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# NOTES ON A NON-ASSOCIATIVE ALGEBRAS WITH EXPONENTIAL FUNCTIONS III

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Abstract. For $\mathbb{F}[e^{\pm x}]_{\{\partial\}}$, all the derivations of the evaluation algebra $\mathbb{F}[e^{\pm x}]_{\{\partial\}}$ is found in the paper (see [16]). For $M = \{\partial_1, \partial_1^2\}$, $Der_{non}(\mathbb{F}[e^{\pm x}]_M))$ of the evaluation algebra $\mathbb{F}[e^{\pm x}, e^{\pm y}]_M$ is found in the paper (see [2]). For $M = \{\partial_1^2, \partial_2^2\}$, we find $Der_{non}(\mathbb{F}[e^{\pm x}, e^{\pm y}]_M))$ of the evaluation algebra $\mathbb{F}[e^{\pm x}, e^{\pm y}]_M$ in this paper.

## 1. Preliminaries

Let $\mathbb{F}$ be a field of characteristic zero (not necessarily algebraically closed). Throughout this paper, $\mathbb{N}$ and $\mathbb{Z}$ will denote the non-negative integers and the integers, respectively. Let $A$ be an associative algebra and $M=\{\delta \mid \delta$ is a mapping from $A$ to itself $\}$. The evaluation algebra $A_M = \{a\delta | a \in A, \delta \in M\}$ with the obvious addition and the multiplication $*$ is defined as follows:

$$a_1\delta_1 * a_2\delta_2 = a_1\delta_1(a_2)\delta_2$$

for any $a_1\delta_1, a_2\delta_2 \in A_M$ (see [1], [3], [4], and [13]). For $A_M$, if $M = \{id\}$, then the ring $A_M = A$ where $id$ is the identity map of $A$. Note that $A_M = \langle A_M, +, *\rangle$ is not an associative ring generally (see [15]). Using the commutator $[,]$ of $A_M$, we can define the semi-Lie ring (see [1]). If the Jacobi identity holds in $A_{M[,]}$, then $A_{M[,]}$ is a Lie ring (see [14]). Generally, $A_{M[,]}$ is not a Lie ring, because of the Jacobi identity. Let $\mathbb{F}[e^{\pm x_1}, e^{\pm x_2}, \ldots, e^{\pm x_n}]$ be a ring in the formal power series ring $\mathbb{F}[[x_1, x_2, \ldots, x_n]]$ (see [6] and [7]). If we take the subalgebra $\mathbb{F}[e^{\pm x}, e^{\pm y}]$ in $\mathbb{F}[[x_1, x_2, \ldots, x_n]]$ and the map $M = \{\partial_1^2, \partial_2^2\}$, then we define the simple evaluation algebra $\mathbb{F}[e^{\pm x}, e^{\pm y}]_M$ (see [5], [9], [10], [11], and [12]). It is well known that the non-associative algebras $\mathbb{F}[e^{\pm x}, e^{\pm y}]_{\{\partial_1^2, \partial_2^2\}}$ and $\mathbb{F}[e^{\pm x}, e^{\pm y}]_{\{\partial_1, \partial_2^2\}}$ are simple (see [1], [3], and [16]). Note that $\mathbb{F}[e^{\pm x}, e^{\pm y}]_{\partial_1^2, \partial_2^2}$ can be decomposed as follows:

$$\mathbb{F}[e^{\pm x}, e^{\pm y}]_{\partial_1^2, \partial_2^2} = A_1 \oplus A_2 \tag{1}$$

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where $A_1$ (resp. $A_2$) is the subalgebra spanned by $\{e^{ax}e^{iy}\partial_1^2 | a, i \in \mathbb{Z}\}$ (resp. $\{e^{bx}e^{jy}\partial_2^2 | b, j \in \mathbb{Z}\}$) in $\mathbb{F}[e^{\pm x}, e^{\pm y}]_{\partial_1^2, \partial_2^2}$.

## 2. Derivations of $\mathbb{F}[e^{\pm x}, e^{\pm y}]_M$

From now on, $M$ denotes the set $\{\partial_1^2, \partial_2^2\}$.

**Lemma 2.1.** *For any $D \in Der_{non}(\mathbb{F}[e^{\pm x}, e^{\pm y}]_M)$ if $D(\partial_1^2) = 0$ and $D(\partial_2^2) = 0$, then we have the followings:*

$$D(e^x e^y \partial_r^2) = (a_{1,1,0} + b_{2,0,1}) e^x e^y \partial_r^2,$$

*where $r \in \{1, 2\}$ with appropriate scalars.*

*Proof.* Let $D$ be the derivation of $\mathbb{F}[e^{\pm x}, e^{\pm y}]_M$ in the lemma. Since $\partial_2^2$ is in the left annihilator of $e^x\partial_1^2$, we have that $\partial_2^2 * D(e^x\partial_1^2) = 0$. This implies that $D(e^x\partial_1^2) = \sum_i a_{1,i,0} e^{ix}\partial_1^2 + \sum_i a_{2,i,0} e^{ix}\partial_2^2$ with appropriate scalars. Since $\partial_1^2$ is a left (multiplicative) identity of $e^x\partial_1^2$ and by assumption, we have that

$$\partial_1^2 * (\sum_i a_{1,i,0} e^{ix}\partial_1^2 + \sum_i a_{2,i,0} e^{ix}\partial_2^2) = \sum_i a_{1,i,0} e^{ix}\partial_1^2 + \sum_i a_{2,i,0} e^{ix}\partial_2^2 \tag{2}$$

with appropriate scalars. By (2), we have that $i = \pm 1$, i.e.,

$$D(e^x\partial_1^2) = a_{1,1,0} e^x\partial_1^2 + a_{1,-1,0} e^{-x}\partial_1^2 + a_{2,1,0} e^x\partial_2^2 + a_{2,-1,0} e^{-x}\partial_2^2.$$

Similarly, we can prove that

$$D(e^y\partial_2^2) = b_{1,0,1} e^y\partial_1^2 + b_{1,0,-1} e^{-y}\partial_1^2 + b_{2,0,1} e^y\partial_2^2 + b_{2,0,-1} e^{-y}\partial_2^2$$

with appropriate scalars. Since $e^y\partial_2^2$ is in the right annihilator of $e^x\partial_1^2$, we have that $a_{2,1,0} = a_{2,-1,0} = 0$, i.e.,

$$D(e^x\partial_1^2) = a_{1,1,0} e^x\partial_1^2 + a_{1,-1,0} e^{-x}\partial_1^2.$$

Symmetrically, we can also prove that $D(e^y\partial_2^2) = b_{2,0,1} e^y\partial_2^2 + b_{2,0,-1} e^{-y}\partial_2^2$. By $D(e^{-x}\partial_1^2 * e^x\partial_1^2) = 0$, we have that $D(e^{-x}\partial_1^2) * e^x\partial_1^2 = -a_{1,1,0}\partial_1^2 - a_{1,-1,0} e^{-2x}\partial_1^2$. This implies that

$$\begin{aligned} D(e^{-x}\partial_1^2) &= -a_{1,1,0} e^{-x}\partial_1^2 - a_{1,-1,0} e^{-3x}\partial_1^2 \\ &\quad + \sum_{i,j} c_{2,i,j} e^{ix} e^{jy} \partial_2^2 \end{aligned} \tag{3}$$

with appropriate scalars. Since $\partial_1^2$ is a left identity of $e^{-x}\partial_1^2$, we have that $a_{1,-1,0} = 0$, i.e., $D(e^x\partial_1^2) = a_{1,1,0} e^x\partial_1^2$. Similarly, we can also prove that $D(e^y\partial_2^2) = b_{2,0,1} e^y\partial_2^2$. Since $\partial_2^2$ is in the left annihilator of $e^{-x}\partial_1^2$, we have that either $j = 0$ or $c_{2,i,j} = 0$. On the other hand, since $\partial_1^2$ is a left identity of $e^{-x}\partial_1^2$, we can prove that $i \in \{1, -1\}$. If $c_{2,i,j} = 0$ holds for all $i, j \in \mathbb{Z}$, then we have that $D(e^{-x}\partial_1^2) = -a_{1,1,0} e^{-x}\partial_1^2$. If $c_{2,i,j} \neq 0$ for some $i, j \in \mathbb{Z}$, then $j = 0$ and we have the following two cases, Case I: $i = 1$ and Case II: $i = -1$.

**Case I.** Let us assume that $i = 1$ holds. We have that

$$D(e^{-x}\partial_1^2) = -a_{1,1,0} e^{-x}\partial_1^2 + c_{2,1,0} e^x\partial_2^2.$$

Since $e^x\partial_1^2$ is in the left annihilator of $e^{-x}\partial_1^2$, we can prove that $c_{2,1,0}=0$.

**Case II.** Let us assume that $i=-1$ holds. We have that

$$D(e^{-x}\partial_1^2) = -a_{1,1,0}e^{-x}\partial_1^2 + c_{2,-1,0}e^{-x}\partial_2^2.$$

By $D(e^x\partial_1^2 * e^{-x}\partial_1^2)=0$, we prove that $c_{2,-1,0}=0$.

Thus, by the cases I and II, we have that $D(e^{-x}\partial_1^2) = -a_{1,1,0}e^{-x}\partial_1^2$. Similarly, we can prove that

$$D(e^x\partial_2^2) = a_{1,1,0}e^x\partial_2^2 \tag{4}$$
$$D(e^{-x}\partial_2^2) = -a_{1,1,0}e^{-x}\partial_2^2.$$

By $D(e^xe^y\partial_1^2 * e^{-x}\partial_2^2) = D(e^y\partial_2^2)$, we have that

$$D(e^xe^y\partial_1^2) * e^{-x}\partial_2^2 = a_{1,1,0}e^y\partial_2^2 + b_{2,0,1}e^y\partial_2^2.$$

This implies that

$$D(e^xe^y\partial_1^2) = (a_{1,1,0}+b_{2,0,1})e^xe^y\partial_1^2 + \sum_{i,j} t_{2,i,j}e^{ix}e^{jy}\partial_2^2 \tag{5}$$

with appropriate scalars. Since $e^y\partial_2^2$ is in the right annihilator of $e^xe^y\partial_1^2$, we also have that $\sum_{i,j} t_{2,i,j}e^{ix}e^{(j+1)y}\partial_2^2 = 0$. This implies that $t_{2,i,j}=0$ for $i,j\in\mathbb{Z}$. Thus, we have that

$$D(e^xe^y\partial_1^2) = (a_{1,1,0}+b_{2,0,1})e^xe^y\partial_1^2. \tag{6}$$

By $D(e^{-x}\partial_2^2 * e^xe^y\partial_2^2) = D(e^y\partial_2^2)$, we have that

$$e^{-x}\partial_2^2 * D(e^xe^y\partial_2^2) = (a_{1,1,0}+b_{2,0,1})e^y\partial_2^2.$$

This implies that

$$D(e^xe^y\partial_2^2) = (a_{1,1,0}+b_{2,0,1})e^xe^y\partial_2^2 + \sum_i u_{1,i,0}e^{ix}\partial_1^2 + \sum_i u_{2,i,0}e^{ix}\partial_2^2 \tag{7}$$

with appropriate scalars. Since $\partial_2^2$ is a left identity of $e^xe^y\partial_2^2$, we have that

$$D(e^xe^y\partial_2^2) = (a_{1,1,0}+b_{2,0,1})e^xe^y\partial_2^2 \tag{8}$$

(6) and (8) are the required forms in the lemma. Therefore we have proven the lemma. □

**Lemma 2.2.** *For any $D\in Der_{non}(\mathbb{F}[e^{\pm x},e^{\pm y}]_M)$ and $e^{ix}e^{jy}\partial_r^2 \in \mathbb{F}[e^{\pm x},e^{\pm y}]_M$, if $D(\partial_1^2)=0$ and $D(\partial_2^2)=0$, then we have the following equality*

$$D(e^{ix}e^{jy}\partial_r^2) = (ia_{1,1,0}+jb_{2,0,1})e^{ix}e^{jy}\partial_r^2$$

*holds where $r\in\{1,2\}$ and $a_{1,1,0}, b_{2,0,1}\in\mathbb{F}$ .*

*Proof.* Let $D$ be the derivation of $\mathbb{F}[e^{\pm x},e^{\pm y}]_M$ in the lemma. Since $D(e^x\partial_1^2 * e^x\partial_1^2) = D(e^{2x}\partial_1^2)$, we have that $D(e^x\partial_1^2) * e^x\partial_1^2 + e^x\partial_1^2 * D(e^x\partial_1^2) = D(e^{2x}\partial_1^2)$. Thus we have that $D(e^{2x}\partial_1^2) = 2a_{1,1,0}e^{2x}\partial_1^2$. By $D(e^x\partial_1^2 * e^{2x}\partial_1^2) = 2D(e^{3x}\partial_1^2)$, we can prove that $D(e^{3x}\partial_1^2) = 3a_{1,1,0}e^{3x}\partial_1^2$. By induction on $i$ of $e^{ix}\partial_1^2$, we also prove that $D(e^{ix}\partial_1^2) = ia_{1,1,0}e^{ix}\partial_1^2$. Similarly, we can prove that $D(e^{jy}\partial_2^2) = jb_{2,0,1}e^{jx}\partial_2^2$. By $D(e^{(i-1)x}\partial_1^2 * e^x e^y \partial_1^2) = D(e^{ix}e^y\partial_1^2)$, we have that

$$\begin{aligned} D(e^{ix}e^y\partial_1^2) &= D(e^{(i-1)x}\partial_1^2) * e^x e^y \partial_1^2 + e^{(i-1)x}\partial_1^2 * D(e^x e^y \partial_1^2) \\ &= (i-1)a_{1,1,0}e^{ix}e^y\partial_1^2 + e^{(i-1)x}\partial_1^2 * \{(a_{1,1,0} + b_{2,0,1})e^x e^y \partial_1^2\} \\ &= (ia_{1,1,0} + b_{2,0,1})e^{ix}e^y\partial_1^2. \end{aligned}$$

Since $D(e^{(j-1)y}\partial_2^2 * e^{ix}e^y\partial_1^2) = D(e^{ix}e^{jy}\partial_1^2)$, we prove that

$$\begin{aligned} D(e^{ix}e^{jy}\partial_1^2) &= D(e^{(j-1)y}\partial_2^2) * e^{ix}e^y\partial_1^2 + e^{(j-1)y}\partial_2^2 * D(e^{ix}e^y\partial_1^2) \\ &= (j-1)b_{2,0,1}e^{ix}e^{jy}\partial_1^2 + e^{(j-1)y}\partial_2^2 * (ia_{1,1,0} + jb_{2,0,1})e^{ix}e^y\partial_1^2 \\ &= (ia_{1,1,0} + jb_{2,0,1})e^{ix}e^{iy}\partial_1^2. \end{aligned}$$

By $D(e^{(i-1)x}\partial_1^2 * e^x e^y \partial_2^2) = D(e^{ix}e^y\partial_2^2)$, we also have that

$$\begin{aligned} D(e^{ix}e^y\partial_2^2) &= D(e^{(i-1)x}\partial_1^2) * e^x e^y \partial_2^2 + e^{(i-1)x}\partial_1^2 * D(e^x e^y \partial_2^2) \\ &= (i-1)a_{1,1,0}e^{(i-1)x}\partial_1^2 * e^x e^y \partial_2^2 \\ &\quad + e^{(i-1)x}\partial_1^2 * (a_{1,1,0} + b_{2,0,1})e^x e^y \partial_2^2 \\ &= (i-1)a_{1,1,0}e^{ix}e^y\partial_2^2 + (a_{1,1,0} + b_{2,0,1})e^{(i)x}e^y\partial_2^2 \\ &= (ia_{1,1,0} + b_{2,0,1})e^{ix}e^y\partial_2^2. \end{aligned}$$

Since $D(e^{(j-1)y}\partial_2^2 * e^{ix}e^y\partial_2^2) = D(e^{ix}e^{jy}\partial_2^2)$, we can prove that

$$\begin{aligned} D(e^{ix}e^{jy}\partial_2^2) &= D(e^{(j-1)y}\partial_2^2) * e^{ix}e^y\partial_2^2 + e^{(j-1)y}\partial_2^2 * D(e^{ix}e^y\partial_2^2) \\ &= (j-1)b_{2,0,1}e^{ix}e^{jy}\partial_2^2 + (ia_{1,1,0} + b_{2,0,1})e^{ix}e^{jy}\partial_2^2 \\ &= (ia_{1,1,0} + jb_{2,0,1})e^{ix}e^{jy}\partial_2^2. \end{aligned}$$

Therefore we have proven the lemma. □

**Lemma 2.3.** *For any $D \in Der_{non}(\mathbb{F}[e^{\pm x}, e^{\pm y}]_M)$, $D(\partial_1^2) = D(\partial_2^2) = 0$ holds.*

*Proof.* Let $D$ be any derivation of $\mathbb{F}[e^{\pm x}, e^{\pm y}]_M$. By $\partial_1^2 * D(\partial_1^2) = 0$ and $\partial_2^2 * D(\partial_1^2) = 0$, we have that $D(\partial_1^2) = c_1\partial_1^2 + c_2\partial_2^2$ for $c_1, c_2 \in \mathbb{F}$. Similarly, we can prove that $D(\partial_2^2) = c_3\partial_1^2 + c_4\partial_2^2$ for $c_3, c_4 \in \mathbb{F}$. By $D(\partial_2^2 * e^x\partial_1^2) = 0$, we have that $\partial_2^2 * D(e^x\partial_1^2) = -c_3e^x\partial_1^2$. This implies that $c_3 = 0$. Symmetrically, we can prove that $c_2 = 0$, i.e., $D(\partial_1^2) = c_1\partial_1^2$ and $D(\partial_2^2) = c_4\partial_2^2$. Since $\partial_1^2$ is a left identity of $e^x\partial_1^2$, we have that

$$c_1e^x\partial_1^2 + \partial_1^2 * D(e^x\partial_1^2) = D(e^x\partial_1^2). \tag{9}$$

If $c_1 \neq 0$, then there is no element $D(e^x\partial_1^2)$ which holds (9). This contradiction implies that $c_1 = 0$. Similarly, we can also prove that $c_4 = 0$. This completes the proof of the lemma. □

**Notes.** For any basis element $e^x e^y \partial_r^2$, $r \in \{1, 2\}$, of $\mathbb{F}[e^{\pm x}, e^{\pm y}]_M$ and for $c_1, c_2 \in \mathbb{F}$, if we define $\mathbb{F}$-linear maps $D_{c_1,c_2}$ as follows:

$$D_{c_1,c_2}(e^{ix}e^{jy}\partial_r^2) = (ic_1 + jc_2)e^{ix}e^{jy}\partial_r^2$$

then $D_{c_1,c_2}$ can be linearly extended to a derivation of $\mathbb{F}[e^{\pm x}, e^{\pm y}]_M$.

**Theorem 2.1.** *$Der_{non}(\mathbb{F}[e^{\pm x}, e^{\pm y}]_M)$ of the non-associative algebra*

$$\mathbb{F}[e^{\pm x}, e^{\pm y}]_M$$

*is generated by $D_{c_1,c_2}$, $c_1, c_2 \in \mathbb{F}$, which are defined in Notes.*

*Proof.* Let $D$ be any derivation of $\mathbb{F}[e^{\pm x}, e^{\pm y}]_M$. By Lemma 2.3, we have that $D(\partial_1^2) = D(\partial_2^2) = 0$. So by Lemma 2.1 and Lemma 2.2, by taking appropriate scalars $c_1$ and $c_2$, we have that $D = D_{c_1,c_2}$ which is defined in Notes. Thus we have proven the theorem. □

**Corollary 2.1.** *For any $D$ in $Der_{non}(\mathbb{F}[e^{\pm x}, e^{\pm y}]_M)$, $D$ is the sum of two outer derivations $D_{c_1\partial_1}$ and $D_{c_2\partial_2}$ where $\partial_1$ and $\partial_2$ are the usual partial derivatives of the $\mathbb{F}$-algebra $\mathbb{F}[e^{\pm x}, e^{\pm y}]$ and $c_1, c_2 \in \mathbb{F}$.*

**Corollary 2.2.** *For any $D$ in $Der_{non}(\mathbb{F}[e^{\pm x}, e^{\pm y}]_M)$, $D(A_1) \subset A_1$ and $D(A_2) \subset A_2$ hold.*

**Proposition 2.1.** *If $M_1$ is either $\{\partial_1\}$ or $\{\partial_1^2\}$, then*

$$Hom_{non}(\mathbb{F}[e^{\pm x}, e^{\pm y}]_M, \mathbb{F}[e^{\pm x}]_{M_1}) = \{0\},$$

*where $0$ is the zero algebra automorphism and $Hom_{non}(\mathbb{F}[e^{\pm x}, e^{\pm y}]_M, \mathbb{F}[e^{\pm x}]_{M_1})$ is the set of all non-associative algebra homomorphisms from the algebra*

$$\mathbb{F}[e^{\pm x}, e^{\pm y}]_M$$

*to the algebra $\mathbb{F}[e^{\pm x}]_{M_1}$.*

*Proof.* Let us assume that there is a non-zero algebra homomorphism $\theta$ from the algebra $\mathbb{F}[e^{\pm x}, e^{\pm y}]_M$ to the algebra $\mathbb{F}[e^{\pm x}]_{M_1}$. Since $\theta$ is injective and the right annihilator of $\mathbb{F}[e^{\pm x}, e^{\pm y}]_M$ is spanned by $\partial_1^2$ and $\partial_2^2$, we can derive a contradiction easily. This completes the proof of the proposition. □

**Corollary 2.3.** *If $M_1$ is either $\{\partial_1\}$ or $\{\partial_1^2\}$, then there is no algebra isomorphism from the algebra $\mathbb{F}[e^{\pm x}, e^{\pm y}]_M$ to the algebra $\mathbb{F}[e^{\pm x}]_{M_1}$.*

**Theorem 2.2.** *There is no algebra isomorphism from the algebra*

$$\mathbb{F}[e^{\pm x}, e^{\pm y}]_{\partial_1,\partial_2^2}$$

*to the algebra $\mathbb{F}[e^{\pm x}, e^{\pm y}]_{\partial_1^2,\partial_2^2}$ as non-associative algebras.*

*Proof.* Let us assume that there is an isomorphism $\theta$ from $\mathbb{F}[e^{\pm x}, e^{\pm y}]_{\partial_1,\partial_2^2}$ to $\mathbb{F}[e^{\pm x}, e^{\pm y}]_{\partial_1^2,\partial_2^2}$ as non-associative algebras. We know that $\theta(\partial_1) = c_1\partial_1^2 + c_2\partial_2^2$ where $c_1, c_2 \in \mathbb{F}$. Let us assume that $c_1$ and $c_2$ are non-zero scalars. This implies that $\theta(\partial_2^2) = c_3\partial_1^2 + c_4\partial_2^2$ such that $c_1c_4 - c_2c_3 \neq 0$ where $c_3, c_4 \in \mathbb{F}$. By $(c_1\partial_1^2 + c_2\partial_2^2) * \theta(e^y\partial_2^1) = 0$, we have that

$$\theta(e^y\partial_1) = \sum_{r,u} C(a_r, i_r, u)e^{a_r x}e^{i_r y}\partial_u^2 \tag{10}$$

such that $c_1a_r + c_2i_r = 0$ for $r$ and $1 \leq u \leq 2$. Since $c_1$ and $c_2$ are non-zero scalars, $a_r$ and $i_r$ are non-zeroes. Since $e^y\partial_1^2$ annihilates itself, $a_r^2 + i_r^2 = 0$. Since $a_r$ and $i_r$ are integers, we have that $i_r = 0$. This contradicts the assumption. Thus either $c_1$ is zero or $c_2$ is zero. Let us assume that $c_2 = 0$, i.e., $\theta(\partial_1) = c_1\partial_1^2$. Similarly, we can prove that $\theta(\partial_2^2) = c_4\partial_2^2$ for $c_4 \in \mathbb{F}^\bullet$. Since $\partial_1$ is a left identity of $e^x\partial_1$ and it is in the right annihilator of $\partial_2^2$, we have that $\theta(e^x\partial_1) = d_1e^x\partial_1^2 + d_2e^x\partial_2^2$ where $d_1, d_2 \in \mathbb{F}$. Similarly, we can prove that $\theta(e^y\partial_2^2) = d_3e^y\partial_1^2 + d_4e^y\partial_2^2$ where $d_3, d_4 \in \mathbb{F}$. Since $e^x\partial_1$ and $e^y\partial_2^2$ annihilates each other, we can prove that $d_2 = d_3 = 0$. By $\theta(e^x\partial_1 * e^x\partial_1) = \theta(e^{2x}\partial_1)$, we have that $\theta(e^{2x}\partial_1) = d_1^2e^{2x}\partial_1^2$. By induction on $e^{nx}\partial_1$, we can prove that $\theta(e^{nx}\partial_1) = d_1^ne^{2x}\partial_1^2$. By $\theta(e^{mx}\partial_1 * e^{nx}\partial_1) = n\theta(e^{(m+n)x}\partial_1)$, we can derive a contradiction easily. Thus there is no isomorphism from $\mathbb{F}[e^{\pm x}, e^{\pm y}]_{\partial_1,\partial_2^2}$ to $\mathbb{F}[e^{\pm x}, e^{\pm y}]_{\partial_1^2,\partial_2^2}$ such that $c_2 = 0$. Thus let us assume that $c_1 = 0$, i.e., $\theta(\partial_1) = c_2\partial_2^2$. Similarly to the proof of $c_2 = 0$ case, we can derive a contradiction easily. Thus there is no isomorphism between them. This completes the proof of the theorem. □

**Open Question.** Find all the derivations and all the non-associative algebra automorphisms of the non-associative algebra $\mathbb{F}[e^{\pm x_1}, e^{\pm x_2}, \ldots, e^{\pm x_n}]_{\partial_1^2,\ldots,\partial_n^2}$ respectively.

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