

ON A VARIANT OF JENSEN'S INEQUALITY FOR FUNCTIONS WITH NONDECREASING INCREMENTS

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ABSTRACT. Refinements of a variant of Jensen's inequality for functions with nondecreasing increments are presented. Obtained results are used to prove refinements of related variants of Čebyšev's inequality and Hölder's inequality.

1. Introduction

In [2] H. D. Brunk investigated an interesting class of multivariate real-valued functions defined as follows:

Definition 1. A real-valued function f on a k -dimensional rectangle $U \subset \mathbb{R}^k$ is said to have nondecreasing increments if

$$f(\xi + h) - f(\xi) \leq f(\eta + h) - f(\eta)$$

whenever $0 \leq h \in \mathbb{R}^k$, $\xi \leq \eta$, $\xi, \eta + h \in U$.

Partial order " \leq " on \mathbb{R}^k is here defined by

$$(x_1, \dots, x_k) \leq (y_1, \dots, y_k) \Leftrightarrow x_1 \leq y_1 \wedge \dots \wedge x_k \leq y_k.$$

In the same paper Brunk gave the following properties of functions with nondecreasing increments:

- (i) A function with nondecreasing increments is not necessarily continuous.
- (ii) If the first partial derivatives of a function $f : U \rightarrow \mathbb{R}$ exist for $x \in U$, then f has nondecreasing increments if and only if each of these partial derivatives is nondecreasing in each argument.
- (iii) If the second partial derivatives of a function $f : U \rightarrow \mathbb{R}$ exist for $x \in U$, then f has nondecreasing increments if and only if each of these partial derivatives is nonnegative.
- (iv) If a function f with nondecreasing increments is continuous for $b \leq x \leq a + b$, where $0 \leq a \in \mathbb{R}^k$, then the function $\Phi : [0, 1] \rightarrow \mathbb{R}$ defined by $\Phi(t) = f(ta + b)$ is convex.

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One very important continuous function with nondecreasing increments is $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$\varphi(x, y) = xy.$$

Another useful continuous function with nondecreasing increments is $\psi : [0, \infty)^k \rightarrow \mathbb{R}$ defined by

$$\psi(\mathbf{x}) = \prod_{i=1}^k x_i.$$

These two functions play a significant role in Section 3: they will help us to obtain refinements of Čebyšev’s inequality (see [8, p. 197]) and also Hölder’s inequality (see [8, p. 112]).

It can be easily seen that if defined on $[a, b] \subset \mathbb{R}$ functions with nondecreasing increments are Wright-convex functions, and as it is well known, the class of convex functions is a proper subclass of the class of Wright-convex functions (see for example [8, p. 7]). Also, the following characterization due to C. T. Ng is known [4]: a function f is Wright convex if and only if it is of the form $f = g + h$, where g is convex and h is additive.

All continuous functions with nondecreasing increments have a useful property: the well known Jensen-Steffensen inequality holds for them (see [5]). Jensen’s inequality for functions with nondecreasing increments can be obtained as a special case of the Jensen-Steffensen inequality, but it is also a simple consequence of its integral variant proved in [2].

Theorem A. *Let $f : U \rightarrow \mathbb{R}$, where $U \subset \mathbb{R}^k$ is a k -dimensional rectangle, be a continuous function with nondecreasing increments, let \mathbf{w} be a nonnegative n -tuple such that $W_n = \sum_{i=1}^n w_i > 0$, and let $\xi^{(i)} \in U$ ($i = 1, \dots, n$) be such that*

$$\xi^{(1)} \leq \dots \leq \xi^{(n)} \quad \text{or} \quad \xi^{(1)} \geq \dots \geq \xi^{(n)}.$$

Then

$$(1) \quad f\left(\frac{1}{W_n} \sum_{i=1}^n w_i \xi^{(i)}\right) \leq \frac{1}{W_n} \sum_{i=1}^n w_i f\left(\xi^{(i)}\right).$$

Remark 1. Let f be as in Theorem A and let \mathbf{w} be a real n -tuple such that

$$w_1 > 0, \quad w_i \leq 0 \quad (i = 2, \dots, n), \quad W_n > 0.$$

Let $\xi^{(i)}$ ($i = 1, \dots, n$) be a sequence of k -tuples in U such that

$$\xi^{(1)} \leq \dots \leq \xi^{(n)} \quad \text{or} \quad \xi^{(1)} \geq \dots \geq \xi^{(n)}$$

and

$$\frac{1}{W_n} \sum_{i=1}^n w_i \xi^{(i)} \in U.$$

Then the reversed inequality (1) is valid. This reversed Jensen’s inequality for functions with nondecreasing increments is a simple consequence of Theorem A (see [6]).

The interested reader can find more details about Jensen's and the Jensen-Steffensen inequality in [8, p. 43-63], and about functions with nondecreasing increments in [5], [6], and [7].

A. McD. Mercer [3] gave the following interesting variant of Jensen's inequality, to which we will refer as to the "Jensen's inequality of Mercer's type".

Theorem B. *Let I be an interval in \mathbb{R} and let $[a, b] \subseteq I$, $a < b$. Let \mathbf{x} be a monotonic n -tuple in $[a, b]^n$ and let \mathbf{p} be a nonnegative n -tuple such that $P_n = \sum_{i=1}^n p_i > 0$. If $f : I \rightarrow \mathbb{R}$ is convex, then*

$$(2) \quad f\left(a + b - \frac{1}{P_n} \sum_{i=1}^n p_i x_i\right) \leq f(a) + f(b) - \frac{1}{P_n} \sum_{i=1}^n p_i f(x_i) .$$

This Mercer's result was generalized in several ways in [1].

The goal of this paper is to give a refinement of the inequality (2) for multivariate real-valued functions and to present several applications of that result. In Section 2 we first give the Jensen's inequality of Mercer's type for functions with nondecreasing increments. Immediately after that we prove its refinement using an index set function. We also show that the monotonicity condition on k -tuples $\xi^{(i)}$ ($i = 1, \dots, n$) can be relaxed on the monotonicity in mean condition. In Section 3 we show how these results can be used to obtain analogous refinements of Čebyšev's inequality (Subsection 3.1) and Hölder's inequality (Subsection 3.2) for monotonic sequences.

2. Main results

Throughout the rest of this paper with U we denote a k -dimensional rectangle $[a_1, b_1] \times \dots \times [a_k, b_k] \subset \mathbb{R}^k$, $a_i < b_i$ ($i = 1, \dots, k$), and with \mathbf{a} and \mathbf{b} two specially chosen k -tuples related to the rectangle U :

$$\mathbf{a} = (a_1, \dots, a_k), \quad \mathbf{b} = (b_1, \dots, b_k).$$

We need the following lemma.

Lemma 1. *Let $f : U \rightarrow \mathbb{R}$ be a function with nondecreasing increments. Then for any $\mathbf{x} \in U$*

$$f(\mathbf{a} + \mathbf{b} - \mathbf{x}) \leq f(\mathbf{a}) + f(\mathbf{b}) - f(\mathbf{x}).$$

Proof. It can be easily seen that $\mathbf{a} \leq \mathbf{x} \leq \mathbf{b}$. If we define

$$\xi = \mathbf{a}, \quad \eta = \mathbf{x}, \quad \mathbf{h} = \mathbf{b} - \mathbf{x},$$

then from Definition 1 we obtain

$$f(\mathbf{a} + \mathbf{b} - \mathbf{x}) - f(\mathbf{a}) \leq f(\mathbf{b}) - f(\mathbf{x}).$$

□

In the next theorem we give the Mercer's type variant of Jensen's inequality for functions with nondecreasing increments.

Theorem 1. Let $f : U \rightarrow \mathbb{R}$ be a continuous function with nondecreasing increments and let $\mathbf{x}^{(i)} \in U$ ($i = 1, \dots, n$) satisfy the condition

$$\mathbf{x}^{(1)} \leq \dots \leq \mathbf{x}^{(n)} \quad \text{or} \quad \mathbf{x}^{(1)} \geq \dots \geq \mathbf{x}^{(n)}.$$

If \mathbf{p} is a nonnegative n -tuple such that $P_n > 0$, then

$$(3) \quad f\left(\mathbf{a} + \mathbf{b} - \frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}^{(i)}\right) \leq f(\mathbf{a}) + f(\mathbf{b}) - \frac{1}{P_n} \sum_{i=1}^n p_i f(\mathbf{x}^{(i)}).$$

If \mathbf{p} is a real n -tuple such that

$$(4) \quad p_1 > 0, \quad p_i \leq 0 \quad (i = 2, \dots, n), \quad P_n > 0,$$

and if

$$(5) \quad \frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}^{(i)} \in U,$$

then (3) remains valid.

Proof. Suppose that \mathbf{p} is a nonnegative n -tuple such that $P_n > 0$. We have

$$f\left(\mathbf{a} + \mathbf{b} - \frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}^{(i)}\right) = f\left(\frac{1}{P_n} \sum_{i=1}^n p_i (\mathbf{a} + \mathbf{b} - \mathbf{x}^{(i)})\right).$$

Since

$$\mathbf{a} + \mathbf{b} - \mathbf{x}^{(1)} \leq \dots \leq \mathbf{a} + \mathbf{b} - \mathbf{x}^{(n)} \quad \text{or} \quad \mathbf{a} + \mathbf{b} - \mathbf{x}^{(1)} \geq \dots \geq \mathbf{a} + \mathbf{b} - \mathbf{x}^{(n)},$$

from Theorem A we obtain

$$f\left(\mathbf{a} + \mathbf{b} - \frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}^{(i)}\right) \leq \frac{1}{P_n} \sum_{i=1}^n p_i f(\mathbf{a} + \mathbf{b} - \mathbf{x}^{(i)}),$$

and if we apply Lemma 1 (3) immediately follows. If \mathbf{p} is a real n -tuple such that (4) and (5) hold, then from Lemma 1 and Remark 1 we have

$$\begin{aligned} f\left(\mathbf{a} + \mathbf{b} - \frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}^{(i)}\right) &\leq f(\mathbf{a}) + f(\mathbf{b}) - f\left(\frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}^{(i)}\right) \\ &\leq f(\mathbf{a}) + f(\mathbf{b}) - \frac{1}{P_n} \sum_{i=1}^n p_i f(\mathbf{x}^{(i)}). \end{aligned}$$

This completes the proof. □

In order to present our next result we need to introduce some more notation.

Let $f : U \rightarrow \mathbb{R}$ and let I be a nonempty finite set of positive integers. Let $\mathbf{X} = \{\mathbf{x}^{(i)} \mid i \in I\}$ be a sequence in U such that $\mathbf{a} \leq \mathbf{x}^{(i)} \leq \mathbf{b}$ ($i \in I$). Let

$\mathbf{p} = \{p_i \mid i \in I\}$ be a real sequence such that $\mathbf{a} + \mathbf{b} - \frac{1}{P_I} \sum_{i \in I} p_i \mathbf{x}^{(i)} \in U$ and $P_I = \sum_{i \in I} p_i \neq 0$. We define the index set function F by

$$F(I) = P_I (f(\mathbf{a}) + f(\mathbf{b})) - \sum_{i \in I} p_i f(\mathbf{x}^{(i)}) - P_I f\left(\mathbf{a} + \mathbf{b} - \frac{1}{P_I} \sum_{i \in I} p_i \mathbf{x}^{(i)}\right).$$

We also define the arithmetic mean \mathbf{A}_I over the index set I by

$$\mathbf{A}_I(\mathbf{X}; \mathbf{p}) = \frac{1}{P_I} \sum_{i \in I} p_i \mathbf{x}^{(i)}.$$

To simplify the notation, in the following we will sometimes write \mathbf{A}_I instead of $\mathbf{A}_I(\mathbf{X}; \mathbf{p})$.

Theorem 2. *Let $f : U \rightarrow \mathbb{R}$ be a continuous function with nondecreasing increments and let I and J be finite nonempty sets of positive integers such that $I \cap J = \emptyset$. Let $\mathbf{p} = \{p_i \mid i \in I \cup J\}$ be a real sequence such that $P_{I \cup J} > 0$, and let $\mathbf{X} = \{\mathbf{x}^{(i)} \mid i \in I \cup J\}$ be a sequence in U such that $\mathbf{A}_S(\mathbf{X}; \mathbf{p}) \in U$ ($S = I, J, I \cup J$).*

If $P_I > 0, P_J > 0$ and if

$$(6) \quad \mathbf{A}_I(\mathbf{X}; \mathbf{p}) \leq \mathbf{A}_J(\mathbf{X}; \mathbf{p}) \quad \text{or} \quad \mathbf{A}_J(\mathbf{X}; \mathbf{p}) \leq \mathbf{A}_I(\mathbf{X}; \mathbf{p}),$$

then

$$(7) \quad F(I \cup J) \geq F(I) + F(J).$$

If $P_I P_J < 0$ and if (6) holds, then (7) is reversed.

Proof. Suppose that $P_I > 0$ and $P_J > 0$. Let

$$\xi^{(1)} = \mathbf{a} + \mathbf{b} - \mathbf{A}_I, \quad \xi^{(2)} = \mathbf{a} + \mathbf{b} - \mathbf{A}_J, \quad w_1 = P_I, \quad w_2 = P_J.$$

We have $\xi^{(1)} \leq \xi^{(2)}$ or $\xi^{(2)} \leq \xi^{(1)}$ and $w_i > 0$ ($i = 1, 2$), so from Theorem A we obtain

$$f\left(\frac{1}{W_2} \sum_{i=1}^2 w_i \xi^{(i)}\right) \leq \frac{1}{W_2} \sum_{i=1}^2 w_i f(\xi^{(i)}),$$

i.e.,

$$(8) \quad \begin{aligned} & f\left(\frac{1}{P_{I \cup J}} (P_I (\mathbf{a} + \mathbf{b} - \mathbf{A}_I) + P_J (\mathbf{a} + \mathbf{b} - \mathbf{A}_J))\right) \\ & \leq \frac{1}{P_{I \cup J}} (P_I f(\mathbf{a} + \mathbf{b} - \mathbf{A}_I) + P_J f(\mathbf{a} + \mathbf{b} - \mathbf{A}_J)). \end{aligned}$$

Since $P_{I \cup J} > 0$, from (8) we have

$$\begin{aligned} & P_{I \cup J} f\left(\mathbf{a} + \mathbf{b} - \frac{1}{P_{I \cup J}} \sum_{i \in I \cup J} p_i \mathbf{x}^{(i)}\right) \\ & \leq P_I f\left(\mathbf{a} + \mathbf{b} - \frac{1}{P_I} \sum_{i \in I} p_i \mathbf{x}^{(i)}\right) + P_J f\left(\mathbf{a} + \mathbf{b} - \frac{1}{P_J} \sum_{i \in J} p_i \mathbf{x}^{(i)}\right). \end{aligned}$$

If we multiply the above inequality by (-1) and add to the both sides

$$P_{I \cup J} (f(\mathbf{a}) + f(\mathbf{b})) - \sum_{i \in I \cup J} p_i f(\mathbf{x}^{(i)}),$$

we obtain

$$\begin{aligned} & P_{I \cup J} (f(\mathbf{a}) + f(\mathbf{b})) - \sum_{i \in I \cup J} p_i f(\mathbf{x}^{(i)}) - P_{I \cup J} f\left(\mathbf{a} + \mathbf{b} - \frac{1}{P_{I \cup J}} \sum_{i \in I \cup J} p_i \mathbf{x}^{(i)}\right) \\ & \geq P_I (f(\mathbf{a}) + f(\mathbf{b})) - \sum_{i \in I} p_i f(\mathbf{x}^{(i)}) - P_I f\left(\mathbf{a} + \mathbf{b} - \frac{1}{P_I} \sum_{i \in I \cup J} p_i \mathbf{x}^{(i)}\right) \\ & \quad + P_J (f(\mathbf{a}) + f(\mathbf{b})) - \sum_{i \in J} p_i f(\mathbf{x}^{(i)}) - P_J f\left(\mathbf{a} + \mathbf{b} - \frac{1}{P_J} \sum_{i \in J} p_i \mathbf{x}^{(i)}\right) \end{aligned}$$

from which (7) immediately follows. In case when $P_I P_J < 0$, for instance $P_I > 0$ and $P_J < 0$, we again define

$$\boldsymbol{\xi}^{(1)} = \mathbf{a} + \mathbf{b} - \mathbf{A}_I, \quad \boldsymbol{\xi}^{(2)} = \mathbf{a} + \mathbf{b} - \mathbf{A}_J, \quad w_1 = P_I, \quad w_2 = P_J,$$

and the reversed (7) follows from Remark 1. □

Remark 2. It can be easily shown that the conditions

$$\mathbf{A}_I \leq \mathbf{A}_J \quad \text{or} \quad \mathbf{A}_J \leq \mathbf{A}_I$$

are equivalent to the conditions

$$\mathbf{A}_I \leq \mathbf{A}_{I \cup J} \quad \text{or} \quad \mathbf{A}_{J \cup I} \leq \mathbf{A}_I.$$

Corollary 1. *Let $f : U \rightarrow \mathbb{R}$ be a continuous function with nondecreasing increments, let $\mathbf{x}^{(i)} \in U$ ($i = 1, \dots, n$) satisfy the condition*

$$\mathbf{x}^{(1)} \leq \dots \leq \mathbf{x}^{(n)} \quad \text{or} \quad \mathbf{x}^{(1)} \geq \dots \geq \mathbf{x}^{(n)},$$

and let \mathbf{p} be a real n -tuple such that $p_1 > 0$. If

$$(9) \quad p_i \geq 0 \quad (i = 2, \dots, n),$$

then

$$(10) \quad 0 \leq F(I_1) \leq \dots \leq F(I_{n-1}) \leq F(I_n),$$

where $I_j = \{1, \dots, j\}$. If

$$(11) \quad p_i \leq 0 \quad (i = 2, \dots, n), \quad P_n > 0,$$

and if

$$\frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}^{(i)} \in U,$$

then

$$(12) \quad 0 \leq F(I_n) \leq F(I_{n-1}) \leq \dots \leq F(I_1).$$

Proof. Without loss of generality we may assume that $\mathbf{x}^{(1)} \leq \dots \leq \mathbf{x}^{(n)}$. Suppose that \mathbf{p} satisfies (9). For such \mathbf{p} we have

$$P_1 = p_1 > 0, \dots, P_n > 0.$$

First we show that $F(\{l\}) \geq 0$ for any $l \in I_n$. From the definition of F we have

$$\begin{aligned} F(\{l\}) &= p_l (f(\mathbf{a}) + f(\mathbf{b})) - p_l f(\mathbf{x}^{(l)}) - p_l f(\mathbf{a} + \mathbf{b} - \mathbf{x}^{(i)}) \\ &= p_l \left[f(\mathbf{a}) + f(\mathbf{b}) - f(\mathbf{x}^{(l)}) - f(\mathbf{a} + \mathbf{b} - \mathbf{x}^{(i)}) \right], \end{aligned}$$

hence by Lemma 1 we obtain $F(\{l\}) \geq 0$. We also have

$$\sum_{i=1}^{n-1} p_i (\mathbf{x}^{(i)} - \mathbf{x}^{(n)}) = P_{n-1} (\mathbf{x}^{(n-1)} - \mathbf{x}^{(n)}) + \sum_{i=1}^{n-2} P_i (\mathbf{x}^{(i)} - \mathbf{x}^{(i+1)}) \leq \mathbf{0}.$$

This means that

$$A_{I_{n-1}} = \frac{1}{P_{n-1}} \sum_{i=1}^{n-1} p_i \mathbf{x}^{(i)} \leq \mathbf{x}^{(n)}.$$

If we let $I = I_{n-1}$ and $J = \{n\}$, then

$$A_I \leq A_J, \quad P_I > 0, P_J > 0, P_{I \cup J} > 0,$$

and all the other conditions of Theorem 2 are also satisfied, so we have

$$F(I_n) \geq F(I_{n-1}) + F(\{n\}) \geq F(I_{n-1}).$$

By iteration we obtain (10), where the case $F(I_1) \geq 0$ follows from Lemma 1. Suppose that \mathbf{p} satisfies the conditions (11) and (5). Again we have

$$P_1 = p_1 > 0, \dots, P_n > 0.$$

Similarly as before we obtain $F(\{l\}) \leq 0$ for any $l \in \{2, \dots, n\}$. If we let $I = I_{n-1}$ and $J = \{n\}$, we have

$$A_I \leq A_J, \quad P_I > 0, P_J < 0, P_{I \cup J} > 0,$$

and from the second part of Theorem 2 we obtain

$$F(I_n) \leq F(I_{n-1}) + F(\{n\}) \leq F(I_{n-1}).$$

In this case it can be proved that

$$A_{I_n} \leq \dots \leq A_{I_2} \leq \mathbf{x}^{(1)} \leq \dots \leq \mathbf{x}^{(n)},$$

and since by the assumption we know that $A_{I_n} \in U$ we may deduce that all terms in (12) are well-defined. We proceed by iteration, and since from the second part of Theorem 1 we also have $F(I_n) \geq 0$ (12) is proved. \square

In [6] J. Pečarić proved that the monotonicity conditions on k -tuples $\boldsymbol{\xi}^{(i)}$ ($i = 1, \dots, n$) in Theorem A can be relaxed on the monotonicity in mean conditions. In the next theorem we show that it can be also done in Corollary 1.

Theorem 3. Let $f : U \rightarrow \mathbb{R}$ be a continuous function with nondecreasing increments and let \mathbf{p} be a real n -tuple such that $p_1 > 0$. Let $\mathbf{x}^{(i)} \in U$ ($i = 1, \dots, n$) satisfy the condition

$$(13) \quad \mathbf{x}^{(n)} \leq \mathbf{A}_{I_2} \leq \dots \leq \mathbf{A}_{I_n} \quad \text{or} \quad \mathbf{x}^{(1)} \geq \mathbf{A}_{I_2} \geq \dots \geq \mathbf{A}_{I_n}.$$

If \mathbf{p} satisfies the condition (9), then (10) holds.

If \mathbf{p} satisfies the conditions (11) and (5), then (12) holds.

Proof. Suppose that \mathbf{p} satisfies the condition (9). If we have $\mathbf{A}_{I_{n-1}} \leq \mathbf{A}_{I_n}$, we know that

$$\sum_{i \in I_n} \frac{P_n - P_{n-1}}{P_n P_{n-1}} p_i \mathbf{x}^{(i)} \leq \frac{p_n}{P_n} \mathbf{x}^{(n)},$$

which can be rewritten as $\mathbf{A}_{I_{n-1}} \leq \mathbf{x}^{(n)}$. Using the substitutions $I = I_{n-1}$ and $J = \{n\}$, from Theorem 2 we obtain

$$F(I_n) \geq F(I_{n-1}),$$

wherefrom we get (10) by iteration. The other case $\mathbf{A}_{I_{n-1}} \geq \mathbf{A}_{I_n}$ is analogous. We can prove the second part of the theorem similarly. \square

Remark 3. Similar assertions can be formulated for P-convex functions (definition of P-convex functions and their properties can be found in [7]).

3. Applications

3.1. A variant of Čebyšev's inequality

In this subsection we show how Corollary 1 can be used to obtain a refinement of the Mercer's type variant of Čebyšev's inequality for two monotonic sequences.

First we introduce some more notation. For a real n -tuple \mathbf{p} and two monotonic real n -tuples $\mathbf{x} \in [a, b]^n$ and $\mathbf{y} \in [c, d]^n$, we define the index set function C by

$$C(I) = P_I(ac + bd) - \sum_{i \in I} p_i x_i y_i - P_I \left(a + b - \frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \left(c + d - \frac{1}{P_I} \sum_{i \in I} p_i y_i \right).$$

The following theorem is valid.

Theorem 4. Let a, b, c and d be real numbers such that $a < b$, $c < d$, and let $\mathbf{x} \in [a, b]^n$ and $\mathbf{y} \in [c, d]^n$. Let \mathbf{p} be a real n -tuple such that $p_1 > 0$.

If \mathbf{p} satisfies the condition (9) and if \mathbf{x} and \mathbf{y} are monotonic in the same direction, then

$$(14) \quad 0 \leq C(I_1) \leq \dots \leq C(I_{n-1}) \leq C(I_n).$$

If \mathbf{x} and \mathbf{y} are monotonic in opposite directions, then all inequalities in (14) are reversed.

If \mathbf{p} satisfies the conditions (11), if

$$\frac{1}{P_n} \sum_{i=1}^n p_i x_i \in [a, b], \quad \frac{1}{P_n} \sum_{i=1}^n p_i y_i \in [c, d],$$

and if \mathbf{x} and \mathbf{y} are monotonic in the same direction, then

$$(15) \quad 0 \leq C(I_n) \leq C(I_{n-1}) \leq \dots \leq C(I_1).$$

If \mathbf{x} and \mathbf{y} are monotonic in opposite directions, then all inequalities in (15) are reversed.

Proof. Suppose that \mathbf{p} satisfies the condition (9) and that the n -tuples \mathbf{x} and \mathbf{y} are monotonic in the same direction. In order to obtain (14) we simply apply Corollary 1 on the function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $\varphi(x, y) = xy$ and on the n -tuples \mathbf{x} , \mathbf{y} and \mathbf{p} . If n -tuples \mathbf{x} and \mathbf{y} are monotonic in opposite directions we apply Corollary 1 on \mathbf{x} , $-\mathbf{y} \in [-d, -c]^n$ and \mathbf{p} , in which case we obtain the reversed (14). The second case can be proved in a similar way. \square

Remark 4. Similarly as in Theorem 3, the “same monotonicity” condition on \mathbf{x} and \mathbf{y} can be replaced with the conditions

$$(16) \quad x_1 \leq \frac{1}{P_2} \sum_{i=1}^2 p_i x_i \leq \dots \leq \frac{1}{P_n} \sum_{i=1}^n p_i x_i,$$

$$(17) \quad y_1 \leq \frac{1}{P_2} \sum_{i=1}^2 p_i y_i \leq \dots \leq \frac{1}{P_n} \sum_{i=1}^n p_i y_i,$$

or with (16) and (17) both reversed, and the “opposite monotonicity” condition with

$$(18) \quad x_1 \leq \frac{1}{P_2} \sum_{i=1}^2 p_i x_i \leq \dots \leq \frac{1}{P_n} \sum_{i=1}^n p_i x_i,$$

$$(19) \quad y_1 \geq \frac{1}{P_2} \sum_{i=1}^2 p_i y_i \geq \dots \geq \frac{1}{P_n} \sum_{i=1}^n p_i y_i,$$

or with (18) and (19) both reversed.

3.2. Two variants of Hölder’s inequality

In this subsection we present two refinements of Hölder’s inequality. The first refinement is for $k \geq 2$ monotonic sequences in which exponents p and q need not to be conjugate. The second refinement is for two monotonic sequences with exponents p, q and r . Here, with U we denote a positive k -dimensional rectangle $[a_1, b_1] \times \dots \times [a_k, b_k]$, where $0 < a_i < b_i$ ($i = 1, \dots, k$). Similarly

as in the previous subsection, for k -tuples $\mathbf{x}^{(i)} \in U$ ($i = 1, \dots, n$) such that $\mathbf{x}^{(1)} \leq \dots \leq \mathbf{x}^{(n)}$ or $\mathbf{x}^{(1)} \geq \dots \geq \mathbf{x}^{(n)}$ we define the index set function H by

$$H(I) = W_I \left(\prod_{j=1}^k a_j + \prod_{j=1}^k b_j \right) - \sum_{i \in I} w_i \prod_{j=1}^k x_j^{(i)} - W_I \prod_{j=1}^k \left(a_j^{p_j} + b_j^{p_j} - \frac{1}{W_I} \sum_{i \in I} w_i \left(x_j^{(i)} \right)^{p_j} \right)^{\frac{1}{p_j}}.$$

Theorem 5. Let $\mathbf{p} \in \langle -\infty, 0 \rangle^k \cup \langle 0, 1 \rangle^k$, let $\mathbf{x}^{(i)} \in U$ ($i = 1, \dots, n$) be such that

$$\mathbf{x}^{(1)} \leq \dots \leq \mathbf{x}^{(n)} \quad \text{or} \quad \mathbf{x}^{(1)} \geq \dots \geq \mathbf{x}^{(n)},$$

and let \mathbf{w} be a real n -tuple such that $w_1 > 0$.

If

$$(20) \quad w_i \geq 0 \quad (i = 2, \dots, n),$$

then

$$(21) \quad 0 \leq H(I_1) \leq \dots \leq H(I_{n-1}) \leq H(I_n).$$

If

$$(22) \quad w_i \leq 0 \quad (i = 2, \dots, n), \quad W_n > 0,$$

and if

$$\frac{1}{W_n} \sum_{i=1}^n w_i \mathbf{x}^{(i)} \in U,$$

then

$$0 \leq H(I_n) \leq H(I_{n-1}) \leq \dots \leq H(I_1).$$

Proof. For some fixed $p_j \in \mathbb{R} \setminus \{0\}$ ($j = 1, \dots, k$) we define the function $\varphi : \langle 0, +\infty \rangle^k \rightarrow \mathbb{R}$ by

$$(23) \quad \varphi(\mathbf{x}) = \prod_{j=1}^k x_j^{\frac{1}{p_j}}.$$

We have

$$\begin{aligned} \varphi_{x_j x_j}(\mathbf{x}) &= \frac{1}{p_j} \left(\frac{1}{p_j} - 1 \right) x_j^{-2} \prod_{l=1}^k x_l^{\frac{1}{p_l}}, \quad (j = 1, \dots, k), \\ \varphi_{x_i x_j}(\mathbf{x}) &= \frac{1}{p_i} \frac{1}{p_j} x_i^{-1} x_j^{-1} \prod_{l=1}^k x_l^{\frac{1}{p_l}}, \quad (i, j = 1, \dots, k), \quad i \neq j, \end{aligned}$$

so it can be easily seen that φ is continuous and that it has nondecreasing increments if and only if p_j ($j = 1, \dots, k$) are all negative or if p_j ($j = 1, \dots, k$) are all positive and not greater than 1. Suppose that $\mathbf{x}^{(1)} \leq \dots \leq \mathbf{x}^{(n)}$ and

$\mathbf{p} \in \langle -\infty, 0 \rangle^k \cup \langle 0, 1 \rangle^k$ (if $\mathbf{x}^{(1)} \geq \dots \geq \mathbf{x}^{(n)}$ the proof is similar). We define the k -tuples $\xi^{(i)}$ ($i = 1, \dots, n$) by

$$\xi_j^{(i)} = \left(x_j^{(i)}\right)^{p_j}, \quad (j = 1, \dots, k).$$

Obviously, if $\mathbf{p} \in \langle -\infty, 0 \rangle^k$ we have $\xi^{(1)} \geq \dots \geq \xi^{(n)}$ and $\xi^{(i)} \in [b_1^{p_1}, a_1^{p_1}] \times \dots \times [b_k^{p_k}, a_k^{p_k}]$ ($i = 1, \dots, n$), and if $\mathbf{p} \in \langle 0, 1 \rangle^k$ we have $\xi^{(1)} \leq \dots \leq \xi^{(n)}$ and $\xi^{(i)} \in [a_1^{p_1}, b_1^{p_1}] \times \dots \times [a_k^{p_k}, b_k^{p_k}]$ ($i = 1, \dots, n$). In both cases we can apply Corollary 1 on $\xi^{(1)}, \dots, \xi^{(n)}$, \mathbf{w} satisfying (20) and φ to obtain (21). The other case can be proved analogously. \square

Corollary 2. *Let \mathbf{w} be a real n -tuple such that $w_1 > 0$ and let $(p, q) \in \langle -\infty, 0 \rangle^2 \cup \langle 0, 1 \rangle^2$. Let $\mathbf{x} \in [a, b]^n$, $0 < a < b$, and $\mathbf{y} \in [c, d]^n$, $0 < c < d$, be two n -tuples monotonic in the same direction.*

If \mathbf{w} satisfies the condition (20), then

$$\begin{aligned} & \left(a^p + b^p - \frac{1}{W_n} \sum_{i=1}^n w_i x_i^p \right)^{\frac{1}{p}} \left(c^q + d^q - \frac{1}{W_n} \sum_{i=1}^n w_i y_i^q \right)^{\frac{1}{q}} \\ (24) \quad & \leq ac + bd - \frac{1}{W_n} \sum_{i=1}^n w_i x_i y_i. \end{aligned}$$

If \mathbf{w} satisfies the conditions (22) and if

$$(25) \quad \frac{1}{W_n} \sum_{i=1}^n w_i x_i \in [a, b], \quad \frac{1}{W_n} \sum_{i=1}^n w_i y_i \in [c, d],$$

then (24) remains valid.

Proof. This variant of Hölder's inequality is a special case of Theorem 5 for $k = 2$. \square

Next we give our second variant of Hölder's inequality. In this case the index set function R is defined by

$$\begin{aligned} R(I) = & W_I (a^r c^r + b^r d^r) - \sum_{i \in I} w_i x_i^r y_i^r \\ & - W_I \left(a^p + b^p - \frac{1}{W_I} \sum_{i \in I} w_i x_i^p \right)^{\frac{r}{p}} \left(c^q + d^q - \frac{1}{W_I} \sum_{i \in I} w_i y_i^q \right)^{\frac{r}{q}}, \end{aligned}$$

where \mathbf{w} is a real n -tuple and the n -tuples $\mathbf{x} \in [a, b]^n$, $0 < a < b$, and $\mathbf{y} \in [c, d]^n$, $0 < c < d$, are monotonic in the same direction.

Theorem 6. *Let p, q be real numbers such that $pq > 0$ and let r be a real number such that $r > 0$ and $r \geq \max\{p, q\}$, or $r < 0$ and $r \leq \min\{p, q\}$. Let $\mathbf{x} \in [a, b]^n$, $0 < a < b$, and $\mathbf{y} \in [c, d]^n$, $0 < c < d$, be two n -tuples monotonic*

in the same direction and let w be a real n -tuple such that $w_1 > 0$.

If w satisfies the condition (20), then

$$(26) \quad 0 \leq R(I_1) \leq \cdots \leq R(I_{n-1}) \leq R(I_n).$$

If w satisfies the conditions (22) and (25), then

$$(27) \quad 0 \leq R(I_n) \leq R(I_{n-1}) \leq \cdots \leq R(I_1).$$

Proof. For some fixed $p, q, r \in \mathbb{R} \setminus \{0\}$ we define the function $\varphi : \langle 0, +\infty \rangle^2 \rightarrow \mathbb{R}$ by

$$\varphi(x, y) = x^{\frac{r}{p}} y^{\frac{r}{q}}.$$

Obviously, φ is continuous and we have

$$\begin{aligned} \varphi_{xx}(x, y) &= \frac{r(r-p)}{p^2} x^{\frac{r}{p}-2} y^{\frac{r}{q}}, \\ \varphi_{xy}(x, y) &= \frac{r^2}{pq} x^{\frac{r}{p}-1} y^{\frac{r}{q}-1}, \\ \varphi_{yy}(x, y) &= \frac{r(r-q)}{q^2} x^{\frac{r}{p}} y^{\frac{r}{q}-2}, \end{aligned}$$

so it can be easily seen that the function φ has nondecreasing increments if and only if $pq > 0$ and $r > 0$, $r \geq \max\{p, q\}$ or $r < 0$, $r \leq \min\{p, q\}$. Suppose that w satisfies the condition (20). Without any loss of generality we may assume that both x and y are increasing. We define

$$\begin{aligned} \xi_i &= x_i^p, \quad (i = 1, \dots, n), \\ \eta_i &= y_i^q, \quad (i = 1, \dots, n). \end{aligned}$$

If p and q are both positive we have

$$\begin{aligned} a^p &\leq \xi_1 \leq \cdots \leq \xi_n \leq b^p, \\ c^q &\leq \eta_1 \leq \cdots \leq \eta_n \leq d^q, \end{aligned}$$

and if they are both negative we have

$$\begin{aligned} b^p &\leq \xi_n \leq \cdots \leq \xi_1 \leq a^p, \\ d^q &\leq \eta_n \leq \cdots \leq \eta_1 \leq c^q. \end{aligned}$$

Suppose that $r > 0$ and $r \geq \max\{p, q\}$ or $r < 0$ and $r \leq \min\{p, q\}$. If we apply Corollary 1 on ξ , η , w and φ we obtain (26). The other case can be proved analogously. \square

Remark 5. Similarly as in the previous subsection, the monotonicity conditions in Theorem 5 and Theorem 6 can be relaxed on the monotonicity in mean conditions.

Corollary 3. Let $x \in [a, b]^n$, $0 < a < b$, and $y \in [c, d]^n$, $0 < c < d$, be two n -tuples monotonic in the same direction. Let w be a real n -tuple with $w_1 > 0$ and satisfying the condition (20) or the conditions (22) and (25). Let p, q be

real numbers such that $pq > 0$ and let r be a real number such that $r > 0$ and $r \geq \max\{p, q\}$. Then

$$(28) \quad \left(a^p + b^p - \frac{1}{W_n} \sum_{i=1}^n w_i x_i^p \right)^{\frac{1}{p}} \left(c^q + d^q - \frac{1}{W_n} \sum_{i=1}^n w_i y_i^q \right)^{\frac{1}{q}} \\ \leq \left(a^r c^r + b^r d^r - \frac{1}{W_n} \sum_{i=1}^n w_i x_i^r y_i^r \right)^{\frac{1}{r}}.$$

If $r < 0$ and $r \leq \min\{p, q\}$ inequality (28) is reversed.

Proof. Suppose that $pq > 0$, $r > 0$ and $r \geq \max\{p, q\}$. From Theorem 6 we have

$$(29) \quad W_n \left(a^p + b^p - \frac{1}{W_n} \sum_{i=1}^n w_i x_i^p \right)^{\frac{r}{p}} \left(c^q + d^q - \frac{1}{W_n} \sum_{i=1}^n w_i y_i^q \right)^{\frac{r}{q}} \\ \leq W_n (a^r c^r + b^r d^r) - \sum_{i=1}^n w_i x_i^r y_i^r.$$

Since $W_n > 0$ and $r > 0$ we immediately obtain (28). If $pq > 0$, $r < 0$ and $r \leq \min\{p, q\}$ then the inequality (29) follows again from Theorem 6, but since r is negative in the end we obtain the reversed inequality (28). \square

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