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A REDUCIBILITY OF EXTON'S TRIPLE HYPERGEOMETRIC SERIES X_2

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ABSTRACT. We aim at presenting an interesting result for a reducibility of Exton's triple hypergeometric series X_2. The identity to be given here is obtained by combining Exton's Laplace integral representation for X_2 and Henrici's formula for the product of three hypergeometric series.

1. Introduction and Preliminaries

In 1982, Exton [2] studied the generalizations of the Horn functions H_3 and H_4 (see [1, p. 225]) and introduced a set of 20 triple hypergeometric series X_1 to X_{20} of which we recall here the definition of X_2:

$$X_2(a,\, b\,;\, c_1,\, c_2,\, c_3\,;\, x,\, y,\, z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a)_{2m+2n+p}\,(b)_p\, x^m\, y^n\, z^p}{(c_1)_m\,(c_2)_n\,(c_3)_p\, m!\, n!\, p!}. \tag{1.1}$$

Exton [2] gave the Laplace integral representation of X_2 as follows:

$$X_2(a,\, b\,;\, c_1,\, c_2,\, c_3\,;\, x,\, y,\, z) = \frac{1}{\Gamma(a)}\int_0^{\infty} e^{-s}\, s^{a-1}\, {}_0F_1(-;\, c_1\,;\, x\, s^2)\, {}_0F_1(-;\, c_2\,;\, y\, s^2)\, {}_1F_1(b\,;\, c_3\,;\, z\, s)\, ds. \tag{1.2}$$

On the other hand, in 1987, Henrici [3] derived the following elegant formula expressing the product of three hypergeometric ${}_0F_1$ functions in terms of a hypergeometric ${}_2F_7$ function:

$$\begin{aligned} &{}_0F_1(-;\, 6c\,;\, x)\, {}_0F_1(-;\, 6c\,;\, \omega\, x)\, {}_0F_1(-;\, 6c\,;\, \omega^2\, x) \\ &= {}_2F_7\left[\begin{matrix} 3c-\frac{1}{4},\ \ 3c+\frac{1}{4} \\ 6c,\ 2c,\ 2c+\frac{1}{3},\ 2c+\frac{2}{3},\ 4c-\frac{1}{3},\ 4c,\ 4c+\frac{2}{3} \end{matrix}\,;\, \left(\frac{4}{9}x\right)^3\right], \end{aligned} \tag{1.3}$$

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where

$$\omega = \exp\left(\frac{2\pi i}{3}\right), \tag{1.4}$$

by consideration of the differential equations satisfied by the functions on either side of (1.3).

In 1990, by making use of certain known transformations of hypergeometric functions, Karlsson and Srivastava [4] proved a general identity having Henrici's (1.3) as a special case.

We aim at presenting an interesting result for a reducibility of Exton's triple hypergeometric series X_2 by combining (1.2) and (1.3).

2. Main result

We will show the following identity:

(2.1)

$$\begin{aligned} &X_2(a,\, 6c - \tfrac{1}{2}\,;\, 6c,\, 6c,\, 12c-1\,;\, 1,\, \omega,\, 4\omega) \\ =&\frac{1}{(1-2\omega)^a} \\ &\times {}_8F_7\left[\begin{matrix} 3c-\frac{1}{4},\, 3c+\frac{1}{4},\, \frac{a}{5},\, \frac{a}{5}+\frac{1}{5},\, \frac{a}{5}+\frac{2}{5},\, \frac{a}{5}+\frac{3}{4},\, \frac{a}{5}+\frac{4}{5},\, \frac{a}{5}+1 \\ 6c,\, 2c,\, 2c+\frac{1}{3},\, 2c+\frac{2}{3},\, 4c-\frac{1}{3},\, 4c,\, 4c+\frac{1}{3} \end{matrix};\left(\frac{4}{1-2\omega}\right)^6\right], \end{aligned}$$

where ω is the same as in (1.4).

Proof. Starting with the left-hand side of (2.1) (denoted by L) and using (1.2), we have

$$\begin{aligned} &L \\ &= X_2(a,\, 6c-\tfrac{1}{2}; 6c,\, 6c,\, 12c-1\,;\, 1,\, \omega,\, 4\omega) \\ &= \frac{1}{\Gamma(a)}\int_0^\infty e^{-s}\, s^{a-1}\, {}_0F_1(-;6c;s^2)\, {}_0F_1(-;6c;\omega\, s^2)\, {}_1F_1(6c-\tfrac{1}{2};12c-1;4\omega s)\, ds, \end{aligned}$$

which, for ${}_1F_1$, upon using Kummer's second theorem (see [5, p. 126]):

$${}_1F_1(\alpha\,;\,2\alpha\,;\,x) = e^{\frac{x}{2}}\, {}_0F_1\left(-;\,\alpha+\frac{1}{2}\,;\,\frac{x^2}{16}\right), \tag{2.2}$$

becomes

$$L = \frac{1}{\Gamma(a)}\int_0^\infty e^{-(1-2\omega)s} s^{a-1}{}_0F_1(-;6c;s^2)\, {}_0F_1(-;\, 6c;\omega\, s^2)\, {}_0F_1(-;6c;\omega^2\, s^2)\, ds.$$

If we substitute $s^2 = t$, we have

$$L = \frac{1}{2\Gamma(a)}\int_0^\infty e^{-(1-2\omega)\sqrt{t}}t^{\frac{1}{2}a-1}\, {}_0F_1(-;6c;t)\, {}_0F_1(-;6c;\omega\, t)\, {}_0F_1(-;6c;\omega^2\, t)\, dt.$$

Now use the result (1.3) and we have

$$L = \frac{1}{2\Gamma(a)} \int_0^\infty e^{-(1-2\omega)\sqrt{s}} s^{\frac{1}{2}a-1}$$

$$\times {}_2F_7 \left[\begin{matrix} 3c-\frac{1}{4}, \quad 3c+\frac{1}{4} \\ 6c, \; 2c, \; 2c+\frac{1}{3}, \; 2c+\frac{2}{3}, \; 4c-\frac{1}{3}, \; 4c, \; 4c+\frac{1}{3} \end{matrix} ; \left(\frac{4s}{9}\right)^3 \right] ds.$$

Now express the ${}_2F_7$ function in the last identity as a series, change the order of integration and summation, evaluate the inner integral with the help of Gamma function and sum the series, we readily arrive at the right-hand side of (2.1). □

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