

LAGUERRE EXPANSIONS AND PRODUCTS OF DISTRIBUTIONS

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ABSTRACT. In this paper we introduce two products of tempered distributions with positive support. These products are based in the Laguerre representation of distributions. We calculate some products as, $[\delta]x_+^\lambda = \delta[x_+^\lambda] = 0$ and $[x_+^\lambda]x_+^\mu = x_+^{\lambda+\mu}$ for appropriate λ and μ .

1. Introduction

Multiplication of distributions is a difficult and involving problem. In general, there does not exist a product of distributions with the classical properties extending the usual product of functions (see L. Schwartz [9] and M. Oberguggenberger [6]). In [1] we introduce a method for multiply tempered distribution based on the representation of \mathcal{S}' by Hermite series. As continuation of that work, here we study products of tempered distributions with positive support, now taking the approximation given by the Laguerre expansion of distributions (see M. Guillemot-Teissiers [3] and A. Duran [2]), which establishes that every $T \in (\mathcal{S}^+)'$ can be represented in the weak sense by a series

$$\sum_{n=0}^{\infty} \langle T, \mathcal{L}_n \rangle \mathcal{L}_n,$$

where $\{\mathcal{L}_n\}$ are the Laguerre functions.

In this context we say that there exists the product $[S]T$ of the tempered distributions with positive support S and T , if $\sum_{k=0}^{\infty} c_k \mathcal{L}_k$ is a tempered distribution where the coefficients c_k are given by

$$(1) \quad c_k = \lim_{m \rightarrow \infty} \sum_{n=0}^m \langle S, \mathcal{L}_n \rangle \langle T, \mathcal{L}_n \mathcal{L}_k \rangle.$$

The product $[S]T$ of S and T , is by definition, $\sum_{k=0}^{\infty} c_k \mathcal{L}_k$. Symmetrically, we define the product $S[T]$.

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This paper is organized as follows: In section 2, we summarize the relevant material on Laguerre functions, tempered distributions with positive support and representation theorems for \mathcal{S}^+ and $(\mathcal{S}^+)'$. In section 3, we introduce the Laguerre products and some properties. In section 4, we calculate the following products: $[T]\delta = eT$, where $e = \sum_{n=0}^{\infty} \langle T, \mathcal{L}_n \rangle$, $[\delta]x_+^\lambda = \delta[x_+^\lambda] = 0$ and $[x_+^\lambda]x_+^\mu = x_+^{\lambda-\mu}$ for appropriate λ and μ . In Appendix, we compile and prove some basic facts about hypergeometric series needed in our work.

2. Laguerre expansion of tempered distributions with positive support

This section collects relevant properties of Laguerre functions and tempered distributions with positive support (see N. Lebedev [5] and L. Schwartz [8]). Let n be a nonnegative integer, the n -th *Laguerre polynomial* L_n is

$$(2) \quad L_n(z) = \sum_{j=0}^n \binom{n}{n-j} \frac{(-z)^j}{j!}.$$

The n -th *Laguerre function* $\mathcal{L}_n(x)$ is

$$(3) \quad \mathcal{L}_n(x) = e^{-\frac{x}{2}} L_n(x).$$

The set of Laguerre functions is an orthonormal basis for $L^2([0, \infty))$ (see for instance [5] and [10]). Expressing any $f \in L^2([0, \infty))$ in this basis we obtain its *Laguerre expansion*, $f = \sum_n f_n \mathcal{L}_n$ with n -th *Laguerre coefficient*

$$f_n = \int_0^\infty f(x) \mathcal{L}_n(x) dx.$$

By the *Laguerre coefficients of f* , denoted by $\mathcal{L}(f)$, we mean the sequence (f_n) .

Throughout the article \mathcal{S} denotes the Schwartz space of rapidly decreasing functions, and \mathcal{S}' its dual, i.e., the space of tempered distributions (see [8]).

The space \mathcal{S}^+ is the set of functions $\phi : [0, \infty) \rightarrow \mathbb{C}$ such that $\phi = \varphi|_{[0, \infty)}$ for some $\varphi \in \mathcal{S}$. The topology of \mathcal{S}^+ is generated by the family of seminorms

$$(4) \quad \|\phi\|_{m,n} = \sup_{x \in [0, \infty)} |x^m D^{(n)} \phi(x)|,$$

where $m, n \in \mathbb{N}$.

We observed that the dual space $(\mathcal{S}^+)'$ can be identified with the space of tempered distributions with positive support.

For $T \in (\mathcal{S}^+)'$, the *Laguerre coefficients of T* are the sequence $(\langle T, \mathcal{L}_n \rangle)$, denoted by $\mathcal{L}(T)$.

In order to characterize the space of tempered distributions with positive support in terms of its Laguerre coefficients, following M. Guillemot [3] and A. Duran [2] we introduce the space \mathfrak{s} of rapidly decreasing sequences and \mathfrak{s}' its dual, i.e., the space of slowly decreasing sequences. We recall that

$$\mathfrak{s} = \{(a_n) \subset \mathbb{C} : \text{for every } p \in \mathbb{N}, \lim_{n \rightarrow \infty} n^p a_n = 0\}.$$

The topology of \mathbf{s} is generated by the seminorms $\|(a_n)\|_p^2 = \sum_{n=0}^{\infty} (1+n)^{2p} |a_n|^2$ for $p \in \mathbb{N}$. The dual of \mathbf{s} is given by

$$\mathbf{s}' = \{(b_n) \subset \mathbb{C} : \text{for some } (C, k) \in \mathbb{R} \times \mathbb{N}, |b_n| \leq C|(1+n)^k| \text{ for all } n \in \mathbb{N}\}.$$

The Laguerre coefficients provide topological isomorphisms between \mathcal{S}^+ and the space of rapidly decreasing sequences and between $(\mathcal{S}^+)'$ and the space of slowly decreasing sequences.

Theorem 1. 1) Let $\phi \in \mathcal{S}^+$ and $a_n = \langle \phi, \mathcal{L}_n \rangle$. Then $(a_n) \in \mathbf{s}$ and $\phi = \sum_n a_n \mathcal{L}_n$. Conversely, $\sum_n a_n \mathcal{L}_n(x) \in \mathcal{S}^+$ if $(a_n) \in \mathbf{s}$.

2) Let $T \in (\mathcal{S}^+)'$ and $b_n = \langle T, \mathcal{L}_n \rangle$. Then $(b_n) \in \mathbf{s}'$ and $T = \sum_n b_n \mathcal{L}_n$. Conversely, $\sum_n b_n \mathcal{L}_n \in \mathcal{S}^+$ if $(b_n) \in \mathbf{s}'$.

Proof. See [2], Theorem 2.8 and 2.9 or [3], p.550. \square

Next, we compute the Laguerre coefficients of some tempered distributions with positive support.

Example 1. The delta distribution.

$$(5) \quad \mathcal{L}(\delta) = (L_n(0) = 1).$$

Example 2. The k -th derivative of the delta distribution.

$$(6) \quad \mathcal{L}(\delta^{(k)}) = \left((-1)^k \mathcal{L}_n^{(k)}(0) = \sum_{m=0}^k \left(\frac{1}{2}\right)^{k-m} \binom{k}{m} \binom{n}{m} \right).$$

Example 3. The Heaviside function

$$(7) \quad \mathcal{L}(H) = (2(-1)^n).$$

Example 4. For complex λ with $\Re \lambda > -1$ the function x_+^λ defines a regular distribution in $(\mathcal{S}^+)'$:

$$(8) \quad \mathcal{L}(x_+^\lambda) = \left(\int_0^\infty x^\lambda \mathcal{L}_n(x) dx \right) = \Gamma(\lambda + 1) 2^{\lambda+1} F(-n, \lambda + 1; 1; 2),$$

where $F(a, b; c; z)$ is the usual hypergeometric function. An easy computation shows that

$$(9) \quad \mathcal{L}(x_+^\lambda) = \left(\sum_{j=0}^n \binom{n}{j} \frac{(-1)^j}{j!} \Gamma(\lambda + j + 1) 2^{\lambda+j+1} \right).$$

3. Laguerre products of distributions

Definition 1. Let S and T be tempered distributions with positive support. Suppose that for all $k \in \mathbb{N} \cup \{0\}$ there exists

$$c_k = \lim_{m \rightarrow \infty} \sum_{n=0}^m \langle S, \mathcal{L}_n \rangle \langle T, \mathcal{L}_n \mathcal{L}_k \rangle$$

and that $(c_k) \in \mathfrak{s}'$. We define the *left Laguerre product* $[S] \cdot T \in \mathcal{S}'$ by

$$(10) \quad [S] \cdot T = \sum_{k=0}^{\infty} c_k \mathcal{L}_k.$$

Definition 2. Let S and T be tempered distributions with positive support. Suppose that for all $k \in \mathbb{N} \cup \{0\}$ there exists

$$d_k = \lim_{m \rightarrow \infty} \sum_{n=0}^m \langle T, \mathcal{L}_n \rangle \langle S, \mathcal{L}_n \mathcal{L}_k \rangle$$

and that $(d_k) \in \mathfrak{s}'$. We define the *right Laguerre product* $S \cdot [T] \in \mathcal{S}'$ by

$$(11) \quad S \cdot [T] = \sum_{k=0}^{\infty} d_k \mathcal{L}_k.$$

Remark 1. (1) The Laguerre representation theorem for $(\mathcal{S}^+)'$ ensures that the above definitions are well posed.

(2) It is clear from the definitions that the Laguerre products satisfies the Leibnitz rule and the commutative rule,

$$[S] \cdot T = T \cdot [S].$$

(3) In Examples 5 and 6 we will show that the products $[H]\delta$ and $\delta[H]$ does not exists and that $[\delta]H = H[\delta] = \delta$. So, the left and right Laguerre products are not commutative.

(4) We have that

$$c_k = \lim_{m \rightarrow \infty} \sum_{i=0}^m \sum_{n=0}^{\infty} C(n, i, k) \langle S, \mathcal{L}_i \rangle \langle T, \mathcal{L}_n \rangle$$

and

$$d_k = \lim_{m \rightarrow \infty} \sum_{n=0}^m \sum_{i=0}^{\infty} C(n, i, k) \langle T, \mathcal{L}_n \rangle \langle S, \mathcal{L}_i \rangle,$$

where $C(n, i, k) = \int_{-\infty}^{\infty} \mathcal{L}_n(x) \mathcal{L}_i(x) \mathcal{L}_k(x) dx$.

The Laguerre products extend the usual product of multipliers of \mathcal{S}^+ by tempered distributions with positive support. We denote by \mathcal{O}_M^+ the space of multipliers of \mathcal{S}^+ .

Theorem 2. Let $T \in (\mathcal{S}^+)'$ and $f \in \mathcal{O}_M^+$. Then $f[T]$ exist and

$$f[T] = [T]f = fT.$$

Proof.

$$\begin{aligned} \langle fT, \mathcal{L}_k \rangle &= \langle T, f\mathcal{L}_k \rangle \\ &= \lim_{m \rightarrow \infty} \left\langle \sum_{n=0}^m \langle T, \mathcal{L}_n \rangle \mathcal{L}_n, f\mathcal{L}_k \right\rangle \end{aligned}$$

$$\begin{aligned}
&= \lim_{m \rightarrow \infty} \sum_{n=0}^m \langle T, \mathcal{L}_n \rangle \langle \mathcal{L}_n, f \mathcal{L}_k \rangle \\
&= \lim_{m \rightarrow \infty} \sum_{n=0}^m \langle T, \mathcal{L}_n \rangle \langle f, \mathcal{L}_n \mathcal{L}_k \rangle \\
&= \langle f[T], \mathcal{L}_k \rangle.
\end{aligned}$$

□

Remark 2. Let $T \in (\mathcal{S}^+)'$ and $f \in \mathcal{O}_M^+$ sometimes the product $[f]T$ does not exist. This is the case of $[H]\delta$ (see Example 5).

4. Some examples of Laguerre products

Example 5. Let $T \in (\mathcal{S}^+)'$. The product $[T]\delta$ exists if and only if $e = \sum_{n=0}^{\infty} \langle T, \mathcal{L}_n \rangle < \infty$, and in this case

$$[T]\delta = e\delta.$$

In fact, we have that

$$\begin{aligned}
\langle [T]\delta, \mathcal{L}_k \rangle &= \lim_{m \rightarrow \infty} \sum_{n=0}^m \langle T, \mathcal{L}_n \rangle \langle \delta, \mathcal{L}_n \mathcal{L}_k \rangle \\
&= \lim_{m \rightarrow \infty} \sum_{n=0}^m \langle T, \mathcal{L}_n \rangle \\
&= \langle e\delta, \mathcal{L}_k \rangle.
\end{aligned}$$

In particular, the products $[H]\delta$ and $[\delta^{(k)}]\delta$, for $k \in \mathbb{N} \cup \{0\}$ does not exist (see (5), (6) and (7)).

Example 6. Let $T \in (\mathcal{S}^+)'$. Then $[T]H = T$.

In fact, we have that

$$\begin{aligned}
\langle [T]H, \mathcal{L}_k \rangle &= \lim_{m \rightarrow \infty} \sum_{n=0}^m \langle T, \mathcal{L}_n \rangle \langle H, \mathcal{L}_n \mathcal{L}_k \rangle \\
&= \lim_{m \rightarrow \infty} \sum_{n=0}^m \langle T, \mathcal{L}_n \rangle \int_0^{\infty} \mathcal{L}_n(t) \mathcal{L}_k(t) dt \\
&= \langle T, \mathcal{L}_k \rangle.
\end{aligned}$$

Example 7. Let $\lambda \in \mathbb{C}$ such that $\Re \lambda > 0$. Then

$$[\delta]x_+^\lambda = \delta[x_+^\lambda] = 0.$$

Let us recall the following formulas involving the generalized hypergeometric function F :

$$(12) \quad \int_0^{\infty} x^\lambda \mathcal{L}_n(x) e^{-\frac{1}{2}x} dx = \Gamma(\lambda + 1) F(-n, \lambda + 1; 1; 1),$$

$$(13) \quad \sum_{n=0}^{\infty} F(-n, \lambda + j + 1; 1; 1) = 0$$

and

$$(14) \quad \sum_{n=0}^{\infty} F(-n, \lambda + 1; 1; 2) = 0.$$

See [4], p.850 for a proof of (12) and the Appendix for proofs of (13) and (14).

In order to prove that $[\delta]x_+^\lambda = 0$, we calculate

$$(15) \quad c_k = \lim_{m \rightarrow \infty} \sum_{n=0}^m \int_0^\infty x^\lambda \mathcal{L}_n(x) \mathcal{L}_k(x) dx.$$

Substituting (3) and (2) into (15) and using (12) we have that

$$(16) \quad c_k = \sum_{j=0}^k \binom{k}{j} \frac{(-1)^j}{j!} \Gamma(\lambda + j + 1) \sum_{n=0}^{\infty} F(-n, \lambda + j + 1; 1; 1).$$

Applying (13) we conclude that $c_k = 0$. Theorem 1 gives $[\delta]x_+^\lambda = 0$.

It remains to prove that $\delta[x_+^\lambda] = 0$, which is clear from (5), (8), and (14).

Example 8. Let $\lambda, \mu \in \mathbb{C}$ such that $\Re\lambda > -1$, $\Re\mu > -1$ and $\Re(\lambda + \mu) > -1$. Then

$$[x_+^\lambda]x_+^\mu = x_+^{\lambda+\mu}.$$

We calculate

$$c_k = \lim_{m \rightarrow \infty} \sum_{n=0}^m \langle x_+^\lambda, \mathcal{L}_n \rangle \langle x_+^\mu, \mathcal{L}_n \mathcal{L}_k \rangle.$$

From (8) we have

$$(17) \quad c_k = \lim_{m \rightarrow \infty} \sum_{n=0}^m \Gamma(\lambda + 1) 2^{\lambda+1} F(-n, \lambda + 1; 1; 2) \langle x_+^\mu, \mathcal{L}_n \mathcal{L}_k \rangle.$$

Substituting (3) and (2) into (17) and using (12) we have that

$$(18) \quad c_k = \Gamma(\lambda + 1) 2^{\lambda+1} \sum_{j=0}^k \binom{k}{j} \frac{(-1)^j}{j!} \Gamma(\mu + j + 1) \cdot \sum_{n=0}^{\infty} F(-n, \lambda + 1; 1; 2) F(-n, \mu + j + 1; 1; 1).$$

We recall that

$$(19) \quad \frac{2^{\mu+j} \Gamma(\mu + j + \lambda + 1) \Gamma(1)}{\Gamma(\lambda + 1) \Gamma(\mu + j + 1)} = \sum_{n=0}^{\infty} F(-n, \lambda + 1; 1; 2) F(-n, \mu + j + 1; 1; 1),$$

see the Appendix for a proof. Substituting (19) into (18) we get

$$(20) \quad c_k = \sum_{j=0}^k \binom{k}{j} \frac{(-1)^j}{j!} \Gamma(\mu + j + \lambda + 1) 2^{\mu+j+\lambda+1}.$$

We conclude that $[x_+^\lambda]x_+^\mu = x_+^{\lambda+\mu}$ from (15) and Theorem 1.

5. Appendix

The *generalized hypergeometric series* are defined by

$$(21) \quad {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{n! (\beta_1)_n (\beta_2)_n \cdots (\beta_q)_n} z^n,$$

where $(\alpha)_n$ is the Pochhammer symbol. The series (21) converges for all $z \in \mathbb{C}$ if $p < q + 1$ and for $|z| < 1$ if $p = q + 1$. In this case, the convergence is absolute in $|z| = 1$ if

$$\Re\left(\sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i\right) > 0.$$

We denote ${}_2F_1(\alpha, \beta; \gamma; z)$ by $F(\alpha, \beta; \gamma; z)$. The following two recurrence relations are very useful.

$$(22) \quad cF(a, b; c; z) - (c - b)F(a, b; c + 1; z) - bF(a, b + 1; c + 1; z) = 0$$

and

$$(23) \quad c(1 - z)F(a, b; c; z) - cF(a - 1, b; c; z) + (c - b)zF(a, b; c + 1; z) = 0.$$

Theorem 3 (Gauss). *Let $\Re(c - b - a) > 0$, $c \neq 0, -1, -2, \dots$. Then*

$$(24) \quad F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.$$

Proof. See [5], p.243. □

Corollary 1. *Let $n \in \mathbb{N} \cup \{0\}$. Then*

$$(25) \quad F(-n, b; c, 1) = \frac{(c - b)_n}{(c)_n}.$$

Theorem 4. *Let $\Re(c + \nu) > 0$, $\Re(b + \nu) > 0$ and $z > 0$. Then*

$$\sum_{n=0}^{\infty} \frac{(-\nu)_n}{n!} F(-n, b; c; z) = \frac{\Gamma(\nu + b)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(\nu + c)} z^\nu.$$

Proof. See [7], Proposition 3. □

5.1. Proof of formula (13)

We observe that $F(0, \lambda + j + 1; 1; 1) = 1$. From (23) we have

$$F(-n - 1, \lambda + j + 1; 1; 1) = -(\lambda + j)F(-n, \lambda + j + 1; 2; 1).$$

Thus

$$\begin{aligned} \sum_{n=0}^{\infty} F(-n, \lambda + j + 1; 1; 1) &= 1 + \sum_{n=0}^{\infty} F(-n - 1, \lambda + j + 1; 1; 1) \\ &= 1 - (\lambda + j) \sum_{n=0}^{\infty} F(-n, \lambda + j + 1; 2; 1). \end{aligned}$$

Taking $\nu = -1$ in Theorem 4 we have

$$\begin{aligned} \sum_{n=0}^{\infty} F(-n, \lambda + j + 1; 2; 1) &= \sum_{n=0}^{\infty} \frac{(-(-1))_n}{n!} F(-n, \lambda + j + 1; 2; 1) \\ &= \frac{1}{(\lambda + j)}. \end{aligned}$$

We conclude that

$$\sum_{n=0}^{\infty} F(-n, \lambda + j + 1; 1; 1) = 0.$$

5.2. Proof of formula (14)

From (22) and Theorem 4 with $\nu = -1$, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} F(-n, \lambda + 1; 1; 2) \\ &= \sum_{n=0}^{\infty} -\lambda F(-n, \lambda + 1; 2; 2) + (\lambda + 1)F(-n, \lambda + 2; 2, 2) \\ &= \left(-\lambda \frac{\Gamma(-1 + \lambda + 1)}{\Gamma(\lambda + 1)} + (\lambda + 1) \frac{\Gamma(-1 + \lambda + 2)}{\Gamma(\lambda + 2)} \right) \frac{\Gamma(2)}{2\Gamma(1)} = 0. \end{aligned}$$

5.3. Proof of formula (19)

By Corollary 1 and Theorem 4, it follows that

$$\begin{aligned} &\sum_{n=0}^{\infty} F(-n, \lambda + 1; 1; 2)F(-n, \mu + j + 1; 1; 1) \\ &= \sum_{n=0}^{\infty} \frac{(-(\mu + j))_n}{n!} F(-n, \lambda + 1; 1; 2) = \frac{\Gamma(\mu + j + \lambda + 1)\Gamma(1)}{\Gamma(\lambda + 1)\Gamma(\mu + j + 1)} 2^{\mu + j}. \end{aligned}$$

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