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SOME RECIPROCAL RELATIONS BETWEEN THE g-UNIFIED AND $^*g$-UNIFIED FIELD TENSORS

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Abstract. In n-dimensional unified field theory(n-UFT), the reciprocal representations between the g-unified field tensor $g_{\lambda\nu}$ and $^*g$-unified field tensor $^*g^{\lambda\nu}$ play essential role in the study of n-UFT. The purpose of the present paper is to obtain some reciprocal relations between g-unified field tensor and $^*g$-unified field tensor.

1. Introduction

Einstein ([5]) proposed a new unified field theory that would include both gravitation and electromagnetism. Characterizing Einstein's unified field theory as a set of geometrical postulates in a 4-dimensional generalized Riemannian space X_4 (i.e., space-time), Hlavatý ([6]) gave the mathematical foundation of the 4-dimensional unified field theory (4-g-UFT) defined by the g-unified field tensor $g_{\lambda\nu}$ for the first time. Generalizing X_4 to the n-dimensional generalized Riemannian manifold X_n, n-dimensional generalization of this theory, the so-called *Einstein's n-dimensional g-unified field theory* (n-g-UFT), had been obtained by a number of mathematicians. Corresponding to n-g-UFT, *n-dimensional $^*g$-unified field theory* (n-$^*g$-UFT) defined by the $^*g$-unified field tensor $^*g^{\lambda\nu}$ such that

$$g_{\lambda\mu}\,{}^*g^{\lambda\nu} = g_{\mu\lambda}\,{}^*g^{\nu\lambda} = \delta_\mu^\nu$$

was introduced recently, and many consequences of this theory have been obtained. In particular, n-$^*g$-UFT is more useful for physical applications than the usual n-g-UFT. The purpose of the present paper is to obtain some reciprocal relations between g-unified field tensor and $^*g$-unified field tensor. The obtained results and discussions in the present paper will be useful for the n-dimensional considerations of the unified field theory.

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2. Preliminary

This section is a brief collection of basic concepts, notations, and results, which are needed in our further considerations in the present paper.

Let X_n be an n-dimensional generalized Riemannian manifold covered by a system of real coordinate neighborhoods $\{U;\, x^\nu\}$, where, here and in the sequel, Greek indices run over the range $\{1, 2, \ldots, n\}$ and follow the summation convention. We assume that on X_n there are two real non-symmetric tensors $g_{\lambda\mu}$ satisfying $\det(g_{\lambda\mu}) \neq 0$ and ${}^*g^{\lambda\nu}$ satisfying the reciprocal relation

$$g_{\lambda\mu}\, {}^*g^{\lambda\nu} = g_{\mu\lambda}\, {}^*g^{\nu\lambda} = \delta_\mu^\nu. \tag{2.1}$$

The tensors $g_{\lambda\mu}$ and ${}^*g^{\lambda\nu}$ will play an essential role in all our subsequent considerations, and for that reason they will be called the *g-unified field tensor* and **g-unified field tensor*, respectively. The g-unified field tensor $g_{\lambda\mu}$ may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \tag{2.2}$$

where we assume that $\det(h_{\lambda\mu}) \neq 0$. Therefore we may define a unique tensor $h^{\lambda\nu} (= h^{\nu\lambda})$ by

$$h_{\lambda\mu}\, h^{\lambda\nu} = \delta_\mu^\nu. \tag{2.3}$$

On the other hand, the ${}^*g$-unified field tensor ${}^*g^{\lambda\nu}$ may be also split into its symmetric part ${}^*h^{\lambda\nu}$ and skew-symmetric part ${}^*k^{\lambda\nu}$:

$${}^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu}, \tag{2.4}$$

where we also assume that $\det({}^*h^{\lambda\nu}) \neq 0$. Therefore we may also define a unique tensor ${}^*h_{\lambda\mu} (= {}^*h_{\mu\lambda})$ by

$${}^*h_{\lambda\mu}\, {}^*h^{\lambda\nu} = \delta_\mu^\nu. \tag{2.5}$$

In the Einstein's usual n-dimensional g-unified field theory (n-g-UFT), the algebraic structure on X_n is imposed by the g-unified field tensor $g_{\lambda\mu}$, and *we use the tensors $h^{\lambda\nu}$ and $h_{\lambda\mu}$ as tensors for raising and/or lowering indices for all tensors defined in n-g-UFT in the usual manner.* Then we may define new tensors by

$$g^{\lambda\mu} = g_{\alpha\beta}\, h^{\lambda\alpha}\, h^{\mu\beta}, \quad k^{\lambda\mu} = k_{\alpha\beta}\, h^{\lambda\alpha}\, h^{\mu\beta}, \quad k_\lambda{}^\nu = k_{\lambda\mu}\, h^{\mu\nu}, \tag{2.6}$$

so that in virtue of (2.2) and (2.3), we obtain

$$g^{\lambda\mu} = h^{\lambda\mu} + k^{\lambda\mu}. \tag{2.7}$$

In n-g-UFT, the differential geometric structure on X_n is imposed by the tensor $g_{\lambda\mu}$ by means of a connection $\Gamma_{\lambda\mu}^\nu$ defined by the Einstein's equations:

$$\partial_\omega\, g_{\lambda\mu} - g_{\alpha\mu}\, \Gamma_{\lambda\omega}^\alpha - g_{\lambda\alpha}\, \Gamma_{\omega\mu}^\alpha = 0 \qquad (\partial_\nu = \frac{\partial}{\partial x^\nu}), \tag{2.8}$$

or equivalently

$$D_\omega\, g_{\lambda\mu} = 2 S_{\omega\mu}{}^\alpha\, g_{\lambda\alpha}, \tag{2.9}$$

where D_ω denotes the symbolic vector of the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$, and $S_{\lambda\mu}{}^\nu$ is the torsion tensor of $\Gamma^\nu_{\lambda\mu}$.

In n-g-UFT, the following quantities are frequently used, where p =1,2,3, ... :

$$(2.10)\quad \begin{aligned}
&(a)\ G = \det(g_{\lambda\mu}),\quad H = \det(h_{\lambda\mu}),\quad T = \det(k_{\lambda\mu}),\\
&(b)\ \tilde{g} = \frac{G}{H},\quad \tilde{k} = \frac{T}{H},\\
&(c)\ K_0 = 1,\qquad K_p = k_{[\alpha_1}{}^{\alpha_1} k_{\alpha_2}{}^{\alpha_2} \dots k_{\alpha_p]}{}^{\alpha_p},\\
&(d)\ {}^{(0)}k_\lambda{}^\nu = \delta^\nu_\lambda,\quad {}^{(p)}k_\lambda{}^\nu = k_\lambda{}^\alpha\, {}^{(p-1)}k_\alpha{}^\nu = {}^{(p-1)}k_\lambda{}^\alpha\, k_\alpha{}^\nu,\\
&(e)\ \overline{K}_0 = 1,\qquad \overline{K}_p = \sum_{x=0}^{p} K_x.
\end{aligned}$$

It should be remarked that the tensor ${}^{(p)}k_{\lambda\nu}$ is symmetric if p is even, and skew-symmetric if p is odd.

On the other hand, in the n-dimensional $^*g$-unified field theory(n-$^*g$-UFT), the role of the unified field tensor is no longer played by $g_{\lambda\mu}$. In n-$^*g$-UFT, the algebraic structure on the same space X_n is imposed by the $^*g$-unified field tensor $^*g^{\lambda\nu}$, and *we use the tensors* $^*h^{\lambda\nu}$ *and* $^*h_{\lambda\mu}$ *as tensors for raising and/or lowering indices for all tensors defined in* n-$^*g$-UFT *in the usual manner.* Then we may also define new tensors by

$$(2.11)\quad {}^*g_{\lambda\mu} = {}^*g^{\alpha\beta}\, {}^*h_{\lambda\alpha}\, {}^*h_{\mu\beta},\quad {}^*k_{\lambda\mu} = {}^*k^{\alpha\beta}\, {}^*h_{\lambda\alpha}\, {}^*h_{\mu\beta},\quad {}^*k_\lambda{}^\nu = {}^*k^{\alpha\nu}\, {}^*h_{\alpha\lambda},$$

so that in virtue of (2.4) and (2.5), we also obtain

$$(2.12)\qquad {}^*g_{\lambda\mu} = {}^*h_{\lambda\mu} + {}^*k_{\lambda\mu}.$$

In n-$^*g$-UFT, the following quantities are also used, where $p = 1, 2, 3, \dots$:

$$(2.13)\quad \begin{aligned}
&(a)\ {}^*G = \det({}^*g_{\lambda\mu}),\quad {}^*H = \det({}^*h_{\lambda\mu}),\quad {}^*T = \det({}^*k_{\lambda\mu}),\\
&(b)\ {}^*\tilde{g} = \frac{{}^*G}{{}^*H},\quad {}^*\tilde{k} = \frac{{}^*T}{{}^*H},\\
&(c)\ {}^{(0)*}k_\lambda{}^\nu = \delta^\nu_\lambda,\quad {}^{(p)*}k_\lambda{}^\nu = {}^*k_\lambda{}^\alpha\, {}^{(p-1)*}k_\alpha{}^\nu = {}^{(p-1)*}k_\lambda{}^\alpha\, {}^*k_\alpha{}^\nu.
\end{aligned}$$

It should be also remarked that the tensor ${}^{(p)*}k_{\lambda\nu}$ is symmetric if p is even, and skew-symmetric if p is odd.

3. The representation of the tensor $^*g_{\lambda\mu}$ in n-g-UFT

In this section, we obtain the generalized representation of the tensor $^*g_{\lambda\mu}$ in terms of the g-unified field tensor $g_{\lambda\mu}$ in n-g-UFT.

Lemma 3.1. *For every integer $p \geq 1$, the tensor ${}^{(p)}k_{\lambda\mu}$ satisfies the following relations in X_n:*

$$(3.1)\quad \begin{aligned} &(a)\ \ {}^{(p)}k_{\lambda\mu}\,{}^*g^{\lambda\nu} = (-1)^{p-1}\sum_{f=1}^{p} {}^{(p-f)}k_\mu{}^\nu + (-1)^p\ h_{\lambda\mu}\,{}^*g^{\lambda\nu},\\ &(b)\ \ {}^{(p)}k_{\mu\lambda}\,{}^*g^{\nu\lambda} = \sum_{f=1}^{p}(-1)^{(f-1)}\ {}^{(p-f)}k_\mu{}^\nu + (-1)^p\ h_{\mu\lambda}\,{}^*g^{\nu\lambda}.\end{aligned}$$

Proof. This assertion $(3.1)(a)$ will be proved by induction on p. Substituting (2.2) into (2.1), we obtain

$$(3.2)\qquad k_{\lambda\mu}\,{}^*g^{\lambda\nu} = \delta_\mu^\nu - h_{\lambda\mu}\,{}^*g^{\lambda\nu}.$$

Hence in virtue of $(2.10)(d)$, the assertion $(3.1)(a)$ holds for the case $p = 1$. Now, assume that $(3.1)(a)$ is true for the case $p = m$, i.e.,

$$(3.3)\qquad {}^{(m)}k_{\lambda\mu}\,{}^*g^{\lambda\nu} = (-1)^{m-1}\sum_{f=1}^{m} {}^{(m-f)}k_\mu{}^\nu + (-1)^m\ h_{\lambda\mu}{}^*\,g^{\lambda\nu}.$$

Multiplying $k^\mu{}_\omega$ to both sides of (3.3), and using $(2.10)(d)$ and (3.2), we obtain

$$\begin{aligned} {}^{(m+1)}k_{\lambda\omega}\,{}^*g^{\lambda\nu} &= (-1)^{m-1}\sum_{f=1}^{m}(-1)\ {}^{(m-f+1)}k_\omega{}^\nu + (-1)^m\ k_{\lambda\omega}\,{}^*g^{\lambda\nu}\\ &= (-1)^m\sum_{f=1}^{m} {}^{(m-f+1)}k_\omega{}^\nu + (-1)^m(\delta_\omega^\nu -\ h_{\lambda\omega}\,{}^*g^{\lambda\nu})\\ &= (-1)^{(m+1)-1}\sum_{f=1}^{m+1} {}^{(m+1-f)}k_\omega{}^\nu + (-1)^{m+1}\ h_{\lambda\omega}\,{}^*g^{\lambda\nu},\end{aligned}$$

which shows that $(3.1)(a)$ holds for the case $p = m + 1$. By the principle of induction, the assertion $(3.1)(a)$ is true for every integer $p \geq 1$. Similarly, we obtain $(3.1)(b)$. □

Theorem 3.2. *The representations of the tensors ${}^*h_{\lambda\mu}$, ${}^*k_{\lambda\mu}$ and ${}^*g_{\lambda\mu}$ in terms of the g-unified field tensor $g_{\lambda\mu}$ in X_n may be given by*

$$(3.4)\quad \begin{aligned} &(a)\ \ {}^*h_{\lambda\mu} = h_{\lambda\mu} - {}^{(2)}k_{\lambda\mu},\\ &(b)\ \ {}^*k_{\lambda\mu} = k_{\lambda\mu} - {}^{(3)}k_{\lambda\mu},\\ &(c)\ \ {}^*g_{\lambda\mu} = g_{\lambda\mu} - {}^{(2)}k_{\lambda\mu} - {}^{(3)}k_{\lambda\mu}.\end{aligned}$$

Proof. When $p = 2$, $(3.1)(a)$ and $(3.1)(b)$ satisfy the following relations:

$$(3.5)\quad \begin{aligned} &(a)\ \ (h_{\lambda\mu} - {}^{(2)}k_{\lambda\mu})\,{}^*g^{\lambda\nu} = k_\mu{}^\nu + \delta_\mu^\nu,\\ &(b)\ \ (h_{\mu\lambda} - {}^{(2)}k_{\mu\lambda})\,{}^*g^{\nu\lambda} = -k_\mu{}^\nu + \delta_\mu^\nu.\end{aligned}$$

Taking the sum of (3.5)(a) and (3.5)(b), and using (2.4), we obtain

$$(h_{\lambda\mu} - {}^{(2)}k_{\lambda\mu})\, {}^*h^{\lambda\nu} = \delta_\mu^\nu,$$

which implies (3.4)(a) in virtue of (2.5). On the other hand, subtracting (3.5)(b) from (3.5)(a), and using (2.4), we obtain

$$(h_{\lambda\mu} - {}^{(2)}k_{\lambda\mu})\, {}^*k^{\lambda\nu} = k_\mu{}^\nu. \tag{3.6}$$

Using (2.11), (3.4)(a) and (3.6), we obtain

$$ {}^*k_\mu{}^\nu = {}^*h_{\mu\alpha}\, {}^*k^{\alpha\nu} = (h_{\mu\alpha} - {}^{(2)}k_{\mu\alpha})\, {}^*k^{\alpha\nu} = k_\mu{}^\nu. \tag{3.7}$$

Next, using (2.11), (2.10)(d), (3.4)(a) and (3.7), we obtain (3.4)(b) by the following step:

$$ {}^*k_{\lambda\mu} = {}^*h_{\mu\alpha}\, {}^*k_\lambda{}^\alpha = (h_{\mu\alpha} - {}^{(2)}k_{\mu\alpha})\, k_\lambda{}^\alpha = k_{\lambda\mu} - {}^{(3)}k_{\lambda\mu}.$$

Finally, adding (3.4)(a) and (3.4)(b), we obtain (3.4)(c). □

Remark 3.3. The representations (3.4)(a), (b), and (c) in X_n were also obtained by Chung ([2], pp. 146–147). However, the proofs of these results in the present paper are more easy and simple than the previous lengthy and complicated proofs.

Theorem 3.4. *Another representations of the tensors* $^*h_{\lambda\mu}$, $^*k_{\lambda\mu}$ *and* $^*g_{\lambda\mu}$ *in* X_n *may be given by*

$$\begin{aligned} &(a)\quad {}^*h_{\lambda\mu} = h^{\alpha\beta}\, g_{\alpha\lambda}\, g_{\beta\mu}, \\ &(b)\quad {}^*k_{\lambda\mu} = k^{\alpha\beta}\, g_{\alpha\lambda}\, g_{\beta\mu}, \\ &(c)\quad {}^*g_{\lambda\mu} = g^{\alpha\beta}\, g_{\alpha\lambda}\, g_{\beta\mu}. \end{aligned} \tag{3.8}$$

Proof. Since (3.4)(a) and (3.4)(b) can be respectively rewritten as

$$ {}^*h^{\lambda\mu} = h_{\lambda\mu} - {}^{(2)}k_{\lambda\mu} = h^{\alpha\beta}\,(h_{\alpha\lambda} + k_{\alpha\lambda})(h_{\beta\mu} + k_{\beta\mu}),$$

$$ {}^*k_{\lambda\mu} = k_{\lambda\mu} - {}^{(3)}k_{\lambda\mu} = k^{\alpha\beta}\,(h_{\alpha\lambda} + k_{\alpha\lambda})(h_{\beta\mu} + k_{\beta\mu}),$$

we obtain (3.8)(a) and (3.8)(b) in virtue of (2.2). Adding (3.8)(a) and (3.8)(b), we obtain (3.8)(c). □

Remark 3.5. The representations (3.8)(a), (b), and (c) will play an important role in the study of n-UFT. For instance, using these representations, we can easily obtain the representation of the tensors $h^{\lambda\mu}$, $k^{\lambda\mu}$ and $g^{\lambda\mu}$ in terms of the $^*g$-unified field tensor $^*g^{\lambda\nu}$. Multiplying $^*g^{\gamma\lambda}\,{}^*g^{\epsilon\mu}$ to both sides of (3.8)(a), (b), and (c), and making use of (1.1), we obtain respectively

$$\begin{aligned} &(a)\quad h^{\lambda\mu} = {}^*h_{\alpha\beta}\, {}^*g^{\lambda\alpha}\, {}^*g^{\mu\beta}, \\ &(b)\quad k^{\lambda\mu} = {}^*k_{\alpha\beta}\, {}^*g^{\lambda\alpha}\, {}^*g^{\mu\beta}, \\ &(c)\quad g^{\lambda\mu} = {}^*g_{\alpha\beta}\, {}^*g^{\lambda\alpha}\, {}^*g^{\mu\beta}. \end{aligned} \tag{3.9}$$

Substituting (2.4) into (3.9), and using (2.11) and (2.13)(c), we obtain

$$(3.10)\qquad \begin{aligned} &(a)\ \ h^{\lambda\mu} = {}^*h^{\lambda\mu} - {}^{(2)*}k^{\lambda\mu},\\ &(b)\ \ k^{\lambda\mu} = {}^*k^{\lambda\mu} - {}^{(3)*}k^{\lambda\mu},\\ &(c)\ \ g^{\lambda\mu} = {}^*g^{\lambda\mu} - {}^{(2)*}k^{\lambda\mu} - {}^{(3)*}k^{\lambda\mu}. \end{aligned}$$

And we can also derive the representations of the determinants $^*G$, $^*H$ and $^*T$, given by (2.13)(a).

Theorem 3.6. *The representations of the determinants $^*G$, $^*H$ and $^*T$ in X_n, given by* (2.13)(a), *may be respectively given by*

$$(3.11)\qquad \begin{aligned} &(a)\ \ {}^*H = \tilde{g}^2\, H,\\ &(b)\ \ {}^*T = \tilde{g}^2\, T,\\ &(c)\ \ {}^*G = \tilde{g}^2\, G. \end{aligned}$$

In particular,

$$(3.12)\qquad \begin{aligned} &(a)\ \ {}^*\tilde{k} = \tilde{k},\\ &(b)\ \ {}^*\tilde{g} = \tilde{g}. \end{aligned}$$

Proof. For any n-square matrices $A = (a^{\lambda\mu})$, $B = (b_{\lambda\mu})$ and $C = (c_{\lambda\mu})$, their matrix product ${}^tBAC = (d_{\lambda\mu})$ is given by

$${}^tBAC = (b_{\alpha\lambda}\, a^{\alpha\beta}\, c_{\beta\mu}),$$

where ${}^tB$ is the transpose of B. From which it follows that since $\det {}^tB = \det B$,

$$(3.13)\qquad \det(b_{\alpha\lambda}\, a^{\alpha\beta}\, c_{\beta\mu}) = \det B\, \det A\, \det C.$$

Since $\det A^{-1} = 1/\det A$, (3.8a) implies

$$(3.14)\qquad {}^*H = \frac{1}{H}G^2 = (\frac{G}{H})^2\, H = \tilde{g}^2\, H,$$

in virtue of (2.10)(a) and (b), (2.13)(a), and (3.13). Hence we obtain (3.11)(a). On the other hand, in virtue of (2.6), (2.10)(a), and (3.13), we obtain

$$(3.15)\qquad \det(k^{\lambda\mu}) = \det(h^{\lambda\alpha}\, k_{\alpha\beta}\, h^{\beta\mu}) = T(\frac{1}{H})^2.$$

Hence in virtue of (3.13) and (3.15), (3.8)(b) implies

$$(3.16)\qquad {}^*T = T(\frac{1}{H})^2\, G^2 = T(\frac{G}{H})^2 = \tilde{g}^2\, T.$$

Hence we obtain (3.11)(b). Similarly, we obtain (3.11)(c). Next, dividing (3.11)(b) and (c) by (3.11)(a), we obtain (3.12)(a) and (b) in virtue of (2.10)(b) and (2.13)(b), respectively. □

Remark 3.7. In 4-g-UFT, the representations (3.11)(a), (b), and (c) are coincident with Chung's 4-dimensional results ([4], pp. 47–48) which were obtained by another method.

4. Some results in even-dimensional X_n

Remark 4.1. From now on, we shall assume that

$$T = \det(k_{\lambda\mu}) \neq 0. \tag{4.1}$$

Hence there exists a unique skew-symmetric tensor $\overline{k}^{\lambda\mu}$ in X_n satisfying

$$k_{\lambda\mu}\,\overline{k}^{\lambda\nu} = \delta_\mu^\nu. \tag{4.2}$$

Since $k_{\lambda\mu}$ is skew-symmetric, and $T \neq 0$, the dimension of X_n is even. That is, n is *even*.

It has been shown by Chung ([4], pp. 143–144) that the following relations hold in even-dimensional X_n:

$$\begin{aligned} &(a)\ \ K_n = \tilde{k}, \qquad K_p = 0\ (p \text{ is odd}),\\ &(b)\ \ \overline{K}_s + K_{s+2} = \overline{K}_{s+2},\\ &(c)\ \ \tilde{g} = \sum_{s=0}^{n} K_s = \overline{K}_n.\\ &(d)\ \ \sum_{s=0}^{n} K_s\ {}^{(n-s)}k_\lambda{}^\nu = 0. \end{aligned} \tag{4.3}$$

Here and in what follows, the index s is assumed to take the values 0,2,4,...,n in the specified range.

In our further considerations in the present paper, we use the following useful abbreviations for any tensor $Z_{\lambda\nu}$, for $p, q = 1, 2, 3, \ldots$

$${}^{(p)}Z_{\lambda\mu} = {}^{(p-1)}k_\lambda{}^\nu\, Z_{\nu\mu}. \tag{4.4}$$

We then have

$${}^{(1)}Z_{\lambda\mu} = Z_{\lambda\mu}, \qquad {}^{(p)}k_\lambda{}^{\nu}\,{}^{(q)}Z_{\nu\mu} = {}^{(p+q)}Z_{\lambda\mu}. \tag{4.5}$$

Theorem 4.2. *The representation of the tensor* $\overline{k}^{\lambda\mu}$ *in* X_n, *given by* (4.2), *may be given by*

$$\overline{k}^{\lambda\mu} = \frac{1}{\tilde{k}} \sum_{s=0}^{n-2} K_s\ {}^{(n-s-1)}k^{\lambda\mu}. \tag{4.6}$$

Proof. Consider a skew symmetric tensor $M^{\lambda\mu}$ uniquely defined by

$$k_{\lambda\nu}\, M^{\lambda\mu} = \delta_\nu^\mu. \tag{4.7}$$

Then in virtue of (2.6) and (4.4), (4.7) implies the following relation:

$${}^{(2)}M^{\nu\mu} = -h^{\nu\mu}. \tag{4.8}$$

Multiplying ${}^{(p-2)}k^\lambda{}_\nu$ to both sides of (4.8), and using (4.5), we obtain

$${}^{(p)}M^{\lambda\mu} = -\,{}^{(p-2)}k^{\lambda\mu} \qquad (p = 2, 3, \ldots). \tag{4.9}$$

Multiplying $M^{\nu\mu}$ to both sides of (4.3)(d), and using (4.4), we obtain

$$\sum_{s=0}^{n} K_s\ {}^{(n-s+1)}M^{\lambda\mu} = \sum_{s=0}^{n-2} K_s\ {}^{(n-s+1)}M^{\lambda\mu} + K_n\, M^{\lambda\mu} = 0. \tag{4.10}$$

Substituting ${}^{(n-s+1)}M^{\lambda\mu}$ from (4.9) into (4.10), and using (4.3)(a), we obtain

$$-\sum_{s=0}^{n-2} K_s\ {}^{(n-s-1)}k^{\lambda\mu} + \tilde{k}\, M^{\lambda\mu} = 0,$$

which implies (4.6), in virtue of (4.2). □

Remark 4.3. In virtue of (3.11)(b) and (4.1), we obtain

$$^*T = \det(^*k_{\lambda\mu}) \neq 0.$$

Hence there exists a unique skew-symmetric tensor $Q^{\lambda\mu}$ in X_n satisfying

$$^*k_{\lambda\mu}\, Q^{\lambda\nu} = \delta^{\nu}_{\mu}. \tag{4.11}$$

Lemma 4.4. *The following recurrence relations in X_n holds:*

$$\begin{aligned} &(a)\quad {}^{(4)}Q^{\mu\nu} = {}^{(2)}Q^{\mu\nu} + h^{\mu\nu}, \\ &(b)\quad {}^{(3)}Q^{\omega\nu} = Q^{\omega\nu} - \overline{k}^{\omega\nu}, \\ &(c)\quad {}^{(p)}Q^{\omega\nu} = {}^{(p-2)}Q^{\omega\nu} + {}^{(p-4)}k^{\omega\nu} \quad (p = 4, 5, \ldots). \end{aligned} \tag{4.12}$$

Proof. Substituting (3.4)(b) into (4.11), we obtain (4.12)(a) in virtue of (4.4). Multiplying $\overline{k}^{\mu\omega}$ to both sides of (4.12)(a), we obtain (4.12)(b) in virtue of (4.2) and (4.4). Multiplying ${}^{(p-4)}k^{\omega}{}_{\mu}$ to both sides of (4.12)(a), we obtain the relation (4.12)(c) in virtue of (4.4). □

Theorem 4.5. *The representation of the tensor $Q^{\lambda\mu}$ in X_n, given by* (4.11), *may be given by*

$$Q^{\lambda\mu} = -\frac{1}{\tilde{g}} \sum_{s=0}^{n-2} \overline{K}_s\ {}^{(n-s-3)}k^{\lambda\mu}, \tag{4.13}$$

where

$$^{(-1)}k^{\lambda\mu} = -\overline{k}^{\lambda\mu}. \tag{4.14}$$

Proof. Multiplying $Q^{\nu\mu}$ to both sides of (4.3)(d), and using (4.4), we obtain

$$\begin{aligned} &\sum_{s=0}^{n} K_s\ {}^{(n-s+1)}Q^{\lambda\mu} \\ &= K_0\ {}^{(n+1)}Q^{\lambda\mu} + K_2\ {}^{(n-1)}Q^{\lambda\mu} + \sum_{s=4}^{n} K_s\ {}^{(n-s+1)}Q^{\lambda\nu} = 0. \end{aligned} \tag{4.15}$$

Substituting ${}^{(n+1)}Q^{\lambda\mu}$ from (4.12)(c) into the first term of (4.15), and using (2.10)(c), we obtain

$$(4.16)\qquad {}^{(n-3)}k^{\lambda\mu} + (1+K_2)\,{}^{(n-1)}Q^{\lambda\mu} + \sum_{s=4}^{n} K_s\ {}^{(n-s+1)}Q^{\lambda\nu} = 0.$$

Substituting again ${}^{(n-1)}Q_{\lambda\mu}$ from (4.12)(c) into (4.16), and using (2.10)(e) and (4.3)(b), we obtain

$${}^{(n-3)}k^{\lambda\mu} + \overline{K}_2\ {}^{(n-5)}k^{\lambda\mu} + (\overline{K}_2 + K_4)\,{}^{(n-3)}Q^{\lambda\mu} + \sum_{s=6}^{n} K_s\ {}^{(n-s+1)}Q^{\lambda\mu} = 0.$$

After $(n-2)/2$ steps of successive repeat substituting for ${}^{(p)}Q^{\lambda\mu}$ from (4.12)(c), we obtain

$$(4.17)\qquad \sum_{s=0}^{n-4} \overline{K}_s\ {}^{(n-s-3)}k^{\lambda\mu} + \overline{K}_{n-2}\ {}^{(3)}Q^{\lambda\mu} + K_n\,Q_{\lambda\mu} = 0,$$

in virtue of (2.10)(e). Substituting (4.12)(b) into (4.17), and using (4.3)(b) and (4.14), we obtain

$$(4.18)\qquad \begin{aligned} &\sum_{s=0}^{n-4} \overline{K}_s\ {}^{(n-s-3)}k^{\lambda\mu} - \overline{K}_{n-2}\,\overline{k}^{\lambda\mu} + \overline{K}_n\,Q^{\lambda\mu} \\ &= \sum_{s=0}^{n-2} \overline{K}_s\ {}^{(n-s-3)}k^{\lambda\mu} + \overline{K}_n\,Q^{\lambda\mu} = 0. \end{aligned}$$

Using (4.3)(c), (4.18) is condensed to (4.13). □

Remark 4.6. For the lower-dimensional cases $n = 2, 4$, we obtain the following Table 1, in virtue of (4.3)(c) and (d), (4.6) and (4.13). When $4\,K = k_{\alpha\beta}\,k^{\alpha\beta} = -{}^{(2)}k_\alpha{}^\alpha$.

n	$\overline{k}^{\lambda\mu}$	$Q^{\lambda\mu}$
2	$\frac{1}{\bar{\bar{k}}}k^{\lambda\mu}$	$-\frac{1}{\tilde{g}\bar{\bar{k}}}k^{\lambda\mu}$
4	$\frac{1}{\bar{\bar{k}}}({}^{(3)}k^{\lambda\mu} + 2Kk^{\lambda\mu})$	$-\frac{1}{\tilde{g}}(k^{\lambda\mu} - \frac{1+2K}{\tilde{k}}({}^{(3)}k^{\lambda\mu} + 2Kk^{\lambda\mu}))$

TABLE 1. For $n = 2, 4$, the representations of $\overline{k}^{\lambda\mu}$ and $Q^{\lambda\mu}$

Example 1. When for some nonzero vector Y^ν, the torsion tensor of the connection $\Gamma^\nu_{\lambda\mu}$

$$S_{\lambda\mu}{}^\nu = k_{\lambda\mu} Y^\nu \tag{4.19}$$

holds, the system of equations (2.8) is equivalent to the followings:

$$\Gamma^\nu_{\lambda\mu} = \{{}_\lambda{}^\nu{}_\mu\} - 2k_{(\lambda}{}^\nu S_{\mu)} + k_{\lambda\mu} Y^\nu, \tag{4.20}$$

$$\nabla_\nu k_{\lambda\mu} = -2k_{\nu[\lambda} Y_{\mu]} + 2{}^{(2)}k_{\nu[\lambda} S_{\mu]}, \tag{4.21}$$

where ∇_ω is the symbolic vector of the covariant derivative with respect to the Christoffel symbols $\{{}_\lambda{}^\nu{}_\mu\}$ defined by $h_{\lambda\mu}$, and $S_\lambda = S_{\lambda\alpha}{}^\alpha$ is the torsion vector (Lee [8], pp. 1049–1050). On the other hand, the Nijenhuis tensor $N_{\lambda\mu}{}^\nu$ may be given by

$$N_{\lambda\mu}{}^\nu = 2(\partial_\alpha k_{[\lambda}{}^\nu) k_{\mu]}{}^\alpha - 2k_\alpha{}^\nu (\partial_{[\mu} k_{\lambda]}{}^\alpha), \tag{4.22}$$

and the symbol ∂ in (4.22) may be replaced by ∇, that is,

$$N_{\lambda\mu}{}^\nu = 2(\nabla_\alpha k_{[\lambda}{}^\nu) k_{\mu]}{}^\alpha - 2k_\alpha{}^\nu (\nabla_{[\mu} k_{\lambda]}{}^\alpha). \tag{4.23}$$

In this case, substituting the condition (4.21) into (4.23), and making use of (3.4)(*b*), the Nijenhuis tensor $N_{\lambda\mu}{}^\nu$ and the torsion vector S_ν are related by

$$N_{\lambda\mu}{}^\nu = 2(k_{\lambda\mu} - {}^{(3)}k_{\lambda\mu}) S^\nu = 2\ {}^*k_{\lambda\mu} S^\nu, \tag{4.24}$$

by a straightforward computation. Hence, multiplying $Q^{\lambda\omega}$ on the both sides of (4.24), and contracting for ω and μ, and making use of (4.13), we can find the representation of the torsion vector S_ν in terms of the g-unified field tensor $g_{\lambda\mu}$:

$$S_\nu = -\frac{1}{2n\tilde{g}} \sum_{s=0}^{n-2} \overline{K}_s\ {}^{(n-s-3)}k^{\lambda\mu} N_{\lambda\mu\nu}. \tag{4.25}$$

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