ has a convergent subsequence $[W, z_{n_k}]$ in $\mathcal{L}_*(n)$ and thus there is a sequence $\{g_n\}$ of projective transformations such that $g_{n_k}(W, z_{n_k})$ converges to $(W', z') \in \mathcal{C}_*(n)$. We may assume g_{n_k} converges to g .

If g is non-singular, then $g(W) = W'$ and $g(z) = \lim g_{n_k}(z_{n_k}) = z'$ must be in $\partial W'$ because $z \in \partial W$. But this contradicts $(W', z') \in \mathcal{C}_*(n)$, so we conclude that the projective transformation g is singular.

Now we show that z is in $\text{Ker}(g)$: Suppose $z \notin \text{Ker}(g)$. Then $g(z) = \lim g_{n_k}(z_{n_k}) = z'$ and thus $z' \in \text{Im}(g)$, which contradicts the fact (2) of Lemma 3.2 : $\text{Im}(g) \cap \text{int}W = \emptyset$.

By (1) of Lemma 3.2, $\text{Ker}(g)$ must contain F and $\text{Ker}(g) \cap W$ is the closure of a face containing \overline{F} , which implies $\text{Ker}(g) \cap W = \overline{F}$ because F is maximal.

By Proposition 3.1, for every closed convex neighborhood U of the face F_z ,

$$W' = \lim g_{n_k}(W) = \lim g_{n_k}(W \cap U),$$

which completes the proof. □

4. Conic faces

Definition 4.1. A k -dimensional face F of an n -dimensional convex body W is called *conic* if there exist $n - k$ supporting hyperplanes H_1, H_2, \dots, H_{n-k} such that

$$H_1 \supseteq H_1 \cap H_2 \supseteq \dots \supseteq H_1 \cap \dots \cap H_{n-k} = \langle F \rangle.$$

Remark 4.2. By definition, every $(n - 1)$ -dimensional face is conic.

The simplest convex bodies with conic faces are convex sums of lower dimensional convex bodies. Here a convex body W is called a *convex sum* of its closed faces F_1 and F_2 , which is denoted by $W = F_1 \dot{+} F_2$, if it is the convex hull of $F_1 \cup F_2$ when we consider W as a bounded set in an affine space $\mathbb{A}^n \subset \mathbb{P}(\mathbb{R}^{n+1})$, i.e., it is the union of all closed line segments joining points in F_1 to points in F_2 in \mathbb{A}^n . Note that if the dimensions of F_1, F_2 and Ω are k_1, k_2 and n respectively, then $n = k_1 + k_2 + 1$.

Theorem 4.3 ([2]). *Let W be a convex body in $\mathbb{P}(\mathbb{R}^{n+1})$ and F be a conic face of W . Then there exist a projective subspace L of $\mathbb{P}(\mathbb{R}^{n+1})$ and a sequence of projective transformations $\{h_i\}$ such that $\{h_i W\}$ converges to $F \dot{+} B$ for some convex body B in L .*

Proof. Let N be the support of F and choose a projective subspace L such that

$$\mathbb{P}(\mathbb{R}^{n+1}) = N + L \text{ and } L \cap W = \emptyset.$$

Fix $f \in F$ and let B be the set of all points b in L such that the line connecting b and f intersects W , that is,

$$B = \{b \in L \mid \overleftrightarrow{bf} \cap W \neq \emptyset\}.$$

Since F is a conic face, B cannot have a complete line and thus it must be a convex body in L . Now consider a projective transformation h which acts trivially on N and as an expansion $2I$ on L :

$$h = \begin{pmatrix} I & O \\ O & 2I \end{pmatrix}$$

Then the sequence of convex bodies $h^n(W)$ converges to $F \dot{+} B$. □

5. Osculating ellipsoids

Definition 5.1. Let W be a convex body and Q an ellipsoid in $P(\mathbb{R}^{n+1})$. We say that W has an *osculating ellipsoid* Q at $p \in \partial W$ if there exists a suitable affine chart with an origin p such that the local boundary equation on some neighborhood of $p = (0, \dots, 0)$ is expressed by $x_n = f(x_1, \dots, x_{n-1})$ satisfying

$$\lim_{(x_1, \dots, x_{n-1}) \rightarrow (0, 0, \dots, 0)} \frac{f(x_1, \dots, x_{n-1})}{x_1^2 + \dots + x_{n-1}^2} = 1$$

and $Q = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n > x_1^2 + \dots + x_{n-1}^2\}$ in this affine chart. We say that W has an *osculating ellipsoid* at p when there is such a boundary point p .

Theorem 5.2 ([2]). *If a convex body W has an osculating ellipsoid Q , then there exists a sequence $\{g_n\} \subset PGL(n + 1, \mathbb{R})$ such that $g_n W$ converges to \overline{Q} , the closure of Q in $P(\mathbb{R}^{n+1})$.*

Proof. If we consider the sequence $h_i = \text{diag}(\lambda_i, \dots, \lambda_i, \lambda_i^2)$ of linear transformations, we can prove that $h_i(W)$ converges to \overline{Q} whenever $\lim_{i \rightarrow \infty} \frac{1}{\lambda_i} = 0$. \square

Lemma 5.3. *Let W be a convex body and f a local defining function of W near a boundary point $z = (z_0, f(z_0))$. If f is twice differentiable at z_0 and $\text{Hess}f(z_0)$ is not degenerate, then W has an osculating ellipsoid at z .*

Proof. By Theorem 5 of [7], the Hessian of a convex function f is a symmetric and positive semi-definite wherever it exists, even in the case that the second differential of f is not of class C^2 or does not exist in a full neighborhood of the point considered. So f has a second order Taylor expansion at z_0 and the Hessian must be positive definite, which implies that W has an osculating ellipsoid at z . \square

We say that a convex body W is called *strictly convex* if ∂W contains no line segment. For such domains we get the following from the above corollary.

Corollary 5.4. *Let W be a strictly convex body and f a local defining function of W . If f is twice differentiable, then W has an osculating ellipsoid.*

Proof. We can find a point $z = (z_0, f(z_0))$ in the boundary of W such that $\text{Hess}f(z_0)$ is not degenerate (see [5]), which implies that W has an osculating ellipsoid at z by Lemma 5.3. \square

6. Stable convex bodies

In this section, we look at some immediate consequences about stable convex bodies from the results in the previous sections. First, we get the following from Theorem 4.3 and Proposition 2.5.

Corollary 6.1. (1) *Let W be a stable convex body in $P(\mathbb{R}^{n+1})$ and L a linear subspace of $P(\mathbb{R}^{n+1})$ such that $L \cap W$ is a r -dimensional section and has a*

conic face F . Then there exists a section which is projectively equivalent to a r -dimensional convex body $F \dot{+} B$ for some suitable convex body B of dimension $r - (\dim(F) + 1)$.

(2) Any stable convex body which is not strictly convex has a triangle as a 2-dimensional section.

(3) Let F be a k -dimensional face of a stable convex body Ω . Then Ω has a $(k + 1)$ -dimensional section S which is projectively equivalent to the convex sum of F and a point.

The definition of stability and Theorem 5.2 immediately imply the following.

Corollary 6.2. *Let W be a stable convex body. If W has an osculating ellipsoid, then W is an ellipsoid.*

As corollaries of Proposition 3.4, we get

Corollary 6.3. *Let W be a stable convex body in $\mathbb{P}(\mathbb{R}^{n+1})$. Then for every maximal face F of W , we can find a sequence of projective transformations $\{g_i\}$ such that $W = \lim_{i \rightarrow \infty} g_i(W \cap U)$ for every closed convex neighborhood U of the face F .*

Proof. From Proposition 3.4, we know there is a convex body $[W'] \in \overline{\{[W]\}}$ such that $W' \in \overline{\{[W \cap U]\}}$ for every closed convex neighborhood U of the face F . But the stability of W implies $\overline{\{[W]\}} = \{[W]\}$ and thus $[W'] = [W]$. \square

If W is strictly convex, every boundary point is itself a maximal face of W , then we get

Corollary 6.4. *Let W be a stable strictly convex body in $\mathbb{P}(\mathbb{R}^{n+1})$. Then for every $z \in \partial W$ we can find a sequence of projective transformations $\{g_i\}$ such that $W = \lim_{i \rightarrow \infty} g_i(W \cap U_z)$ for every closed convex neighborhood U_z of z .*

7. 2-dimensional stable convex bodies

Benzécri tried to classify all minimal convex bodies in $\mathbb{P}(\mathbb{R}^3)$. Even though he made a wrong classification (Proposition 11 in §5.3 of [2]), we can immediately get a classification of 2-dimensional stable convex bodies from his idea. But it is not clear if every minimal convex body is stable in $\mathbb{P}(\mathbb{R}^3)$ or not.

Proposition 7.1. *Let W be a convex body in $\mathbb{P}(\mathbb{R}^3)$ and f be a local defining function of W near a boundary point $z = (z_0, f(z_0))$. If f is twice differentiable at z_0 and $f''(z_0) > 0$, then W has an osculating ellipse at z .*

Proof. We may assume that $z = 0$, $f(z) = 0$, $f'(z) = 0$ and $f''(z) = 2$ by normalizing. Then we get the following inequality

$$\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = \lim_{x \rightarrow 0} \frac{f'(x)}{2x} = \lim_{x \rightarrow 0} \frac{f''(x)}{2} = 1,$$

which implies that W has an osculating ellipse at z . \square

Theorem 7.2. *Let W be a stable convex body in $P(\mathbb{R}^3)$. Then W is either a triangle or a strictly convex body with C^1 boundary. Furthermore*

- (1) *if ∂W has a singular point, that is, it is not differentiable at some point, then W is a triangle,*
- (2) *if the boundary is twice differentiable, then W is an ellipse,*
- (3) *if W is neither an ellipse nor a triangle, then W is a strictly convex body and*
 - (i) *the boundary curve is of class C^1 ,*
 - (ii) *the boundary curve fails to be twice differentiable on a dense subset,*
 - (iii) *the boundary curve is twice differentiable a.e.,*
 - (iv) *the boundary curve has a vanishing second derivative if exists.*

Proof. (1) Let z be a singular point. Then z is a conic face of W and thus there is a 1-dimensional face F of W such that $W = \{z\} + F$ by Corollary 6.1, which proves that W is a triangle.

(2) If ∂W is twice differentiable, then we can find a point $p \in \partial W$ having non-zero 2nd derivative and thus W has an osculating ellipse by Proposition 7.1. From the stability of W we can conclude that W must be an ellipse (Corollary 6.2).

(3) If W has a 1-dimensional face F in the boundary, then F is a conic face and thus W must be a triangle. So W is strictly convex, if W is not a triangle. By (1) the boundary is of class C^1 if W is not a triangle, since a differentiable convex function is continuously differentiable. This proves (i). If the boundary is twice differentiable on some open subset U of ∂W , then W must be an ellipse by Corollary 5.4 and 6.2. So every open subset of the boundary must have a point where ∂W is not twice differentiable since W is not an ellipse, which proves (ii). The properties (iii) is obvious because every convex differentiable function is twice differentiable a.e. The properties (iv) is trivially proved by Proposition 7.1 because W is not ellipse.

□

8. Stable strictly convex bodies

In the previous section, we saw the necessary condition for boundary smoothness of 2-dimensional stable strictly convex bodies. For higher dimensional cases, we can prove the same result.

Theorem 8.1. *Let W be a stable strictly convex body in $P(\mathbb{R}^{n+1})$. Then*

- (1) *its boundary is at least C^1 ,*
- (2) *either it is conic or the boundary fails to be twice differentiable on a dense subset,*

- (3) *if it is not conic, its boundary is twice differentiable a.e. and furthermore the Hessian is degenerate everywhere the boundary is twice differentiable.*

Proof.

- (1) If the boundary has a singular point, then W has a triangular section by Corollary 6.1, which contradicts the strict convexity of W .
- (2) If the boundary is twice differentiable on some open subset U of ∂W , then W has an osculating ellipsoid by Corollary 5.4. From the stability of W we can conclude that W must be an ellipsoid.
- (3) Since every C^1 convex function is twice differentiable a.e., the boundary of W is twice differentiable a.e. But if W is not an ellipsoid, then the Hessian must be degenerate wherever it exists by Corollary 5.3 and Corollary 6.2.

□

9. Examples of non-stable strictly convex bodies

There are infinitely many non-conic stable strictly convex bodies as we can see in [3, 6, 9]. In this section, we will show the smoothness condition in Theorem 8.1 is not a sufficient condition for a convex body to be a stable domain.

Choose any two stable non-conic strictly convex bodies W_1 and W_2 in $P(\mathbb{R}^3)$ which are not projectively equivalent to each other, that is, there is no projective transformation g satisfying $gW_1 = W_2$. By applying projective transformation to one of them, we may assume that there are two points $\{p, q\}$ in $\partial W_1 \cap \partial W_2$ where the tangent lines to W_1 and W_2 are the same:

Lemma 9.1. *Let W_1 and W_2 be strictly convex bodies in $P(\mathbb{R}^3)$. Then there is a projective transformation h such that there are two points $\{p, q\}$ in $\partial h(W_1) \cap \partial W_2$ where the tangent lines to $h(W_1)$ and W_2 are the same.*

Proof. We want to construct h .

Choose two points $p_1 \in \partial W_1$ and $p_2 \in \partial W_2$. If l_1 and l_2 are tangent lines at p_1 and p_2 , we can find a projective transformation g such that

$$g(l_1) = l_2 \text{ and } g(p_1) = p_2.$$

Both $g(W_1)$ and W_2 have exactly one point $q_i \in \partial W_i$ where the tangent line is parallel to l_2 . Put l'_1 and l'_2 the tangent lines at q_1 and q_2

Consider the affine space $E = P(\mathbb{R}^3) \setminus l_2$ with an origin q_2 such that y -axis is the line $\overleftrightarrow{p_2 q_2}$ and x -axis is the tangent line l'_2 at q_2 . If q_1 is expressed by (x, y) in this affine chart and τ_1 is the affine transformation which is a translation by $(0, -y)$, then $\tau_1(l'_1) = l'_2$. So $\tau_1 g(W_1)$ and W_2 have the same tangent lines

$$l_2 = g(l_1) = \tau_1 g(l_1) \text{ and } l'_2 = \tau_1(l'_1).$$

Now consider the affine transformation τ_2 which is a translation by $(-x, 0)$. Then

$$\tau_2\tau_1(q_1) = q_2 \text{ and } \tau_2(l'_2) = l'_2.$$

So $\tau_2\tau_1g(W_1)$ and W_2 meet at p_2 and q_2 , and have the same tangent lines

$$l_2 = g(l_1) = \tau_1g(l_1) = \tau_2\tau_1g(l_1)$$

and

$$l'_2 = \tau_2(l'_2) = \tau_2\tau_1(l'_1).$$

By taking $h := \tau_2\tau_1g$, $p := p_2$, and $q := q_2$, we get our desired result. \square

The line segment l connecting p and q divide each W_i into two convex bodies W_{i1} and W_{i2} with $W_i = W_{i1} \cup W_{i2} = l$ for each $i = 1, 2$. Let

$$W = W_{11} \cup W_{22}.$$

Proposition 9.2. *W is a strictly convex body whose boundary satisfies:*

- (1) ∂W is C^1 ,
- (2) ∂W fails to be twice differentiable on a dense subset,
- (3) ∂W is twice differentiable a.e. and its second derivative vanishes if exists.

But W is not stable.

Proof. The first statement is immediate from the construction of W . To prove that W is not stable, we will show

$$\{[W_1], [W_2]\} \subset \overline{\{[W]\}}.$$

Choose z_i in the boundary of W such that z_i is included in the component of $\partial W \setminus \{p, q\}$ which is a part of ∂W_i . Choose closed convex neighborhoods U_1 and U_2 of z_1 and z_2 such that $U_i \cap W$ is contained entirely in W_{ii} .

By Corollary 6.4 and the stability of W_1 and W_2 , we can find a sequence of projective transformations $\{g_{in}\}$ such that

$$W_i = \lim_{n \rightarrow \infty} g_{in}(W_i \cap U_i) \text{ for } i = 1, 2.$$

We may assume that g_{in} converges to g_i by taking a subsequence if necessary. Obviously g_i is a singular projective transformation for each $i = 1, 2$. The kernel $Ker(g_i)$ of g_i must be the tangent line at $\{z_i\}$, because $W_i = \lim_{n \rightarrow \infty} g_{in}(W_i \cap V_i)$ for every closed convex neighborhood V_i of z_i , especially for $V_i \subset U_i$. So $W_i = \lim_{n \rightarrow \infty} g_{in}(W)$ by Proposition 3.1 and this means $[W_i] \in \overline{\{[W]\}}$. \square

10. More questions

A domain Ω in $P(\mathbb{R}^{n+1})$ is called *quasi-homogeneous* if there is a compact subset $K \subset \text{int}(\Omega)$ and a subgroup G of $\text{Aut}_{\text{proj}}(\Omega)$ such that $GK = \Omega$, that is, G acts on Ω syndetically. Quasi-homogeneous domains are very special among all projective domains.

Consider the following diagram:

$$\begin{array}{ccc} \mathcal{C}_*(n) & \xrightarrow{\rho_{\mathcal{C}}} & \mathcal{C}(n) \\ \downarrow \pi^* & & \downarrow \pi \\ \mathcal{L}_*(n) & \xrightarrow{\rho_{\mathcal{L}}} & \mathcal{L}(n) \end{array}$$

Note that all maps are quotient maps. By the quasi-homogeneity of Ω , there is a compact set $K \subset \Omega$ such that for all $x \in \Omega$ there is $g \in \text{Aut}_{\text{proj}}(\Omega)$ with $gx \in K$. So

$$\rho_{\mathcal{L}}^{-1}(\pi(\overline{\Omega})) = \pi^*({(\overline{\Omega}, k) \mid k \in K}).$$

Since the set $\{(\overline{\Omega}, k) \mid k \in K\}$ is compact in $\mathcal{C}_*(n)$, $\pi^*({(\overline{\Omega}, k) \mid k \in K})$ is compact in $\mathcal{L}_*(n)$, which implies that $\pi^*({(\overline{\Omega}, k) \mid k \in K})$ is closed in $\mathcal{L}_*(n)$ because $\mathcal{L}_*(n)$ is a Hausdorff space, which implies that $\pi(\overline{\Omega})$ is closed in $\mathcal{L}(n)$. This is Benzécri's way of proving the fact: *the closure of any quasi-homogeneous properly convex domain $\Omega \subset P(\mathbb{R}^{n+1})$ is a stable convex body.*

We can ask if the converse of the above statement is true or not, that is, does every stable convex body have a quasi-homogeneous interior? This question was proposed first by Benzécri in his paper [2]. Here are some of his questions:

- (i) Which convex bodies are stable(minimal, quasi-homogeneous)?
- (ii) Is there any stable convex body which is not quasi-homogeneous?
- (iii) Is there any minimal convex body which is not stable?

References

- [1] Y. Benoist, *Convex divisibles II*, Duke Math. J. **120** (2003), 97–120.
- [2] J. P. Benzécri, *Sur les variétés localement affines et projectives*, Bull. Soc. Math. France **88** (1960), 229–332.
- [3] W. M. Goldman, *Geometric structures on manifolds and varieties of representations*, Contemp. Math. **74** (1988), 169–198.
- [4] ———, *Projective Geometry on Manifolds*, Lecture Notes (v.0.3)-Mathematics 748B Spring (1988).
- [5] K. Jo, *Quasi-homogeneous domains and convex affine manifolds*, Topology Appl. **134** (2003), no 2. 123–146.
- [6] J. L. Koszul, *Deformation des connexions localement plats*, Ann. Inst. Fourier **18** (1968), 103–114.
- [7] R. T. Rockafellar, *Second derivatives in convex analysis*, Nonlinear analysis and convex analysis: Proceedings of the international conference on Niigata, Japan, 28-31 July (1998), 3–7.
- [8] J. Vey, *Sur les automorphismes affines des ouverts convexes saillants*, Ann. Scuola Norm. Sup. Pisa **24**(3) (1970), 641–665.
- [9] E. B. Vinberg and V. G. Kats, *Quasi-homogeneous cones*, Math. Notes **1** (1967), 231–235.(translated from Math. Zametki **1** (1967), 347–354.)

DIVISION OF LIBERAL ARTS AND SCIENCES
MOKPO NATIONAL MARITIME UNIVERSITY
MOKPO 530-729, KOREA
E-mail address: khjo@mmu.ac.kr