

A NOTE ON LACUNARY STATISTICALLY CONVERGENT DOUBLE SEQUENCES OF FUZZY NUMBERS

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ABSTRACT. In this paper we introduce and study lacunary statistical convergence for double sequences of fuzzy numbers and we shall also present some inclusion theorems.

1. Introduction

The concept of fuzzy sets was first introduced by Zadeh [19]. Bounded and convergent sequences of fuzzy numbers were introduced by Matloka [9]. Matloka has shown that every convergent sequences of fuzzy numbers is bounded. Later on sequences of fuzzy numbers have been discussed by Nanda [11], Nuray and Savas [13], Nuray [12], Kwon [8], Mursaleen and Basarir [10] and many others.

The natural density of a set A of positive integers is defined by

$$\delta(A) = \lim_n \frac{1}{n} |\{k \leq n : k \in A\}|,$$

where $|\{k \leq n : k \in A\}|$ denotes the number of elements of $A \subseteq \mathbb{N}$ does not exceeding n . It is clear that any finite subset of \mathbb{N} have zero natural density and $\delta(A^c) = 1 - \delta(A)$. If a property $P(k)$ holds for all $k \in A$ with $\delta(A) = 1$, we say that P holds for almost all k , that is a.a.k.

The concept of statistical convergence was introduced by Fast [4], Buck [2], Schoenberg [17] studied statistical convergence as a summability method and listed some of elementary properties of statistical convergence.

A sequence (x_k) is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$. In this case we write $S - \lim x_k = L$.

By the convergence of a double sequence we mean the convergence in Pringsheim's sense [14]. A double sequence $x = (x_{k\ell})$ has Pringsheim limit L (denoted by $P - \lim x = L$) if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_{k\ell} - L| < \varepsilon$ whenever $k, \ell \geq N$.

Some works in the direction of double sequence is due to Altay and Başar [1] and Tripathy [18].

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Let $C(\mathbb{R}^n) = \{A \subset \mathbb{R}^n : A \text{ compact and convex}\}$. The space $C(\mathbb{R}^n)$ has a linear structure induced by the operations $A + B = \{a + b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$ for $A, B \in C(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. The Hausdorff distance between A and B of $C(\mathbb{R}^n)$ is defined as:

$$\delta_\infty(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\}.$$

It is well known that $(C(\mathbb{R}^n), \delta_\infty)$ is a complete (not separable) metric space.

A fuzzy number is a function X from \mathbb{R}^n to $[0, 1]$ which is normal, fuzzy convex, upper semi-continuous and the closure of $\{x \in \mathbb{R}^n : X(x) > 0\}$ is compact. These properties imply that for each $0 < \alpha \leq 1$, the α -level set $[X]^\alpha = \{x \in \mathbb{R}^n : X(x) \geq \alpha\}$ is a nonempty compact convex subset of \mathbb{R}^n , with support $X^0 = \{x \in \mathbb{R}^n : X(x) > 0\}$. Let $L(\mathbb{R}^n)$ denote the set of all fuzzy numbers. The linear structure of $L(\mathbb{R}^n)$ induces the addition $X + Y$ and the scalar multiplication λX , $\lambda \in \mathbb{R}$, in terms of α -level sets, by

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha \quad [\lambda X]^\alpha = \lambda [X]^\alpha$$

for each $0 \leq \alpha \leq 1$.

Define, for each $1 \leq q < \infty$,

$$d_q(X, Y) = \left(\int_0^1 \delta_\infty(X^\alpha, Y^\alpha)^q d\alpha \right)^{1/q}$$

and $d_\infty = \sup_{0 \leq \alpha \leq 1} \delta_\infty(X^\alpha, Y^\alpha)$, where δ_∞ is the Hausdorff metric. Clearly $d_\infty(X, Y) = \lim_{q \rightarrow \infty} d_q(X, Y)$ with $d_q \leq d_r$ if $q \leq r$. Moreover d_q is a complete, separable and locally compact metric space [3]. Throughout the paper, d will denote d_q with $1 \leq q < \infty$.

Definition 1.1. Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K(n, m)$ be the numbers of (i, j) in K such that $i \leq n$ and $j \leq m$. Then the two-dimensional analogue of natural density can be defined as follows.

The lower asymptotic density of a set $K \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{\delta}_2(K) = \liminf_{n, m} \frac{K(n, m)}{nm}.$$

In case the sequence $(K(n, m)/nm)$ has a limit in Pringsheim's sense then we say that K has a double natural density and is defined as

$$\lim_{n, m} \frac{K(n, m)}{nm} = \delta_2(K).$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathbb{N}\}$. Then

$$\delta_2(K) = \lim_{n, m} \frac{K(n, m)}{nm} \leq \lim_{n, m} \frac{\sqrt{n}\sqrt{m}}{nm} = 0,$$

i.e., the set K has double natural density zero, while the set $\{(i, 2j) : i, j \in \mathbb{N}\}$ has double natural density $1/2$ (see [15]).

Definition 1.2. A double sequence $X = (X_{k\ell})_{k\ell=0}^\infty$ of fuzzy numbers is said to be convergent in the Pringsheim's sense or P -convergent if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(X_{k\ell}, X_0) < \varepsilon$ whenever $k, \ell \geq N$ and we denote by $P - \lim X = X_0$. The number X_0 is called the Pringsheim limit of $X_{k\ell}$, see [15].

Definition 1.3. A double sequence $X = (X_{k\ell})$ is bounded if there exists a positive number M such that $d(X_{k\ell}, X_0) < M$ for all k and ℓ , i.e. if

$$\|X\|_{(\infty,2)} = \sup_{k,\ell} d(X_{k\ell}, X_0) < \infty.$$

We will denote the set of all bounded double sequence by $\ell_\infty^2(F)$, see [15].

Definition 1.4. A sequence $X = (X_{k\ell})$ of fuzzy numbers is said to be statistically convergent to a fuzzy number X_0 if for every $\varepsilon > 0$,

$$P - \lim_{n,m} \frac{1}{mn} |\{k \leq m \text{ and } \ell \leq n : d(X_{k\ell}, X_0) \geq \varepsilon\}| = 0.$$

In this case we write $st_2 - \lim X_{k,\ell} = X_0$, and we denote the set of all statistically convergent double sequences by st_2 , see [15].

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio k_r/k_{r-1} will be abbreviated by q_r . The space of lacunary strongly convergent sequences was defined by Freedman et al. [5] as follows :

$$N_\theta = \left\{ x = (x_k) : \lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0 \text{ for some } L \right\}.$$

Later on lacunary sequence have been discussed in (see Firdy and Orhan [6], [7], Savaş and Patterson [16]).

The following concepts are due to Savaş and Patterson [16] .

By a double lacunary sequence, we mean an increasing sequence $\theta_{r,s} = \{(k_r, \ell_s)\}$ of positive integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \rightarrow \infty \text{ as } r \rightarrow \infty$$

and

$$\ell_0 = 0, \quad \bar{h}_s = \ell_s - \ell_{s-1} \rightarrow \infty \text{ as } s \rightarrow \infty.$$

Notations: $k_{r,s} = k_r \ell_s, h_{r,s} = h_r \bar{h}_s$

Throughout this paper, the intervals determined by $\theta_{r,s}$ will be denoted by

$$I_{r,s} = \{(k, \ell) : k_{r-1} < k \leq k_r \text{ or } \ell_{s-1} < \ell \leq \ell_s\},$$

and the ratios $\frac{k_r}{k_{r-1}}, \frac{\ell_s}{\ell_{s-1}}$ will be abbreviated by q_r, \bar{q}_s , respectively and $q_{r,s} = q_r \bar{q}_s$.

We define the lacunary statistical analogue for double sequences $X = (X_{k\ell})$ of fuzzy numbers.

Definition 1.5. A sequence $X = (X_{kl})$ of fuzzy numbers is said to be lacunary statistically convergent to a fuzzy number X_0 if for every $\varepsilon > 0$,

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k, \ell) \in I_{r,s} : d(X_{kl}, X_0) \geq \varepsilon\}| = 0.$$

In this case we write $s_{\theta t_2} - \lim X_{kl} = X_0$, and we denote the set of all statistically convergent double sequences by $s_{\theta t_2}$.

2. Main results

In this section we prove the results of this paper.

Theorem 2.1. *Let (X_{kl}) and (Y_{kl}) be double sequences of fuzzy numbers.*

- (a) *If $s_{\theta t_2} - \lim X_{kl} = X_0$ and $c \in \mathbb{R}$, then $s_{\theta t_2} - \lim cX_{kl} = cX_0$.*
- (b) *If $s_{\theta t_2} - \lim X_{kl} = X_0$ and $s_{\theta t_2} - \lim Y_{kl} = Y_0$, then $s_{\theta t_2} - \lim (X_{kl} + Y_{kl}) = X_0 + Y_0$.*

Proof. Let $\alpha \in [0, 1]$, $c \in \mathbb{R}$ and $\varepsilon > 0$. Let X_{kl}^α , Y_{kl}^α , X_0^α and Y_0^α be α -level sets of X_{kl} , Y_{kl} , X_0 and Y_0 , respectively. Then since $\delta_\infty(cX_{kl}^\alpha, cX_0^\alpha) = |c| \delta_\infty(X_{kl}^\alpha, X_0^\alpha)$ we have $d(cX_{kl}, cX_0) = |c| d(X_{kl}, X_0)$ and this gives

$$\begin{aligned} & \frac{1}{h_{r,s}} |\{(k, \ell) \in I_{r,s} : d(cX_{kl}, cX_0) \geq \varepsilon\}| \\ & \leq \frac{1}{h_{r,s}} \left| \left\{ (k, \ell) \in I_{r,s} : d(X_{kl}, X_0) \geq \frac{\varepsilon}{|c|} \right\} \right|. \end{aligned}$$

Now we write,

$$\delta_\infty(X_{kl}^\alpha + Y_{kl}^\alpha, X_0^\alpha + Y_0^\alpha) \leq \delta_\infty(X_{kl}^\alpha, X_0^\alpha) + \delta_\infty(Y_{kl}^\alpha, Y_0^\alpha).$$

Thus, by Minkowski's inequality, d is a metric we have

$$d(X_{kl} + Y_{kl}, X_0 + Y_0) \leq d(X_{kl}, X_0) + d(Y_{kl}, Y_0).$$

Therefore

$$\begin{aligned} & \frac{1}{h_{r,s}} |\{(k, \ell) \in I_{r,s} : d(X_{kl} + Y_{kl}, X_0 + Y_0) \geq \varepsilon\}| \\ & \leq \frac{1}{h_{r,s}} |\{(k, \ell) \in I_{r,s} : d(X_{kl}, X_0) + d(Y_{kl}, Y_0) \geq \varepsilon\}| \\ & \leq \frac{1}{h_{r,s}} \left| \left\{ (k, \ell) \in I_{r,s} : d(X_{kl}, X_0) \geq \frac{\varepsilon}{2} \right\} \right| \\ & \quad + \frac{1}{h_{r,s}} \left| \left\{ (k, \ell) \in I_{r,s} : d(Y_{kl}, Y_0) \geq \frac{\varepsilon}{2} \right\} \right|. \end{aligned}$$

This completes the proof. □

Theorem 2.2. *For any double lacunary sequence $\theta_{r,s}$, $s_{\theta t_2} \subset st_2$ if $\limsup_r q_r < \infty$ and $\limsup_s \bar{q}_s < \infty$.*

Proof. Suppose $\limsup_r q_r < \infty$ and $\limsup_s \bar{q}_s < \infty$. Then there exists $H > 0$ such that $q_r < H$ and $\bar{q}_s < H$ for all r and s . Suppose that $X_{kl} \xrightarrow{P} X_0(st_2)$ and

$$N_{r,s} = |\{(k, \ell) \in I_{r,s} : d(X_{kl}, X_0) \geq \varepsilon\}|.$$

By the definition of $X_{kl} \xrightarrow{P} X_0(st_2)$ given $\varepsilon > 0$ there exists $r_0 \in \mathbb{N}$ such that $\frac{N_{r,s}}{h_{r,s}} < \varepsilon$ for all r and $s > r_0$. Let

$$M = \max \{N_{r,s} : 1 \leq r \leq r_0 \text{ and } 1 \leq s \leq r_0\}.$$

Let n and m be such that $k_{r-1} < m \leq k_r$ and $\ell_{s-1} < n \leq \ell_s$. Therefore we obtain the following:

$$\begin{aligned} & \frac{1}{mn} |\{k \leq m \text{ and } \ell \leq n : d(X_{kl}, X_0) \geq \varepsilon\}| \\ & \leq \frac{1}{k_{r-1}\ell_{s-1}} |\{k \leq k_r \text{ and } \ell \leq \ell_s : d(X_{kl}, X_0) \geq \varepsilon\}| \\ & = \frac{1}{k_{r-1}\ell_{s-1}} \left\{ \sum_{i,j=1,1}^{r,s} N_{i,j} \right\} \\ & \leq \frac{Mr_0^2}{k_{r-1}\ell_{s-1}} + \varepsilon H^2 \end{aligned}$$

on the result follows immediately. □

Theorem 2.3. *For any double lacunary sequence $\theta_{r,s}$, $st_2 \subseteq s\theta t_2$ if $\liminf q_r > 1$ and $\liminf \bar{q}_s > 1$.*

Proof. Suppose $\liminf q_r > 1$ and $\liminf \bar{q}_s > 1$, then there exists $\delta > 0$ such that both $q_r > 1 + \delta$ and $\bar{q}_s > 1 + \delta$. This implies $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$ and $\frac{\bar{h}_s}{\ell_s} \geq \frac{\delta}{1+\delta}$.

If $X_{kl} \rightarrow X_0(st_2)$ then for each $\varepsilon > 0$ and for sufficiently large r and s , we have

$$\begin{aligned} & \frac{1}{k_{r,s}} |\{k \leq k_r \text{ and } \ell \leq \ell_s : d(X_{kl}, X_0) \geq \varepsilon\}| \\ & \geq \frac{1}{k_{r,s}} |\{(k, \ell) \in I_{r,s} : d(X_{kl}, X_0) \geq \varepsilon\}| \\ & \geq \left(\frac{\delta}{1+\delta}\right)^2 \frac{1}{h_{r,s}} |\{(k, \ell) \in I_{r,s} : d(X_{kl}, X_0) \geq \varepsilon\}| \end{aligned}$$

which proves the theorem. □

Theorem 2.4. *For any double lacunary sequence $\theta_{r,s}$, $st_2 = s\theta t_2$ if $1 < \liminf q_r \leq \limsup q_r < \infty$ and $1 < \liminf \bar{q}_s \leq \limsup \bar{q}_s < \infty$.*

Theorem 2.5. *If $\{X_{kl}\} \in st_2 \cap s\theta t_2$, then we have $s\theta t_2 - \lim X_{kl} = s - \lim X_{kl}$.*

Proof. Suppose that $st_2 - \lim X_{k\ell} = X_0$ and $s_{\theta}t_2 - \lim X_{k\ell} = X'_0$ and that $X_0 \neq X'_0$. Then, we have $d(X_0, X'_0) > 0$, $\frac{d(X_0, X'_0)}{2} > \varepsilon > 0$, we have

$$P - \lim_{m,n} \frac{1}{mn} |\{k \leq m \text{ and } \ell \leq n : d(X_{k\ell}, X_0) \geq \varepsilon\}| = 1.$$

Let us now consider the $k_p\ell_v$ -th term of the following expression.

$$\begin{aligned} & \frac{1}{mn} |\{k \leq m \text{ and } \ell \leq n : d(X_{k\ell}, X_0) \geq \varepsilon\}| \\ &= \frac{1}{k_p\ell_v} |\{(k, \ell) \in \cup_{r,s}^{p,v} I_{r,s} : d(X_{k\ell}, X'_0) \geq \varepsilon\}| \\ (2.1) \quad &= \frac{1}{k_p\ell_v} \sum_{r,s=1,1}^{p,v} |\{(k, \ell) \in I_{r,s} : d(X_{k\ell}, X'_0) \geq \varepsilon\}| \\ &\leq \frac{1}{\sum_{r,s=1,1}^{p,v} h_{r,s}} \sum_{r,s=1,1}^{p,v} h_{r,s} t_{r,s}, \end{aligned}$$

where $t_{r,s} = \frac{1}{h_{r,s}} |\{(k, \ell) \in I_{r,s} : d(X_{k\ell}, X'_0) \geq \varepsilon\}|$ is a Pringsheim null sequence, since $st_2 - \lim X_{k\ell} = X'_0$. Since $\theta_{r,s}$ is a double lacunary sequence and

$$\frac{1}{\sum_{r,s=1,1}^{p,v} h_{r,s}} \sum_{r,s=1,1}^{p,v} h_{r,s} t_{r,s}$$

is a regular weighted mean transform of $(t_{r,s})$ also tends zero. Since (2.1) is a subsequence of $\left\{ \frac{1}{mn} |\{k \leq m \text{ and } \ell \leq n : d(X_{k\ell}, X_0) \geq \varepsilon\}| \right\}_{m,n=1}^{\infty}$, we have

$$\frac{1}{mn} |\{k \leq m \text{ and } \ell \leq n : d(X_{k\ell}, X'_0) \geq \varepsilon\}| \nrightarrow 1$$

which contradicts to the fact that $X_0 \neq X'_0$. \square

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References

- [1] B. Altay and F. Başar, *Some new spaces of double sequences*, J. Math. Anal. Appl. **309** (2005), no. 1, 70–90.
- [2] R. C. Buck, *Generalized asymptotic density*, Amer. J. Math. **75** (1953), 335–346.
- [3] P. Diamond and P. Kloeden, *Metric spaces of fuzzy sets*, Fuzzy Sets and Systems **35** (1990), 241–249.
- [4] H. Fast, *Sur la convergence statistique*, Colloquium Math. **2** (1951), 241–244.
- [5] A. R. Freedman, J. J. Sember, and M. Raphael, *Some Cesàro-type summability spaces*, Proc. London Math. Soc. **37** (1978), no. 3, 508–520.
- [6] J. A. Fridy and C. Orhan, *Lacunary statistical convergence*, Pacific J. Math. **160** (1993), no. 1, 43–51.
- [7] ———, *Lacunary statistical summability*, J. Math. Anal. Appl. **173** (1993), no. 2, 497–504.

- [8] J. S. Kwon, *On statistical and p -Cesàro convergence of fuzzy numbers*, Korean J. Comput. Appl. Math. **7** (2000), no. 1, 195–203.
- [9] M. Matloka, *Sequences of fuzzy numbers*, BUSEFAL **28** (1986), 28–37.
- [10] M. Mursaleen and M. Başarır, *On some new sequence spaces of fuzzy numbers*, Indian J. Pure and Appl. Math. **34** (2003), no. 9, 1351–1357.
- [11] S. Nanda, *On sequences of fuzzy numbers*, Fuzzy Sets and Systems **33** (1989), 123–126.
- [12] F. Nuray, *Lacunary statistical convergence of sequences of fuzzy numbers*, Fuzzy Sets and Systems **99** (1998), 353–356.
- [13] F. Nuray and E. Savas, *Statistical convergence of fuzzy numbers*, Math. Slovaca **45** (1995), no. 3, 269–273.
- [14] A. Pringsheim, *Zur Theorie der zweifach unendlichen Zahlenfolgen*, Math. Ann. **53** (1900), no. 3, 289–321.
- [15] E. Savaş and M. Mursaleen, *On statistically convergent double sequences of fuzzy numbers*, Inform. Sci. **162** (2004), no. 3-4, 183–192.
- [16] E. Savaş and R. F. Patterson, *Lacunary statistical convergence of multiple sequences*, Appl. Math. Lett. **20** (2007), no. 9, 964–970.
- [17] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959), 361–375.
- [18] B. C. Tripathy, *Statistically convergent double sequences*, Tamkang J. Math. **34** (2003), no. 3, 231–237.
- [19] L. A. Zadeh, *Fuzzy sets*, Inform and Control **8** (1965), 338–353.

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