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GENERALIZED VARIATIONAL-LIKE INEQUALITIES WITH COMPOSITELY MONOTONE MULTIFUNCTIONS

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Abstract. In this paper, we introduce two classes of generalized variational-like inequalities with compositely monotone multifunctions in Banach spaces. Using the KKM-Fan lemma and the Nadler's result, we prove the existence of solutions for generalized variational-like inequalities with compositely relaxed $\eta - \alpha$ monotone multifunctions in reflexive Banach spaces. On the other hand we also derive the solvability of generalized variational-like inequalities with compositely relaxed $\eta - \alpha$ semimonotone multifunctions in arbitrary Banach spaces by virtue of the Kakutani-Fan-Glicksberg fixed-point theorem. The results presented in this paper extend and improve some earlier and recent results in the literature.

1. Introduction

Variational inequality theory has become very effective and quite powerful tool in the study of a large number of problems arising in differential equations, mechanics, contact problems in elasticity, optimization and control problems, management science, operations research, general equilibrium problems in economics and transportation, unilateral, obstacle, moving, etc. Because of their important applicability, variational inequality problems have been extensively studied and generalized in various directions by many authors for a long time. For more details, the reader is refereed to [1–4, 6–11, 13–15, 17–21] and the references therein.

It is well-known that the monotonicity has always been being an effective and important tool in the study of variational inequalities. Because of its useful applicability, the monotonicity has been given many important generalizations by some authors, for example, quasimonotonicity, pseudomonotonicity, relaxed monotonicity, p-monotonicity, semimonotonicity, relaxed $\eta - \alpha$ monotonicity,

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and relaxed $\eta - \alpha$ semimonotonicity; see [2–9, 14–16, 19–21] and the references therein. In 1997, Verma [14] studied a class of nonlinear variational inequalities with p-monotone and p-Lipschitz mappings in reflexive Banach spaces and gave some existence theorems of solutions. Subsequently, Chen [3] introduced a class of variational inequalities with semimonotone mappings in nonreflexive Banach spaces and obtained existence theorems of solutions by using the Kakutani-Fan-Glicksberg fixed-point theorem. Recently, Fang and Huang [6] introduced two concepts of relaxed $\eta - \alpha$ monotonicity and relaxed $\eta - \alpha$ semimonotonicity as well as two classes of variational-like inequalities with relaxed $\eta - \alpha$ monotone mappings and relaxed $\eta - \alpha$ semimonotone mappings. Using the KKM technique, they proved the existence of solutions for variational-like inequalities with relaxed $\eta - \alpha$ monotone mappings in reflexive Banach spaces. Moreover, they also derived the solvability of variational-like inequalities with relaxed $\eta - \alpha$ semimonotone mappings in arbitrary Banach spaces by means of the Kakutani-Fan-Glicksberg fixed-point theorem.

In this paper, we introduce two classes of generalized variational-like inequalities with compositely monotone multifunctions in Banach spaces. Using the KKM-Fan lemma and the Nadler's result, we prove the existence of solutions for generalized variational-like inequalities with compositely relaxed $\eta - \alpha$ monotone multifunctions in reflexive Banach spaces. On the other hand, we also derive the solvability of generalized variational-like inequalities with compositely relaxed $\eta - \alpha$ semimonotone multifunctions in arbitrary Banach spaces by virtue of the Kakutani-Fan-Glicksberg fixed-point theorem. The results presented in this paper extend and improve some earlier and recent results in the literature including [2, 3, 6, 8, 13–15].

Throughout this paper, we will denote by "$\to$" and "$\rightharpoonup$" the strong convergence and weak convergence, respectively.

2. Generalized variational-like inequalities with compositely relaxed $\eta - \alpha$ monotone multifunctions

In this section, suppose that X is a real Banach space with dual space X^* and that K is a nonempty closed convex subset of X. Let us denote by 2^X and 2^{X^*} the collection of all nonempty subsets of X and the collection of all nonempty subsets of X^*, respectively.

Definition 2.1. Let $A : X^* \to X^*$ and $\eta : K \times K \to X$ be two mappings, let $V : K \to 2^K$ and $H : K \times K \to 2^{X^*}$ be two vector multifunctions, and let $\alpha : X \to R$ be a real function with $\alpha(tx) = t^p \alpha(x)$, $\forall t > 0$, $x \in X$, where $p > 1$ is a constant. Then H and V are said to be compositely relaxed $\eta - \alpha$ monotone with respect to A if for each $x_1, x_2 \in K$,

(1) $\langle A\xi_1 - A\xi_2, \eta(x_1, x_2)\rangle \geq \alpha(x_1 - x_2), \quad \forall z_i \in V(x_i),\ \xi_i \in H(x_i, z_i),\ i = 1, 2.$

Remark 2.1. Fang and Huang [6] introduced and considered the following concept of relaxed $\eta - \alpha$ monotonicity:

A mapping $T : K \to X^*$ is said to be relaxed $\eta - \alpha$ monotone if there exist a mapping $\eta : K \times K \to X$ and a real function $\alpha : X \to R$ with $\alpha(tx) = t^p\alpha(x)$ for all $t > 0$ and $x \in X$ such that

$$\langle Tx - Ty, \eta(x,y)\rangle \geq \alpha(x-y), \quad \forall x, y \in K,$$

where $p > 1$ is a constant.

Obviously, it is easy to see that our compositely relaxed $\eta - \alpha$ monotonicity is a set-valued version generalization of Fang and Huang's relaxed $\eta - \alpha$ monotonicity.

Definition 2.2. Let $f : K \to R \cup \{+\infty\}$ be a function.

(i) f is said to be convex if for each $x, y \in K$ and $t \in [0,1]$, one has

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y);$$

(ii) f is said to be concave, if the function $-f$ is convex.

Lemma 2.1 (See [4]). *Let X, Y and Z be real topological vector spaces, K and C be nonempty subsets of X and Y, respectively. Let $H : K \times C \to 2^Z$, $V : K \to 2^C$ be multivalued maps. If both H and V are upper semicontinuous with compact values, then the multivalued map $T : K \to 2^Z$ defined by*

$$T(x) = \bigcup_{z \in V(x)} H(x,z) = H(x, V(x))$$

is upper semicontinuous with compact values.

Lemma 2.2 (Nadler's theorem [12]). *Let $(Y, \|\cdot\|)$ be a normed vector space and $\widetilde{H}(\cdot,\cdot)$ be a Hausdorff metric on the collection $CB(Y)$ of all nonempty, closed and bounded subsets of Y, induced by a metric d in terms of $d(u,v) = \|u - v\|$, which is defined by*

$$\widetilde{H}(\Delta, \Lambda) = \max(\sup_{u\in\Delta}\inf_{v\in\Lambda}\|u-v\|, \sup_{v\in\Lambda}\inf_{u\in\Delta}\|u-v\|),$$

for Δ and Λ in $CB(Y)$. If Δ and Λ are compact sets in Y, then for each $u \in \Delta$, there exists $v \in \Lambda$ such that

$$\|u - v\| \leq \widetilde{H}(\Delta, \Lambda).$$

Definition 2.3. (i) See [17]. Let $T : K \to X^*$ and $\eta : K \times K \to X$ be two mappings. T is said to be η-hemicontinuous if, for any fixed $x, y \in K$, the mapping $f : [0,1] \to (-\infty, +\infty)$ defined by $f(t) = \langle T(x + t(y-x)), \eta(y,x)\rangle$ is continuous at 0^+;

(ii) A nonempty compact-valued multifunction $T : K \to 2^{X^*}$ is called $\widetilde{H}$-uniformly continuous if for any given $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in K$ with $\|x - y\| < \delta$ there holds

$$\widetilde{H}(Tx, Ty) < \varepsilon,$$

where $\widetilde{H}$ is the Hausdorff metric defined on $CB(X^*)$.

Let D be a nonempty subset of a topological vector space Y. A multivalued map $F : D \to 2^Y$ is called a KKM map if for each finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq D$,

$$\mathrm{co}\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i),$$

where $\mathrm{co}\{x_1, x_2, \ldots, x_n\}$ denotes the convex hull of $\{x_1, x_2, \ldots, x_n\}$.

Lemma 2.3 (KKM-Fan's lemma [5]). *Let D be an arbitrary nonempty subset of a Hausdorff topological vector space Y. Let the multivalued mapping $F : D \to 2^Y$ be a KKM map such that $F(x)$ is closed for all $x \in D$ and is compact for at least one $x \in D$. Then*

$$\bigcap_{x \in D} F(x) \neq \emptyset.$$

Theorem 2.1. *Let K be a nonempty, closed and convex subset of a real Banach space X. Let $A : X^* \to X^*$ be a continuous mapping, $f : K \to R \cup \{+\infty\}$ be a proper convex function and $\eta : K \times K \to X$ be a mapping such that (a) $\langle A\xi, \eta(\cdot, y)\rangle : K \to R$ is convex for each $(\xi, y) \in X^* \times K$ fixed, and (b) $\langle A\xi, \eta(x, x)\rangle = 0$, $\forall(\xi, x) \in X^* \times K$. Let $V : K \to 2^X$ and $H : K \times K \to 2^{X^*}$ be two upper semicontinuous mappings with compact values such that H and V are compositely relaxed $\eta - \alpha$ monotone with respect to A. If the multivalued map $T : K \to 2^{X^*}$ defined by*

$$T(x) = \bigcup_{z \in V(x)} H(x, z) = H(x, V(x))$$

is $\widetilde{H}$-uniformly continuous, then the following are equivalent:

(i) *there exist $x_0 \in K$, $z_0 \in V(x_0)$ and $\xi_0 \in H(x_0, z_0)$ such that*

$$\langle A\xi_0, \eta(y, x_0)\rangle + f(y) - f(x_0) \geq 0, \quad \forall y \in K; \tag{2}$$

(ii) *there exists $x_0 \in K$ such that*

$$\langle A\xi, \eta(y, x_0)\rangle + f(y) - f(x_0) \geq \alpha(y - x_0), \quad \forall y \in K,\ z \in V(y),\ \xi \in H(y, z). \tag{3}$$

Proof. Suppose that there exist $x_0 \in K$, $z_0 \in V(x_0)$ and $\xi_0 \in H(x_0, z_0)$ such that

$$\langle A\xi_0, \eta(y, x_0)\rangle + f(y) - f(x_0) \geq 0, \quad \forall y \in K.$$

Since H and V are compositely relaxed $\eta - \alpha$ monotone with respect to A, we have

$$\langle A\xi - A\xi_0, \eta(y, x_0)\rangle \geq \alpha(y - x_0)$$

for all $y \in K$, $z \in V(y)$ and $\xi \in H(y, z)$, which hence implies that

$$\begin{aligned}\alpha(y - x_0) &\leq \alpha(y - x_0) + \langle A\xi_0, \eta(y, x_0)\rangle + f(y) - f(x_0)\\ &\leq \langle A\xi, \eta(y, x_0)\rangle + f(y) - f(x_0)\end{aligned}$$

for all $y \in K$, $z \in V(y)$ and $\xi \in H(y,z)$; that is,

$$\langle A\xi, \eta(y,x_0)\rangle + f(y) - f(x_0) \geq \alpha(y - x_0)$$

for all $y \in K$, $z \in V(y)$ and $\xi \in H(y,z)$.

Conversely, suppose that there exists $x_0 \in K$ such that

$$\langle A\xi, \eta(y,x_0)\rangle + f(y) - f(x_0) \geq \alpha(y - x_0)$$

for all $y \in K$, $z \in V(y)$ and $\xi \in H(y,z)$. For any given $y \in K$, we know that $y_t = ty + (1-t)x_0 \in K$, $\forall t \in (0,1)$ since K is convex. Replacing y by y_t in the left-hand side of the above inequality, one deduces from assumptions (a)-(b) that for each $\xi_t \in T(y_t) = H(y_t, V(y_t))$

$$\begin{aligned} t^p\alpha(y - x_0) &= \alpha(t(y - x_0)) = \alpha(y_t - x_0) \\ &\leq \langle A\xi_t, \eta(y_t, x_0)\rangle + f(y_t) - f(x_0) \\ &= \langle A\xi_t, \eta(ty + (1-t)x_0, x_0)\rangle + f(ty + (1-t)x_0) - f(x_0) \\ &\leq t\langle A\xi_t, \eta(y,x_0)\rangle + (1-t)\langle A\xi_t, \eta(x_0,x_0)\rangle \\ &\quad + tf(y) + (1-t)f(x_0) - f(x_0) \\ &= t[\langle A\xi_t, \eta(y,x_0)\rangle + f(y) - f(x_0)], \end{aligned} \tag{4}$$

which hence implies that

$$\langle A\xi_t, \eta(y,x_0)\rangle + f(y) - f(x_0) \geq t^{p-1}\alpha(y - x_0), \quad \forall \xi_t \in T(y_t),\ t \in (0,1). \tag{5a}$$

We remark that according to Lemma 2.1 the multivalued mapping $T : K \to 2^{X^*}$ defined by

$$T(x) = \bigcup_{z \in V(x)} H(x,z) = H(x, V(x))$$

is upper semicontinuous with compact values. Hence $T(y_t)$ and $T(x_0)$ are compact, and from Lemma 2.2 it follows that for each fixed $\xi_t \in T(y_t)$ there exists an $\zeta_t \in T(x_0)$ such that

$$||\xi_t - \zeta_t|| \leq \widetilde{H}(T(y_t), T(x_0)).$$

Since $T(x_0)$ is compact, without loss of generality, we may assume that $\zeta_t \to \xi_0 \in T(x_0)$ as $t \to 0^+$. Since T is $\widetilde{H}$-uniformly continuous and $||y_t - x_0|| = t||y - x_0|| \to 0$ as $t \to 0^+$, so $\widetilde{H}(T(y_t), T(x_0)) \to 0$ as $t \to 0^+$. Thus one has

$$\begin{aligned} ||\xi_t - \xi_0|| &\leq ||\xi_t - \zeta_t|| + ||\zeta_t - \xi_0|| \\ &\leq \widetilde{H}(T(y_t), T(x_0)) + ||\zeta_t - \xi_0|| \to 0 \quad \text{as } t \to 0^+. \end{aligned}$$

Note that A is continuous. Hence $A\xi_t \to A\xi_0$ as $t \to 0^+$. Thus we obtain

$$\begin{aligned} |\langle A\xi_t, \eta(y,x_0)\rangle - \langle A\xi_0, \eta(y,x_0)\rangle| &= |\langle A\xi_t - A\xi_0, \eta(y,x_0)\rangle| \\ &\leq ||A\xi_t - A\xi_0||||\eta(y,x_0)|| \to 0 \quad \text{as } t \to 0^+. \end{aligned}$$

Consequently, from $(5a)$ we deduce that for any given $y \in K$

$$\langle A\xi_0, \eta(y,x_0)\rangle + f(y) - f(x_0) \geq 0.$$

Next, we claim that there holds

$$\langle A\xi_0, \eta(v, x_0)\rangle + f(v) - f(x_0) \geq 0, \quad \forall v \in K.$$

Indeed, let v be an arbitrary element in K and set $v_t = tv + (1-t)x_0$ for each $t \in (0, 1)$. Then one has $\|y_t - v_t\| = t\|y - v\| \to 0$ as $t \to 0^+$. Hence from the $\widetilde{H}$-uniform continuity of T it follows that $\widetilde{H}(Ty_t, Tv_t) \to 0$ as $t \to 0^+$. Let $\{\xi_t\}_{t\in(0,1)}$ be the net chosen as above such that $\xi_t \to \xi_0$ as $t \to 0^+$. Since Ty_t and Tv_t are compact, from Lemma 2.2 it follows that for each fixed $\xi_t \in Ty_t$ there exists a $\gamma_t \in Tv_t$ such that

$$\|\xi_t - \gamma_t\| \leq \widetilde{H}(Ty_t, Tv_t).$$

Consequently

$$\begin{aligned}\|\gamma_t - \xi_0\| &\leq \|\xi_t - \gamma_t\| + \|\xi_t - \xi_0\| \\ &\leq \widetilde{H}(Ty_t, Tv_t) + \|\xi_t - \xi_0\| \to 0 \quad \text{as } t \to 0^+.\end{aligned}$$

Note that A is continuous. Thus letting $t \to 0^+$, we obtain

$$\begin{aligned}|\langle A\gamma_t, \eta(v, x_0)\rangle - \langle A\xi_0, \eta(v, x_0)\rangle| &= |\langle A\gamma_t - A\xi_0, \eta(v, x_0)\rangle| \\ &\leq \|A\gamma_t - A\xi_0\|\|\eta(v, x_0)\| \to 0.\end{aligned}$$

Replacing y, y_t and ξ_t in $(5a)$ by v, v_t and γ_t, respectively, one deduces that

$$\langle A\gamma_t, \eta(v, x_0)\rangle + f(v) - f(x_0) \geq t^{p-1}\alpha(v - x_0), \quad \forall t \in (0, 1).$$

Letting $t \to 0^+$ we immediately get

$$\langle A\xi_0, \eta(v, x_0)\rangle + f(v) - f(x_0) \geq 0.$$

Thus according to the arbitrariness of v the assertion is valid.

Since $\xi_0 \in T(x_0) = \bigcup_{z\in V(x_0)} H(x_0, z) = H(x_0, V(x_0))$, there exists $z_0 \in V(x_0)$ such that $\xi_0 \in H(x_0, z_0)$. Therefore, (i) holds. This completes the proof. □

Remark 2.2. Theorem 2.1 generalizes Theorem 2.1 of Fang and Huang [6], Theorem 2.1 of Verma [14] and Theorem 2.1 of Verma [15].

Theorem 2.2. *Let K be a nonempty, bounded, closed and convex subset of a real reflexive Banach space X, and let X^* be the dual space of X. Suppose there hold the following:*

(i) *for each $x \in K$, $z \in V(x)$ and $\xi \in H(x, z)$ fixed, $\langle A\xi, \eta(x, \cdot)\rangle : K \to R$ is weakly upper semicontinuous;*

(ii) *$\langle A\xi, \eta(x, x)\rangle = 0$ for each $x \in K$ and $\xi \in X^*$;*

(iii) *for each $(\xi, y) \in X^* \times K$ fixed, $\langle A\xi, \eta(\cdot, y)\rangle : K \to R$ is affine;*

(iv) *$f : K \to R \cup \{+\infty\}$ is a proper, affine and lower semicontinuous function;*

(v) *$A : X^* \to X^*$ is continuous, and $\alpha : X \to R$ is weakly lower semicontinuous.*

Suppose additionally that $V : K \to 2^X$ and $H : K \times K \to 2^{X^*}$ are two upper semicontinuous mappings with compact values such that H and V are compositely relaxed $\eta - \alpha$ monotone with respect to A. If the multivalued map $T : K \to 2^{X^*}$ defined by

$$T(x) = \bigcup_{z \in V(x)} H(x, z) = H(x, V(x))$$

is $\widetilde{H}$-uniformly continuous, then there exist $\hat{x} \in K$, $\hat{z} \in V(\hat{x})$ and $\hat{\xi} \in H(\hat{x}, \hat{z})$ such that

$$\langle A\hat{\xi}, \eta(y, \hat{x})\rangle + f(y) - f(\hat{x}) \geq 0, \quad \forall y \in K.$$

Proof. First we claim that for every finite subset E of K, there exist $\bar{x} \in \mathrm{co}E$, $\bar{z} \in V(\bar{x})$ and $\bar{\xi} \in H(\bar{x}, \bar{z})$ such that

$$\langle A\bar{\xi}, \eta(y, \bar{x})\rangle + f(y) - f(\bar{x}) \geq 0, \quad \forall y \in \mathrm{co}E.$$

Indeed, let us define a vector multifunction $F : coE \to 2^{\mathrm{co}E}$ as follows:

$$F(y) = \{x \in \mathrm{co}E : \exists z \in V(x),\ \xi \in H(x, z) \text{ such that}$$
$$\langle A\xi, \eta(y, x)\rangle + f(y) - f(x) \geq 0\}, \quad \forall y \in coE.$$

From assumption (ii), one has $F(y) \neq \emptyset$ since $y \in F(y)$. The set $F(y)$ is also closed. Indeed, let $\{x_n\} \subseteq F(y)$ such that $x_n \to x$ as $n \to \infty$. Hence, for each n there exist $z_n \in V(x_n)$ and $\xi_n \in H(x_n, z_n)$ such that

$$\langle A\xi_n, \eta(y, x_n)\rangle + f(y) - f(x_n) \geq 0.$$

Since V is upper semicontinuous with compact values, $V(\mathrm{co}E)$ is compact. Therefore, without loss of generality one deduces that $z_n \to z \in V(x)$ as $n \to \infty$. On the other hand, since H is upper semicontinuous with compact values, $H(\mathrm{co}E, V(\mathrm{co}E))$ is compact. It follows without loss of generality that $\xi_n \to \xi \in H(x, z)$. Now, let $\{y_1, y_2, \dots, y_n\} \subseteq \mathrm{co}E$ and let us verify that $\mathrm{co}\{y_1, y_2, \dots, y_n\} \subseteq \bigcup_{i=1}^n F(y_i)$. Let $x \in \mathrm{co}\{y_1, y_2, \dots, y_n\}$, $x = \sum_{i=1}^n \lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$. Utilizing assumptions (ii)-(iv), we obtain

$$\begin{aligned}
0 &= \langle A\xi, \eta(x, x)\rangle + f(x) - f(x) \\
&= \langle A\xi, \eta(\sum_{i=1}^n \lambda_i y_i, x)\rangle + f(\sum_{i=1}^n \lambda_i y_i) - f(x) \\
&= \sum_{i=1}^n \lambda_i \langle A\xi, \eta(y_i, x)\rangle + \sum_{i=1}^n \lambda_i f(y_i) - f(x) \\
&= \sum_{i=1}^n \lambda_i [\langle A\xi, \eta(y_i, x)\rangle + f(y_i) - f(x)].
\end{aligned}$$

This shows that

$$\sum_{i=1}^n \lambda_i [\langle A\xi, \eta(y_i, x)\rangle + f(y_i) - f(x)] = 0.$$

Therefore, there exists $i \in \{1, 2, \ldots, n\}$ such that

$$\langle A\xi, \eta(y_i, x)\rangle + f(y_i) - f(x) \geq 0.$$

Hence $x \in F(y_i) \subseteq \bigcup_{j=1}^{n} F(y_j)$. Consequently, from Lemma 2.3, we know that $\bigcap_{y \in \mathrm{co}E} F(y) \neq \emptyset$.

Let $\bar{x} \in \bigcap_{y \in \mathrm{co}E} F(y)$. Then for each fixed $y \in \mathrm{co}E$ there exists $\xi_y \in T\bar{x} = H(\bar{x}, V(\bar{x}))$ such that

$$\langle A\xi_y, \eta(y, \bar{x})\rangle + f(y) - f(\bar{x}) \geq 0.$$

Let $y_t = \bar{x} + t(y - \bar{x}), \forall t \in (0,1)$. Then, observe that

$$\begin{aligned} &\langle A\xi_y, \eta(y_t, \bar{x})\rangle + f(y_t) - f(\bar{x}) \\ &= \langle A\xi_y, \eta(\bar{x} + t(y - \bar{x}), \bar{x})\rangle + f(\bar{x} + t(y - \bar{x})) - f(\bar{x}) \\ &= t\langle A\xi_y, \eta(y, \bar{x})\rangle + (1-t)\langle A\xi_y, \eta(\bar{x}, \bar{x})\rangle \\ &\quad + tf(y) + (1-t)f(\bar{x}) - f(\bar{x}) \\ &= t[\langle A\xi_y, \eta(y, \bar{x})\rangle + f(y) - f(\bar{x})]. \end{aligned}$$

Hence

$$\langle A\xi_y, \eta(y_t, \bar{x})\rangle + f(y_t) - f(\bar{x}) \geq 0.$$

Since H and V are compositely relaxed $\eta - \alpha$ monotone with respect to A, we have

$$\langle A\xi_t - A\xi_y, \eta(y_t, \bar{x})\rangle \geq \alpha(y_t - \bar{x}) = t^p\alpha(y - \bar{x}), \quad \forall \xi_t \in Ty_t,\ t \in (0,1),$$

and so

$$\begin{aligned} t^p\alpha(y - \bar{x}) &\leq t^p\alpha(y - \bar{x}) + \langle A\xi_y, \eta(y_t, \bar{x})\rangle + f(y_t) - f(\bar{x}) \\ &\leq \langle A\xi_t, \eta(y_t, \bar{x})\rangle + f(y_t) - f(\bar{x}). \end{aligned}$$

Thus

$$\langle A\xi_t, \eta(y_t, \bar{x})\rangle + f(y_t) - f(\bar{x}) \geq t^p\alpha(y - \bar{x}), \quad \forall \xi_t \in Ty_t,\ t \in (0,1). \tag{5b}$$

Now observe that

$$\begin{aligned} &\langle A\xi_t, \eta(y_t, \bar{x})\rangle + f(y_t) - f(\bar{x}) \\ &= \langle A\xi_t, \eta((1-t)\bar{x} + ty, \bar{x})\rangle + f((1-t)\bar{x} + ty) - f(\bar{x}) \\ &= (1-t)\langle A\xi_t, \eta(\bar{x}, \bar{x})\rangle + t\langle A\xi_t, \eta(y, \bar{x})\rangle + (1-t)f(\bar{x}) + tf(y) - f(\bar{x}) \\ &= t[\langle A\xi_t, \eta(y, \bar{x})\rangle + f(y) - f(\bar{x})], \end{aligned}$$

which together with (5b), implies that

$$\langle A\xi_t, \eta(y, \bar{x})\rangle + f(y) - f(\bar{x}) \geq t^{p-1}\alpha(y - \bar{x}), \quad \forall \xi_t \in Ty_t,\ t \in (0,1). \tag{5c}$$

We remark that according to Lemma 2.1 the multivalued mapping $T : K \to 2^{X^*}$ defined by

$$Tx = \bigcup_{z \in V(x)} H(x, z) = H(x, V(x))$$

is upper semicontinuous with compact values. Hence $T(y_t)$ and $T(\bar{x})$ are compact, and from Lemma 2.2 it follows that for each fixed $\xi_t \in T(y_t)$ there exists an $\zeta_t \in T(\bar{x})$ such that

$$||\xi_t - \zeta_t|| \leq \widetilde{H}(T(y_t), T(\bar{x})).$$

Since $T(\bar{x})$ is compact, without loss of generality, we may assume that $\zeta_t \to \bar{\xi} \in T(\bar{x})$ as $t \to 0^+$. Since T is $\widetilde{H}$-uniformly continuous and $||y_t - \bar{x}|| = t||y - \bar{x}|| \to 0$ as $t \to 0^+$, so $\widetilde{H}(T(y_t), T(\bar{x})) \to 0$ as $t \to 0^+$. Thus one has

$$\begin{aligned} ||\xi_t - \bar{\xi}|| &\leq ||\xi_t - \zeta_t|| + ||\zeta_t - \bar{x}|| \\ &\leq \widetilde{H}(T(y_t), T(\bar{x})) + ||\zeta_t - \bar{\xi}|| \to 0 \quad \text{as } t \to 0^+. \end{aligned}$$

Note that A is continuous. Therefore letting $t \to 0^+$, we obtain

$$\begin{aligned} |\langle A\xi_t, \eta(y, \bar{x})\rangle - \langle A\bar{\xi}, \eta(y, \bar{x})\rangle| &= |\langle A\xi_t - A\bar{\xi}, \eta(y, \bar{x})\rangle| \\ &\leq ||A\xi_t - A\bar{\xi}||||\eta(y, \bar{x})|| \to 0. \end{aligned}$$

This together with (5c), implies that

$$\langle A\bar{\xi}, \eta(y, \bar{x})\rangle + f(y) - f(\bar{x}) \geq 0.$$

Next we claim that there holds

$$\langle A\bar{\xi}, \eta(v, \bar{x})\rangle + f(v) - f(\bar{x}) \geq 0, \quad \forall v \in \mathrm{co}E.$$

Indeed, let v be an arbitrary element in co$E$ and set $v_t = tv + (1-t)\bar{x}$ for each $t \in (0,1)$. Then one has $||y_t - v_t|| = t||y - v|| \to 0$ as $t \to 0^+$. Hence from the $\widetilde{H}$-uniform continuity of T it follows that $\widetilde{H}(Ty_t, Tv_t) \to 0$ as $t \to 0^+$. Let $\{\xi_t\}_{t\in(0,1)}$ be the net chosen as above such that $\xi_t \to \bar{\xi}$ as $t \to 0^+$. Since Ty_t and Tv_t are compact, from Lemma 2.2 it follows that for each fixed $\xi_t \in Ty_t$ there exists a $\gamma_t \in Tv_t$ such that

$$||\xi_t - \gamma_t|| \leq \widetilde{H}(Ty_t, Tv_t).$$

Consequently

$$\begin{aligned} ||\gamma_t - \bar{\xi}|| &\leq ||\xi_t - \gamma_t|| + ||\xi_t - \bar{\xi}|| \\ &\leq \widetilde{H}(Ty_t, Tv_t) + ||\xi_t - \bar{\xi}|| \to 0 \quad \text{as } t \to 0^+. \end{aligned}$$

Note that A is continuous. Thus letting $t \to 0^+$, we obtain

$$\begin{aligned} |\langle A\gamma_t, \eta(v, \bar{x})\rangle - \langle A\bar{\xi}, \eta(v, \bar{x})\rangle| &= |\langle A\gamma_t - A\bar{\xi}, \eta(v, \bar{x})\rangle| \\ &\leq ||A\gamma_t - A\bar{\xi}||||\eta(v, \bar{x})|| \to 0. \end{aligned}$$

Replacing y, y_t and ξ_t in (5c) by v, v_t and γ_t, respectively, one has

$$\langle A\gamma_t, \eta(v, \bar{x})\rangle + f(v) - f(\bar{x}) \geq t^{p-1}\alpha(v - \bar{x}), \quad \forall t \in (0,1),$$

which hence implies that

$$\langle A\bar{\xi}, \eta(v, \bar{x})\rangle + f(v) - f(\bar{x}) \geq 0.$$

Thus, according to the arbitrariness of v the assertion is valid.

Since $\bar{\xi} \in T(\bar{x}) = \bigcup_{z \in V(\bar{x})} H(\bar{x}, z) = H(\bar{x}, V(\bar{x}))$, it follows that there exists $\bar{z} \in V(\bar{x})$ such that $\bar{\xi} \in H(\bar{x}, \bar{z})$. Therefore, the original assertion is valid.

Now, by Theorem 2.1 we conclude that for every finite subset E of K, there exists $\bar{x} \in \mathrm{co}E$ such that

$$\langle A\xi, \eta(y, \bar{x})\rangle + f(y) - f(\bar{x}) \geq \alpha(y - \bar{x}), \quad \forall y \in \mathrm{co}E,\ z \in V(y),\ \xi \in H(y, z).$$

Second, we claim that there exists $\hat{x} \in K$ such that

$$\langle A\xi, \eta(y, \hat{x})\rangle + f(y) - f(\hat{x}) \geq \alpha(y - \hat{x}), \quad \forall y \in K,\ z \in V(y),\ \xi \in H(y, z).$$

Indeed, since X is reflexive and K is a nonempty, bounded, closed and convex subset of X, so K is compact with respect to the weak topology of X. Let $\Im$ be the family of all finite subsets of K. For each $E \in \Im$, consider the following set:

$$\begin{aligned} M_E = \{x \in K : \langle A\xi, \eta(y, x)\rangle + f(y) - f(x) \geq \alpha(y - x),& \\ \forall y \in \mathrm{co}E,\ z \in V(y),\ \xi \in H(y, z)\}.& \end{aligned}$$

Then one has $M_E \neq \emptyset$ for each $E \in \Im$. We shall prove that $\bigcap_{E \in \Im} \overline{M}_E^w \neq \emptyset$, where $\overline{M}_E^w$ denotes the closure of E with respect to the weak topology of X. For this, it suffices to show that the family $\{\overline{M}_E^w\}_{E \in \Im}$ has the finite intersection property. Let $E, F \in \Im$ and set $G = E \cup F \in \Im$. Then $M_G \subseteq M_E \cap M_F$ and it follows that $\overline{M}_E^w \cap \overline{M}_F^w \neq \emptyset$. This shows that the family $\{\overline{M}_E^w\}_{E \in \Im}$ has the finite intersection property. Since K is compact with respect to the weak topology of X, it follows that $\bigcap_{E \in \Im} \overline{M}_E^w \neq \emptyset$. Let $\hat{x} \in \bigcap_{E \in \Im} \overline{M}_E^w$ and for an arbitrary $y \in K$ fixed, consider $F = \{y, \hat{x}\}$. Since $\hat{x} \in \overline{M}_F^w$, there exists $\{x_n\} \subseteq \overline{M}_F^w$ such that $\{x_n\} \subseteq K$, $x_n \rightharpoonup \hat{x}$ and for each n

$$\langle A\xi, \eta(v, x_n)\rangle + f(v) - f(x_n) \geq \alpha(v - x_n), \quad \forall v \in \mathrm{co}F,\ z \in V(v),\ \xi \in H(v, z).$$

In particular, whenever $v = y$, one derives for each n

$$\langle A\xi, \eta(y, x_n)\rangle + f(y) - f(x_n) \geq \alpha(y - x_n), \quad \forall z \in V(y),\ \xi \in H(y, z).$$

Since $f : K \to R \cup \{+\infty\}$ is a proper, affine and lower semicontinuous function, f is weakly lower semicontinuous. Note that $\alpha : X \to R$ is weakly lower semicontinuous, and that for each $x \in K$, $z \in V(x)$ and $\xi \in H(x, z)$ fixed, $\langle A\xi, \eta(x, \cdot)\rangle : K \to R$ is weakly upper semicontinuous. Thus we conclude that for each $y \in K$, $z \in V(y)$ and $\xi \in H(y, z)$ fixed,

$$\begin{aligned} \alpha(y - \hat{x}) &\leq \liminf_{n \to \infty} \alpha(y - x_n) \\ &\leq \limsup_{n \to \infty} \alpha(y - x_n) \\ &\leq \limsup_{n \to \infty} [\langle A\xi, \eta(y, x_n)\rangle + f(y) - f(x_n)] \\ &\leq \limsup_{n \to \infty} \langle A\xi, \eta(y, x_n)\rangle + f(y) - \liminf_{n \to \infty} f(x_n) \\ &\leq \langle A\xi, \eta(y, \hat{x})\rangle + f(y) - f(\hat{x}), \end{aligned}$$

that is,

$$\langle A\xi, \eta(y, \hat{x})\rangle + f(y) - f(\hat{x}) \geq \alpha(y - \hat{x}), \quad \forall y \in K, \ z \in V(y), \ \xi \in H(y, z).$$

Thus, the assertion is proved.

Now by Theorem 2.1 we infer that there exist $\hat{x} \in K$, $\hat{z} \in V(\hat{x})$ and $\hat{\xi} \in H(\hat{x}, \hat{z})$ such that

$$\langle A\hat{\xi}, \eta(y, \hat{x})\rangle + f(y) - f(\hat{x}) \geq 0, \quad \forall y \in K.$$

This completes the proof. □

If K is unbounded, then we have the following theorem under certain coercivity condition:

Theorem 2.3. *Let K be a nonempty, unbounded, closed and convex subset of a real reflexive Banach space X, and let X^* be the dual space of X. Suppose there hold the following:*

(i) *for each $x \in K$, $z \in V(x)$ and $\xi \in H(x, z)$ fixed, $\langle A\xi, \eta(x, \cdot)\rangle : K \to R$ is weakly upper semicontinuous;*

(ii) *$\langle A\xi, \eta(x, x)\rangle = 0$ for each $x \in K$ and $\xi \in X^*$;*

(iii) *for each $(\xi, y) \in X^* \times K$ fixed, $\langle A\xi, \eta(\cdot, y)\rangle : K \to R$ is affine;*

(iv) *$f : K \to R \cup \{+\infty\}$ is a proper, affine and lower semicontinuous function;*

(v) *$A : X^* \to X^*$ is continuous, and $\alpha : X \to R$ is weakly lower semicontinuous.*

Suppose additionally that $V : K \to 2^X$ and $H : K \times K \to 2^{X^*}$ are two upper semicontinuous mappings with compact values such that H and V are compositely relaxed $\eta - \alpha$ monotone with respect to A. If the multivalued map $T : K \to 2^{X^*}$ defined by

$$T(x) = \bigcup_{z \in V(x)} H(x, z) = H(x, V(x))$$

is $\widetilde{H}$-uniformly continuous such that H and V are η-coercive with respect to A and f; i.e., there exist $x_0 \in K$, $z_0 \in V(x_0)$ and $\xi_0 \in H(x_0, z_0)$ such that

$$\inf_{\xi \in T(x)} \frac{\langle A\xi_0 - A\xi, \eta(x_0, x)\rangle + f(x) - f(x_0)}{||\eta(x_0, x)||} \to +\infty$$

as $||x|| \to +\infty$, then there exist $\hat{x} \in K$, $\hat{z} \in V(\hat{x})$ and $\hat{\xi} \in H(\hat{x}, \hat{z})$ such that

$$\langle A\hat{\xi}, \eta(y, \hat{x})\rangle + f(y) - f(\hat{x}) \geq 0, \quad \forall y \in K.$$

Proof. Let

$$K_r = \{y \in K : ||y|| \leq r\}.$$

Consider the problem of finding $x_r \in K_r$, $z_r \in V(x_r)$ and $\xi_r \in H(x_r, z_r)$ such that

$$\langle A\xi_r, \eta(v, x_r)\rangle + f(v) - f(x_r) \geq 0, \quad \forall v \in K_r. \tag{6}$$

One can readily see that all conditions of Theorem 2.1 are fulfilled for nonempty, bounded, closed and convex subset $K_r = K \cap B_r$, where $B_r = \{x \in X : \|x\| \leq r\}$. Thus according to Theorem 2.2 we know that problem (6) has one solution; that is, there exist $x_r \in K_r$, $z_r \in V(x_r)$ and $\xi_r \in H(x_r, z_r)$ such that inequality (6) holds. Choose $r > \|x_0\|$ with x_0 as in the coercivity condition. Then we have

$$\langle A\xi_r, \eta(x_0, x_r)\rangle + f(x_0) - f(x_r) \geq 0.$$

Moreover,

$$\begin{aligned}
&\langle A\xi_r, \eta(x_0, x_r)\rangle + f(x_0) - f(x_r) \\
= &- \langle A\xi_0 - A\xi_r, \eta(x_0, x_r)\rangle + f(x_0) - f(x_r) + \langle A\xi_0, \eta(x_0, x_r)\rangle \\
\leq &- \langle A\xi_0 - A\xi_r, \eta(x_0, x_r)\rangle + f(x_0) - f(x_r) + \|A\xi_0\|\|\eta(x_0, x_r)\| \\
= &\|\eta(x_0, x_r)\| \cdot [-\frac{\langle A\xi_0 - A\xi_r, \eta(x_0, x_r)\rangle + f(x_r) - f(x_0)}{\|\eta(x_0, x_r)\|} + \|A\xi_0\|].
\end{aligned}$$

Now, if $\|x_r\| = r$ for all r, we may choose r large enough such that the above inequality and the η-coercivity of H and V with respect to A and f imply that

$$\langle A\xi_r, \eta(x_0, x_r)\rangle + f(x_0) - f(x_r) < 0,$$

which contradicts

$$\langle A\xi_r, \eta(x_0, x_r)\rangle + f(x_0) - f(x_r) \geq 0.$$

Hence there exists r such that $\|x_r\| < r$. For any $y \in K$, we can choose $\epsilon > 0$ small enough such that

$$\epsilon < 1 \quad \text{and} \quad x_r + \epsilon(y - x_r) \in K_r.$$

It follows from (6) that

$$\begin{aligned}
&\epsilon[\langle A\xi_r, \eta(y, x_r)\rangle + f(y) - f(x_r)] \\
&= (1 - \epsilon)\langle A\xi_r, \eta(x_r, x_r)\rangle + \epsilon\langle A\xi_r, \eta(y, x_r)\rangle \\
&\quad + (1 - \epsilon)f(x_r) + \epsilon f(y) - f(x_r) \\
&= \langle A\xi_r, \eta(x_r + \epsilon(y - x_r), x_r)\rangle + f(x_r + \epsilon(y - x_r)) - f(x_r) \\
&\geq 0.
\end{aligned}$$

This implies that

$$\langle A\xi_r, \eta(y, x_r)\rangle + f(y) - f(x_r) \geq 0$$

for all $y \in K$, and so problem (2) has a solution. This completes the proof. □

Remark 2.3. Theorems 2.2 and 2.3 generalize Theorems 2.2 and 2.3 of Fang and Huang [6], the known results of Hartman and Stampacchia [10] and the corresponding results of [8, 13, 15].

3. Generalized variational-like inequalities with compositely relaxed $\eta - \alpha$ semimonotone multifunctions

Throughout this section, let X be an arbitrary Banach space with its dual space X^*, let X^{**} denote the dual space of X^*, and let K be a nonempty closed convex subset of X^{**}. Let us denote by 2^{X^*} and $2^{X^{**}}$ the collection of all nonempty subsets of X^* and the collection of all nonempty subsets of X^{**}, respectively.

Definition 3.1. Let $A : K \times X^* \to X^*$ and $\eta : K \times K \to X^{**}$ be two mappings, let $V : K \to 2^{X^{**}}$ and $H : K \times K \to 2^{X^*}$ be two vector multifunctions, and let $\alpha : X^{**} \to R$ be a real function with $\alpha(tx) = t^p\alpha(x)$, $\forall t > 0$, $x \in X^{**}$, where $p > 1$ is a constant. Then H and V are said to be compositely relaxed $\eta - \alpha$ semimonotone with respect to A if the following conditions hold:

(a) for each fixed $y \in K$, $A(y, \cdot) : X^* \to X^*$ is continuous, and H and V are compositely relaxed $\eta - \alpha$ monotone with respect to $A(y, \cdot)$; i.e., for each $x_1, x_2 \in K$,

$$\langle A(y, \xi_1) - A(y, \xi_2), \eta(x_1, x_2)\rangle \geq \alpha(x_1 - x_2),$$
$$\forall z_i \in V(x_i),\ \xi_i \in H(x_i, z_i),\ i = 1, 2;$$

(b) for each fixed $\xi \in X^*$, $A(\cdot, \xi) : K \to X^*$ is completely continuous; i.e., for any net $\{x_\beta\}$, x_β converges to x_0 in $\sigma(X^{**}, X^*)$ implies that $\{A(x_\beta, \xi)\}$ converges to $A(x_0, \xi)$ in the norm topology of X^*.

The above concept is a set-valued version generalization of the following relaxed $\eta - \alpha$ semimonotonicity.

Definition 3.2 (See Definition 3.1 [6]). Let $\eta : K \times K \to X^{**}$ be a mapping and let $\alpha : X^{**} \to R$ be a function with $\alpha(tx) = t^p\alpha(x)$ for all $t > 0$ and $x \in X^{**}$, where $p > 1$ is a constant. A mapping $\hat{A} : K \times K \to X^*$ is said to be relaxed $\eta - \alpha$ semimonotone if the following conditions hold:

(a) for each fixed $x \in K$, $\hat{A}(x, \cdot)$ is relaxed $\eta - \alpha$ monotone; i.e.,

$$\langle \hat{A}(x, y) - \hat{A}(x, v), \eta(y, v)\rangle \geq \alpha(y - v), \quad \forall y, v \in K;$$

(b) for each fixed $v \in K$, $\hat{A}(\cdot, v)$ is completely continuous; i.e., for any net $\{x_\beta\}$, x_β converges to x_0 in $\sigma(X^{**}, X^*)$ implies that $\{\hat{A}(x_\beta, v)\}$ converges to $\hat{A}(x_0, v)$ in the norm topology of X^*.

Let $A : K \times X^* \to X^*$ and $\eta : K \times K \to X^{**}$ be two mappings, $f : K \to R \cup \{+\infty\}$ be a proper convex lower semicontinuous function, and $V : K \to 2^{X^{**}}$ and $H : K \times K \to 2^{X^*}$ be two vector multifunctions. We consider the following problem: Find $\hat{x} \in K$, $\hat{z} \in V(\hat{x})$ and $\hat{\xi} \in H(\hat{x}, \hat{z})$ such that

$$\langle A(\hat{x}, \hat{\xi}), \eta(v, \hat{x})\rangle + f(v) - f(\hat{x}) \geq 0, \quad \forall v \in K. \tag{7}$$

Theorem 3.1. *Let X be a real Banach space and let $K \subseteq X^{**}$ be a nonempty, bounded, closed and convex subset. Let $V : K \to 2^{X^{**}}$ and $H : K \times K \to 2^{X^*}$*

be finite-dimensional upper semicontinuous mappings with compact values; i.e., for any finite-dimensional subspace $L \subseteq X^{**}$, $V : K_L \to 2^{X^{**}}$ *and* $H : K_L \times K_L \to 2^{X^*}$ *are upper semicontinuous mappings with compact values where* $K_L = K \cap L$. *Suppose there hold the following:*

(i) $\eta(x, y) + \eta(y, x) = 0$ *for all* $x, y \in K$;

(ii) *for each* $y, v \in K$ *and* $\xi \in X^*$ *fixed, the mapping* $x \mapsto \langle A(y, \xi), \eta(x, v)\rangle$ *is affine and lower semicontinuous*;

(iii) $f : K \to R \cup \{+\infty\}$ *is a proper, affine and lower semicontinuous function*;

(iv) $\alpha : X^{**} \to R$ *is convex and lower semicontinuous.*

Suppose additionally that H *and* V *are compositely relaxed* $\eta - \alpha$ *semimonotone with respect to* A. *If the multivalued map* $T : K \to 2^{X^*}$ *defined by*

$$T(x) = \bigcup_{z \in V(x)} H(x, z) = H(x, V(x))$$

is $\widetilde{H}$*-uniformly continuous, then there exist* $\hat{x} \in K$, $\hat{z} \in V(\hat{x})$ *and* $\hat{\xi} \in H(\hat{x}, \hat{z})$ *such that*

$$\langle A(\hat{x}, \hat{\xi}), \eta(v, \hat{x})\rangle + f(v) - f(\hat{x}) \geq 0, \quad \forall v \in K.$$

Proof. Let $L \subseteq X^{**}$ be a finite-dimensional subspace with $K_L = K \cap L \neq \emptyset$. For each $y \in K$, consider the following problem: Find $x_0 \in K_L$, $z_0 \in V(x_0)$ and $\xi_0 \in H(x_0, z_0)$ such that

$$\langle A(y, \xi_0), \eta(v, x_0)\rangle + f(v) - f(x_0) \geq 0, \quad \forall v \in K_L. \tag{8}$$

Observe that $K_L \subseteq L$ is bounded, closed and convex, $A(y, \cdot) : X^* \to X^*$ is continuous, and H and V are compositely relaxed $\eta - \alpha$ monotone with respect to $A(y, \cdot)$. Since assumptions (i) and (ii) guarantee that conditions (i)-(iii) in Theorem 2.2 are fulfilled, from Theorem 2.2 it follows that problem (8) has a solution; that is, there exist $\hat{x} \in K$, $\hat{z} \in V(\hat{x})$ and $\hat{\xi} \in H(\hat{x}, \hat{z})$ such that inequality (8) holds.

Now, define a set-valued mapping $\Gamma : K_L \to 2^{K_L}$ as follows:

$$\begin{aligned}\Gamma(y) = \{x \in K_L :& \exists \bar{z} \in V(x),\ \bar{\xi} \in H(x, \bar{z}) \text{ such that} \\ & \langle A(y, \bar{\xi}), \eta(v, x)\rangle + f(v) - f(x) \geq 0,\ \forall v \in K_L\}, \quad \forall y \in K_L.\end{aligned}$$

It follows from Theorem 2.1 that, for each fixed $y \in K_L$,

$$\begin{aligned}&\{x \in K_L :\ \exists \bar{z} \in V(x),\ \bar{\xi} \in H(x, \bar{z}) \text{ such that} \\ &\qquad \langle A(y, \bar{\xi}), \eta(v, x)\rangle + f(v) - f(x) \geq 0,\ \forall v \in K_L\} \\ &= \{x \in K_L : \langle A(y, \xi), \eta(v, x)\rangle + f(v) - f(x) \geq \alpha(v - x), \\ &\qquad \forall v \in K_L, z \in V(v), \xi \in H(v, z)\}.\end{aligned}$$

Since every convex lower semicontinuous function in Banach spaces is weakly lower semicontinuous, the proper convex lower semicontinuity of f and α and assumption (ii) imply that $\Gamma : K_L \to 2^{K_L}$ has nonempty, bounded, closed and convex values. We also know that Γ is upper semicontinuous by the complete continuity of $A(\cdot,\xi)$. By the Kakutani-Fan-Glicksberg fixed-point theorem, Γ has a fixed point $x_0 \in K_L$, i.e., $x_0 \in \Gamma(x_0)$. Consequently, there exist $z_0 \in V(x_0)$ and $\xi_0 \in H(x_0, z_0)$ such that

$$\langle A(x_0,\xi_0), \eta(v, x_0)\rangle + f(v) - f(x_0) \geq 0, \quad \forall v \in K_L. \tag{9}$$

Let

$$\mho = \{L \subseteq X^{**} : L \text{ is finite dimensional with } K \cap L \neq \emptyset\}$$

and let

$$W_L = \{x \in K : \langle A(x,\xi), \eta(v,x)\rangle + f(v) - f(x) \geq \alpha(v - x), \\ \forall v \in K_L, z \in V(v), \xi \in H(v,z)\}$$

for all $L \in \mho$. By (9) and Theorem 2.1, we know that W_L is nonempty and bounded. Denote by $\overline{W}_L$ the $\sigma(X^{**}, X^*)$-closure of W_L in X^{**}. Then, $\overline{W}_L$ is $\sigma(X^{**}, X^*)$-compact in X^{**}.

For any $L_i \in \mho, i = 1, 2, \ldots, N$, we know that $W_{\cap_i L_i} \subseteq \cap_i W_{L_i}$, so $\{\overline{W}_L : L \in \mho\}$ has the finite intersection property. Therefore, it follows that

$$\bigcap_{L\in\mho} \overline{W}_L \neq \emptyset.$$

Let $\hat{x} \in \bigcap_{L\in\mho} \overline{W}_L \neq \emptyset$. We claim that there exist $\hat{z} \in V(\hat{x})$ and $\hat{\xi} \in H(\hat{x}, \hat{z})$ such that

$$\langle A(\hat{x},\hat{\xi}), \eta(v,\hat{x})\rangle + f(v) - f(\hat{x}) \geq 0, \quad \forall v \in K.$$

Indeed, for each $v \in K$, let $L \in \mho$ be such that $v \in K_L$ and $\hat{x} \in K_L$. Then, there exists a net $\{x_\beta\} \in W_L$ such that x_β converges to $\hat{x}$ in $\sigma(X^{**}, X^*)$, which implies by the definition of W_L that

$$\langle A(x_\beta,\xi), \eta(v,x_\beta)\rangle + f(v) - f(x_\beta) \geq \alpha(v - x_\beta), \quad \forall z \in V(v), \xi \in H(v,z).$$

It follows that

$$\langle A(\hat{x},\xi), \eta(v,\hat{x})\rangle + f(v) - f(\hat{x}) \geq \alpha(v - \hat{x}), \quad \forall v \in K, z \in V(v), \xi \in H(v,z),$$

by the complete continuity of $A(\cdot, v)$ and the proper convex lower semicontinuity of f and α. Therefore according to Theorem 2.1 there exist $\hat{z} \in V(\hat{x})$ and $\hat{\xi} \in H(\hat{x}, \hat{z})$ such that

$$\langle A(\hat{x},\hat{\xi}), \eta(v,\hat{x})\rangle + f(v) - f(\hat{x}) \geq 0, \quad \forall v \in K.$$

This completes the proof. □

Theorem 3.2. *Let X be a real Banach space and let $K \subseteq X^{**}$ be a nonempty, unbounded, closed and convex subset. Let $V : K \to 2^{X^{**}}$ and $H : K \times K \to 2^{X^*}$ be finite-dimensional upper semicontinuous mappings with compact values. Suppose there hold the following:*

(i) *$\eta(x,y) + \eta(y,x) = 0$ for all $x, y \in K$;*

(ii) *for each $y, v \in K$ and $\xi \in X^*$ fixed, the mapping $x \mapsto \langle A(y,\xi), \eta(x,v)\rangle$ is affine and lower semicontinuous;*

(iii) $f : K \to R \cup \{+\infty\}$ *is a proper, affine and lower semicontinuous function;*

(iv) $\alpha : X^{**} \to R$ *is convex and lower semicontinuous.*

Suppose additionally that H *and* V *are compositely relaxed* $\eta - \alpha$ *semimonotone with respect to* A. *If the multivalued map* $T : K \to 2^{X^*}$ *defined by*

$$T(x) = \bigcup_{z \in V(x)} H(x,z) = H(x, V(x))$$

is $\widetilde{H}$*-uniformly continuous such that*

(v) *there exists a point* $x_0 \in K$ *such that*

$$\liminf_{\|x\| \to \infty} \inf_{\xi \in T(x)} [\langle A(x,\xi), \eta(x,x_0)\rangle + f(x) - f(x_0)] > 0,$$

then there exist $\hat{x} \in K$, $\hat{z} \in V(\hat{x})$ *and* $\hat{\xi} \in H(\hat{x},\hat{z})$ *such that*

$$\langle A(\hat{x},\hat{\xi}), \eta(v,\hat{x})\rangle + f(v) - f(\hat{x}) \geq 0, \quad \forall v \in K.$$

Proof. Denote by B_r the closed ball with radius r and center at 0 in X^{**}. First consider the problem of finding $x_r \in K_r$, $z_r \in V(x_r)$ and $\xi_r \in H(x_r, z_r)$ such that

$$\langle A(x_r,\xi_r), \eta(v,x_r)\rangle + f(v) - f(x_r) \geq 0, \quad \forall v \in K_r, \tag{10}$$

where $K_r = \{x \in K : \|x\| \leq r\} = K \cap B_r$. By Theorem 3.1 problem (10) has a solution; that is, there exist $x_r \in K_r$, $z_r \in V(x_r)$ and $\xi_r \in H(x_r, z_r)$ such that inequality (10) holds.

Let r be large enough such that $x_0 \in B_r$. Therefore,

$$\langle A(x_r,\xi_r), \eta(x_0,x_r)\rangle + f(x_0) - f(x_r) \geq 0. \tag{11}$$

From condition (v) it follows that $\{x_r\}$ is bounded. Indeed, if this was false, we may assume without loss of generality that $\|x_r\| \to \infty$ as $r \to \infty$. Now we derive from (11)

$$\begin{aligned}
&\inf_{\xi \in T(x_r)} [\langle A(x_r,\xi), \eta(x_r,x_0)\rangle + f(x_r) - f(x_0)] \\
&\leq \langle A(x_r,\xi_r), \eta(x_r,x_0)\rangle + f(x_r) - f(x_0) \\
&\leq 0,
\end{aligned}$$

which hence implies that

$$\liminf_{\|x_r\| \to \infty} \inf_{\xi \in T(x_r)} [\langle A(x_r,\xi), \eta(x_r,x_0)\rangle + f(x_r) - f(x_0)] \leq 0.$$

This contradicts condition (v). So, we may assume that x_r converges to $\hat{x}$ in $\sigma(X^{**}, X^*)$ as $r \to \infty$. On the other hand, it follows from Theorem 2.1 that

$$\langle A(x_r,\xi), \eta(v,x_r)\rangle + f(v) - f(x_r) \geq \alpha(v - x_r), \ \forall v \in K, \ z \in V(v), \ \xi \in H(v,z).$$

Letting $r \to \infty$, we have

$$\langle A(\hat{x},\xi), \eta(v,\hat{x})\rangle + f(v) - f(\hat{x}) \geq \alpha(v - \hat{x}), \ \forall v \in K, \ z \in V(v), \ \xi \in H(v,z).$$

Again from Theorem 2.1 we know that there exist $\hat{z} \in V(\hat{x})$ and $\hat{\xi} \in H(\hat{x}, \hat{z})$ such that

$$\langle A(\hat{x}, \hat{\xi}), \eta(v, \hat{x})\rangle + f(v) - f(\hat{x}) \geq 0, \ \forall v \in K.$$

This completes the proof. □

Remark 3.1. Theorems 3.1 and 3.2 improve and generalize Theorems 3.1 and 3.2 of Fang and Huang [6], and Theorems 2.1 to 2.6 of Chen [3].

Remark 3.2. We may raise an open question concerning with proofs of main theorems in the paper. Can we replace the affinity condition of the function f in Theorems 2.2, 2.3, 3.1, and 3.2 by the convexity condition?

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