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ENERGY FINITE SOLUTIONS OF ELLIPTIC EQUATIONS ON RIEMANNIAN MANIFOLDS

Seok Woo Kim and Yong Hah Lee

Abstract. We prove that for any continuous function f on the s-harmonic ($1 < s < \infty$) boundary of a complete Riemannian manifold M, there exists a solution, which is a limit of a sequence of bounded energy finite solutions in the sense of supremum norm, for a certain elliptic operator $\mathcal{A}$ on M whose boundary value at each s-harmonic boundary point coincides with that of f.

If $E_1, E_2, \ldots, E_l$ are s-nonparabolic ends of M, then we also prove that there is a one to one correspondence between the set of bounded energy finite solutions for $\mathcal{A}$ on M and the Cartesian product of the sets of bounded energy finite solutions for $\mathcal{A}$ on E_i which vanish at the boundary ∂E_i for $i = 1, 2, \ldots, l$.

1. Introduction

In 1975, Yau [13] proved that every positive harmonic function on a complete Riemannian manifold with nonnegative Ricci curvature must be constant. Later, Li-Tam [9] studied various spaces of harmonic functions on a complete Riemannian manifold with nonnegative sectional curvature outside a compact set. In particular, Li-Tam proved that the dimension of the space of positive harmonic functions on such a manifold is equal to the number of ends of the manifold. Such a theory is well developed by works of Donnelly [1], Grigor'yan [2], Holopainen [5], Li-Tam [10], Sung-Tam-Wang [12], present authors [6], [7], and others.

The main purpose of this paper is to prove that bounded energy finite solutions of a nonlinear elliptic operator on a complete Riemannian manifold can be represented in terms of bounded energy finite solutions of such an operator on each end of the manifold. In particular, we give a one to one correspondence between the totality of bounded energy finite solutions of a nonlinear elliptic operator on a complete Riemannian manifold and the Cartesian product of

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those on ends of the manifold. In this paper, we consider a nonlinear elliptic operator $\mathcal{A}$ of type s in the following setting:

Let Ω be an open subset of a complete Riemannian manifold M and $W^{1,s}(\Omega)$, $(1 < s < \infty)$ be the Sobolev space of all functions $u \in L^s(\Omega)$ whose distributional gradient ∇u also belongs to $L^s(\Omega)$. We equip $W^{1,s}(\Omega)$ with the norm $\|u\|_{1,s} = \|u\|_s + \|\nabla u\|_s$. The subspace $W_0^{1,s}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,s}(\Omega)$. We shall pay attention to functionals associated with $\mathbf{F} : T\Omega \to \mathbf{R}$, where

($A1$) the mapping $\mathbf{F}_x = \mathbf{F}|_{T_xM} : T_xM \to \mathbf{R}$ is strictly convex and differentiable for all $x \in \Omega$, and the mapping $x \mapsto \mathbf{F}_x(\xi)$ is measurable whenever ξ is;

($A2$) there exist constants $0 < C_1 \le C_2 < \infty$ such that
$$C_1|\xi|^s \le \mathbf{F}_x(\xi) \le C_2|\xi|^s$$
for all $x \in \Omega$ and $\xi \in T_xM$.

Let $\mathcal{A}_x(\xi) = (A^1(\xi), A^2(\xi), \dots, A^n(\xi))$ be defined by
$$A^i(\xi) = \frac{\partial}{\partial \xi^i}\mathbf{F}_x(\xi) \quad \text{for } i = 1, 2, \dots, n.$$
Then $\mathcal{A}$ also satisfies the following properties: (See [11].)

($A3$) the mapping $\mathcal{A}_x = \mathcal{A}|_{T_xM} : T_xM \to T_xM$ is continuous for a.e. $x \in \Omega$, and the mapping $x \mapsto \mathcal{A}_x(\xi)$ is a measurable vector field whenever ξ is;

for a.e. $x \in \Omega$ and for all $\xi \in T_xM$

($A4$) $\langle \mathcal{A}_x(\xi), \xi\rangle \ge C_1|\xi|^s$;

($A5$) $|\mathcal{A}_x(\xi)| \le C_2|\xi|^{s-1}$;

($A6$) $\langle \mathcal{A}_x(\xi_1) - \mathcal{A}_x(\xi_2), \xi_1 - \xi_2\rangle > 0$ whenever $\xi_1 \ne \xi_2$.

A function u in $W^{1,s}_{\text{loc}}(\Omega)$ is a solution (supersolution, subsolution, respectively) of the equation

$$-\text{div}\mathcal{A}_x(\nabla u) = 0 \ (\ge 0, \le 0, \text{respectively}) \tag{1}$$

in Ω if
$$\int_\Omega \langle \mathcal{A}_x(\nabla u), \nabla\phi\rangle = 0 \ (\ge 0, \le 0, \text{respectively})$$
for any (nonnegative, respectively) $\phi \in C_0^\infty(\Omega)$. A function u is said to be $\mathcal{A}$-harmonic (of type s) if u is a continuous solution of the equation (1). In the typical case $\mathcal{A}_x(\xi) = \xi|\xi|^{s-2}$, $\mathcal{A}$-harmonic functions are called s-harmonic and, in particular, if $s = 2$, we obtain harmonic functions. In fact, if we write
$$\mathbf{J}(u, \Omega) = \int_\Omega \mathbf{F}_x(\nabla u),$$
then given $f \in W^{1,s}(\Omega)$, each $\mathcal{A}$-harmonic function h with $h - f \in W_0^{1,s}(\Omega)$ minimizes the energy functional $\mathbf{J}$ in $\mathcal{U} = \{u \in W^{1,s}(\Omega) : u - f \in W_0^{1,s}(\Omega)\}$.

(See [11].) In the case when an $\mathcal{A}$-harmonic function u on Ω has finite energy, i.e., $\mathbf{J}(u,\Omega) < \infty$, we say that u is an energy finite $\mathcal{A}$-harmonic function on Ω. An important property is the comparison principle as follows: If $u \in W^{1,s}(\Omega)$ is a supersolution and $v \in W^{1,s}(\Omega)$ is a subsolution on an open set Ω and $\min\{u-v,0\} \in W_0^{1,s}(\Omega)$, then $u \geq v$ a.e. in Ω. In particular, if both u and v are $\mathcal{A}$-harmonic in a bounded set Ω and $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω. (See [3].) We now introduce additional assumptions on $\mathbf{F}$ as follows:

($A7$) $\mathcal{A}_x(\lambda\xi) = \lambda|\lambda|^{s-2}\mathcal{A}_x(\xi)$ whenever λ is a nonzero real number; for any $\xi_1, \xi_2 \in T_xM$

($A8$) in case $2 \leq s < \infty$,

$$\mathbf{F}_x\Big(\frac{\xi_1+\xi_2}{2}\Big) + \mathbf{F}_x\Big(\frac{\xi_1-\xi_2}{2}\Big) \leq \frac{1}{2}(\mathbf{F}_x(\xi_1) + \mathbf{F}_x(\xi_2));$$

in case $1 < s \leq 2$,

$$\mathbf{F}_x\Big(\frac{\xi_1+\xi_2}{2}\Big)^{\tilde{s}} + \mathbf{F}_x\Big(\frac{\xi_1-\xi_2}{2}\Big)^{\tilde{s}} \leq \Big(\frac{1}{2}(\mathbf{F}_x(\xi_1) + \mathbf{F}_x(\xi_2))\Big)^{\tilde{s}},$$

where $\tilde{s} = 1/(s-1)$.

Using Clarkson's inequality, the assumption ($A8$) holds in the typical case $\mathbf{F}(\xi) = \frac{1}{s}|\xi|^s$, i.e., the s-harmonic case. (See [4].)

In this paper, we solve the Dirichlet problem on the s-harmonic boundary of a complete Riemannian manifold as follows:

Theorem 1.1. *Let M be a complete Riemannian manifold and $\mathcal{A}$ be an elliptic operator on M satisfying ($A1$), ($A2$), ($A7$), and ($A8$). Then for any continuous function f on the s-harmonic boundary Δ_M of M, there exists an $\mathcal{A}$-harmonic function h on M, which is a limit of a sequence of bounded energy finite $\mathcal{A}$-harmonic functions in the sense of supremum norm, such that for each $\mathbf{x} \in \Delta_M$,*

$$h(\mathbf{x}) = f(\mathbf{x}).$$

In particular, in the case that Δ_M consists of finitely many points $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots$, $\mathbf{x}_l$, for given real numbers $a_1, a_2, \ldots, a_l \in \mathbf{R}$, there exists a bounded energy finite $\mathcal{A}$-harmonic function h on M such that

$$h(\mathbf{x}_i) = a_i \quad for \quad i = 1, 2, \ldots, l. \tag{2}$$

Conversely, each bounded energy finite $\mathcal{A}$-harmonic function h on M is uniquely determined by the values in (2).

We also give a precise relation between the totality of all bounded energy finite $\mathcal{A}$-harmonic functions on the manifold and the Cartesian product of those on its ends as follows:

Theorem 1.2. *Let M be a complete Riemannian manifold and $\mathcal{A}$ be an elliptic operator on M satisfying ($A1$), ($A2$), ($A7$), and ($A8$). Let $E_1, E_2, \ldots, E_l$,*

$l \geq 1$, be s-nonparabolic ends of M, whose s-harmonic boundaries have finite cardinality. Then there exists a one to one correspondence

$$\Phi : \mathcal{HBD}_{\mathcal{A}}(M) \to \prod_{i=1}^{l} \mathcal{HBD}_{\mathcal{A}}(E_i, \partial E_i),$$

where $\mathcal{HBD}_{\mathcal{A}}(X)$ and $\mathcal{HBD}_{\mathcal{A}}(X, \partial X)$ denote the set of bounded energy finite $\mathcal{A}$-harmonic functions on X and the subset of elements of $\mathcal{HBD}_{\mathcal{A}}(X)$ vanishing at ∂X, respectively.

2. Preliminaries

Let M be a complete Riemannian manifold. Let $\mathcal{BD}_s(M)$ be the set of bounded continuous functions u whose distributional gradient ∇u belongs to $L^s(M)$. Then $\mathcal{BD}_s(M)$ forms an algebra over the real number with the usual addition and multiplication of functions and scalar multiplication defined pointwise. We equip $\mathcal{BD}_s(M)$ with the following topology: For a sequence $\{f_n\}$ in $\mathcal{BD}_s(M)$ and a function f on M, we say that $\{f_n\}$ converges to f in $\mathcal{BD}_s(M)$ if $\{f_n\}$ is uniformly bounded on M, $\lim_{n\to\infty} \sup_K |f_n - f| = 0$ for each compact subset K of M, and

$$\lim_{n\to\infty} \int_M |\nabla(f_n - f)|^s = 0.$$

We denote by $\mathcal{BD}_{s,0}(M)$ the closure of the set of all compactly supported smooth functions in $\mathcal{BD}_s(M)$. It is easy to see that $\mathcal{BD}_{s,0}(M)$ is not only a subalgebra but also an ideal of $\mathcal{BD}_s(M)$. We denote by $\mathcal{HBD}_{\mathcal{A}}(M)$ the subset of all bounded energy finite $\mathcal{A}$-harmonic functions in $\mathcal{BD}_s(M)$, where $\mathcal{A}$ is an elliptic operator on M satisfying $(A1), (A2), (A7)$, and $(A8)$.

For a complete Riemannian manifold M, there exists a locally compact Hausdorff space $\hat{M}$, called the Royden s-compactification of M, which contains M as an open dense subset. In particular, every function $f \in \mathcal{BD}_s(M)$ can be extended to a continuous function, denoted again by f, on $\hat{M}$ and the class of such extended functions separates points in $\hat{M}$. Moreover, $\hat{M}$ is unique up to a homeomorphism fixing M. We call the subset $\partial\hat{M} = \hat{M} \setminus M$ the Royden s-boundary of M. Throughout this paper, for a subset A of M, we denote by $\hat{A}$ the closure of A in $\hat{M}$. We define the s-harmonic boundary of M by

$$\Delta_M = \{\mathbf{x} \in \partial\hat{M} : f(\mathbf{x}) = 0 \text{ for all } f \in \mathcal{BD}_{s,0}(M)\}.$$

M is said to be s-nonparabolic if M has positive s-capacity, i.e., there exists a compact subset $K \subset M$ such that

$$\mathrm{Cap}_s(K, \infty, M) = \inf_{\phi} \int_M |\nabla\phi|^s > 0,$$

where the infimum is taken over all compactly supported smooth functions ϕ with $\phi = 1$ on K. Otherwise, M is called s-nonparabolic.

The following two lemmas give some useful properties related to bounded energy finite $\mathcal{A}$-harmonic functions and s-harmonic boundary. (See [8] for the proof.)

Lemma 2.1. *Let M be a complete Riemannian manifold. Then M is s-parabolic if and only if s-harmonic boundary Δ_M is empty.*

Lemma 2.2. *Let M be a complete s-nonparabolic Riemannian manifold. Then we have the followings:*

(i) (*Maximum principle*) *If h is a function in $\mathcal{HBD}_{\mathcal{A}}(M)$ such that*

$$a \le h \le b \quad on \quad \Delta_M$$

for some constants $a \le b$, then $a \le h \le b$ on M.

(ii) (*Duality relation*) $\mathcal{BD}_{s,0}(M) = \{f \in \mathcal{BD}_s(M) : f = 0 \;\; on \;\; \Delta_M\}$.

(iii) (*Royden decomposition*) *For each $f \in \mathcal{BD}_s(M)$, there exist unique $h \in \mathcal{HBD}_{\mathcal{A}}(M)$ and $g \in \mathcal{BD}_{s,0}(M)$ such that $f = h + g$.*

3. s-harmonic boundary and energy finite $\mathcal{A}$-harmonic functions on ends

Let o be a fixed point of M. If $\mathbf{n}(r)$ denotes the number of unbounded components of $M \setminus B_r(o)$, then it is easy to prove that $\mathbf{n}(r)$ is nondecreasing in $r > 0$. Let $\lim_{r\to\infty} \mathbf{n}(r) = k$, where k may be infinity. Then we say that the number of ends of M is k. In particular, if k is finite, there exists $r_0 > 0$ such that $\mathbf{n}(r) = k$ for all $r \ge r_0$. In this case, there exist mutually disjoint unbounded components $E_1, E_2, \dots, E_k$ of $M \setminus B_{r_0}(o)$ and we call each E_i an end of M. All ends are divided into two classes by the following s-parabolicity: We say that E is s-nonparabolic if E has positive s-capacity, i.e.,

$$\mathrm{Cap}_s(\partial E, \infty, E) = \inf_u \int_E |\nabla u|^s > 0,$$

where the infimum is taken over all compactly supported smooth functions u with $u = 1$ on ∂E. Otherwise, E is called an s-parabolic end. If $\mathcal{A}$ is an elliptic operator on M satisfying $(A1), (A2), (A7)$, and $(A8)$, then by the quasi-minimizing property of $\mathcal{A}$-harmonic functions, the s-nonparabolicity of E means that for some $r_1 \ge r_0$, there exists a continuous function u_E, called an $\mathcal{A}$-harmonic measure of E, on E such that

$$\begin{cases} \mathcal{A} u_E = 0 & \text{in } E \setminus \overline{B}_{r_1}(o); \\ u_E = 0 & \text{on } B_{r_1}(o) \cap E; \\ \sup_{E \setminus B_{r_1}(o)} u_E = 1. & \end{cases}$$

For an end E of M, we define the sets $\mathcal{BD}_s(E)$ and $\mathcal{BD}_{s,0}(E)$ in the same way as $\mathcal{BD}_s(M)$ and $\mathcal{BD}_{s,0}(M)$, respectively. Let $\tilde{E}$ be the Royden s-compactification of E and $\partial\tilde{E} = \tilde{E} \setminus E$ be the Royden s-boundary of E. For a subset

A of E, we denote by $\tilde{A}$ the closure of A in $\tilde{E}$. Then the s-harmonic boundary of E is the set

$$\Delta_E = \{\mathbf{x} \in \partial\tilde{E} : f(\mathbf{x}) = 0 \ \text{ for all } \ f \in \mathcal{BD}_{s,0}(E)\}.$$

On the other hand, we define the set $\mathcal{HBD}_{\mathcal{A}}(E, \partial E)$, similarly as $\mathcal{HBD}_{\mathcal{A}}(M)$, by the totality of bounded energy finite $\mathcal{A}$-harmonic functions on E vanishing on ∂E.

Lemma 3.1. *Let Ω be a subset of an end E of a complete Riemannian manifold. Suppose that $\tilde{\Omega} \cap \Delta_E = \emptyset$. Then there exists a function w in $\mathcal{BD}_{s,0}(E)$ such that $w > 1$ on $\tilde{\Omega}$.*

Proof. If $z \in \tilde{\Omega} \cap \partial\tilde{E}$, then $z \notin \Delta_E$ by the assumption. Thus by definition of Δ_E, there exists a function $f \in \mathcal{BD}_{s,0}(E)$ such that $f(z) \neq 0$. Let us define a function w_z on $\tilde{E}$ by

$$w_z(x) = \frac{2|f(x)|}{|f(z)|}.$$

Clearly w_z is a nonnegative function in $\mathcal{BD}_{s,0}(E)$ satisfying $w_z(z) = 2$. From the compactness of $\tilde{\Omega} \cap \partial\tilde{E}$, there exist finitely many points $z_1, z_2, \ldots, z_k$ in $\tilde{\Omega} \cap \partial\tilde{E}$ such that

$$\tilde{\Omega} \cap \partial\tilde{E} \subset \bigcup_{j=1}^{k} \{x \in \tilde{E} : w_{z_j}(x) > 1\}.$$

Now suppose that $z \in \tilde{\Omega} \setminus \cup_{j=1}^{k} \{x \in \tilde{E} : w_{z_j}(x) > 1\}$. Define a function $w_z \in \mathcal{BD}_s(E)$ such that

$$\begin{cases} \mathcal{A}w_z = 0 & \text{in } B_2(z) \setminus \overline{B}_1(z); \\ w_z = 2 & \text{on } \overline{B}_1(z); \\ w_z = 0 & \text{on } E \setminus B_2(z). \end{cases}$$

Then $w_z \in \mathcal{BD}_{s,0}(E)$. From the compactness of $\tilde{\Omega} \setminus \cup_{j=1}^{k} \{x \in \tilde{E} : w_{z_j}(x) > 1\}$, there exist finitely many points $z_{k+1}, z_{k+2}, \ldots, z_{k+m}$ in $\tilde{\Omega} \setminus \cup_{j=1}^{k} \{x \in \tilde{E} : w_{z_j}(x) > 1\}$ such that

$$\Big(\tilde{\Omega} \setminus \bigcup_{j=1}^{k} \{x \in \tilde{E} : w_{z_j}(x) > 1\}\Big) \subset \bigcup_{j=k+1}^{k+m} \{x \in \tilde{E} : w_{z_j}(x) > 1\}.$$

In particular, $w = \sum_{j=1}^{k+m} w_{z_j}$ belongs to $\mathcal{BD}_{s,0}(E)$ and $w > 1$ on $\tilde{\Omega}$. □

We have the maximum principle on each end similarly as on the whole manifold as follows:

Lemma 3.2 (Maximum principle). *Let E be an end of a complete Riemannian manifold and h be a bounded energy finite $\mathcal{A}$-harmonic function defined on E which may not belong to $\mathcal{HBD}_{\mathcal{A}}(E, \partial E)$. Suppose that for some constants $a \leq b$,*

$$a \leq h \leq b \quad \text{on} \quad \Delta_E \cup \partial E.$$

Then $a \leq h \leq b$ on E.

Proof. It suffices to prove that $h \leq b$ on E. Otherwise, we may assume that there exist a point $x_0 \in E$ and a constant c such that $h(x_0) > c > b$. Let Ω be the component of $\{x \in E : h(x) > c\}$ containing x_0. Then $\tilde{\Omega} \cap \Delta_E = \emptyset$. Thus by Lemma 3.1, we can find a function $w \in \mathcal{BD}_{s,0}(E)$ such that $w > 1$ on $\tilde{\Omega}$. Hence there exists a sequence $\{g_n\}$ of compactly supported functions converging to w in $\mathcal{BD}_s(E)$. We denote by $\overline{w}$ and $\overline{g}_n$ the symmetric extension of $w|_\Omega$ and $g_n|_\Omega$ to Ω_d, respectively, where Ω_d denotes the double of Ω. Clearly, each $\overline{g}_n$ is compactly supported on Ω_d, and $\{\overline{g}_n\}$ converges to $\overline{w}$ in $\mathcal{BD}_s(\Omega_d)$. Therefore, $\overline{w} \in \mathcal{BD}_{s,0}(\Omega_d)$ and $\overline{w} > 1$ on Ω_d. Since $\mathcal{BD}_{s,0}(\Omega_d)$ is an ideal of $\mathcal{BD}_s(\Omega_d)$, $1 = (1/\overline{w})\overline{w} \in \mathcal{BD}_{s,0}(\Omega_d)$. Hence there exists a sequence $\{\phi_n\}$ of compactly supported functions converging to 1 in $\mathcal{BD}_s(\Omega_d)$. Then $\{h\phi_n\}$ converges to h in $\mathcal{BD}_s(\Omega)$. Putting $v = h - c$, the function $v\phi_n$ vanishes on $\partial(\Omega \cap K_n)$, where K_n is the support of ϕ_n in Ω_d. Hence we get

$$\mathbf{J}(h, \Omega) = \int_\Omega \langle \mathcal{A}(\nabla h), \nabla v \rangle = \lim_{n\to\infty} \int_\Omega \langle \mathcal{A}(\nabla h), \nabla(v\phi_n) \rangle = 0.$$

This implies that $h \equiv c$ on Ω, which contradicts $h(x_0) > c$. □

We also have the duality relation between $\mathcal{BD}_{s,0}(E)$ and Δ_E as follows:

Lemma 3.3. *Let E be an end of a complete Riemannian manifold. Then*

$$\mathcal{BD}_{s,0}(E) = \{f \in \mathcal{BD}_s(E) : f = 0 \quad on \ \Delta_E\}.$$

Proof. Let $f \in \mathcal{BD}_s(E)$ and $f = 0$ on Δ_E. Let $\{h_n\}_{n\in\mathbf{N}}$ be a sequence of continuous functions on M such that

$$\begin{cases} \mathcal{A}h_n = 0 & \text{in } B_n(o) \cap E; \\ h_n = 0 & \text{on } \partial E; \\ h_n = f & \text{on } E \setminus B_n(o). \end{cases}$$

Obviously, $h_n \in \mathcal{BD}_s(E)$ and $|h_n| \leq \sup_E |f|$ on E, hence the sequence $\{h_n\}$ is equicontinuous. By Ascoli's theorem, there exists a subsequence $\{h_{n_k}\}$ converging uniformly to a limit function h on any compact subset of E. Then h is an $\mathcal{A}$-harmonic function by the result of [3].

On the other hand, there exists a constant $\alpha < \infty$ such that

$$\alpha = \inf_\eta \mathbf{J}(f - \eta, E),$$

where the infimum is taken over all smooth functions η on E such that η is compactly supported on E with $\eta = f$ on ∂E. By the minimizing property of $\mathcal{A}$-harmonic functions, we get

$$\lim_{n\to\infty} \mathbf{J}(h_n, E) = \alpha.$$

In case $2 \le s < \infty$, by $(A8)$,

$$\begin{aligned} \alpha &\le \mathbf{J}((h_n + h_m)/2, E) \\ &\le \mathbf{J}((h_n + h_m)/2, E) + \mathbf{J}((h_n - h_m)/2, E) \\ &\le 2^{-1}(\mathbf{J}(h_n, E) + \mathbf{J}(h_m, E)) \to \alpha \ \text{ as } \ n, m \to \infty. \end{aligned}$$

In case $1 < s \le 2$, by $(A8)$,

$$\begin{aligned} \alpha^{\tilde{s}} &\le \mathbf{J}((h_n + h_m)/2, E)^{\tilde{s}} \\ &\le \mathbf{J}((h_n + h_m)/2, E)^{\tilde{s}} + \mathbf{J}((h_n - h_m)/2, E)^{\tilde{s}} \\ &\le (2^{-1}(\mathbf{J}(h_n, E) + \mathbf{J}(h_m, E)))^{\tilde{s}} \to \alpha^{\tilde{s}} \ \text{ as } \ n, m \to \infty, \end{aligned}$$

where $\tilde{s} = 1/(s-1)$. These imply that

$$\mathbf{J}(h_n - h_m, E) \to 0 \ \text{ as } \ n, m \to \infty.$$

Hence, $\{h_{n_k}\}$ converges to h in $\mathcal{BD}_s(E)$. Since $\{f - h_{n_k}\}$ is a sequence of compactly supported continuous functions and it converges to $f-h$ in $\mathcal{BD}_s(E)$, $f - h \in \mathcal{BD}_{s,0}(E)$. By Lemma 3.3 together with the definition of f, $h = 0$ on Δ_E. From Lemma 3.2, we get $h \equiv 0$ on E. This implies that $f \in \mathcal{BD}_{s,0}(E)$. □

Following the proof of Lemma 3.3, we get a decomposition theorem which is the end version of the Royden decomposition theorem as follows:

Proposition 3.4. *For each $f \in \mathcal{BD}_s(E)$, there exist unique $h \in \mathcal{HBD}_{\mathcal{A}}(E, \partial E)$ and $g \in \mathcal{BD}_{s,0}(E)$ such that $f = h + g$.*

Proof. To prove the uniqueness of the decomposition, suppose that there exist $h' \in \mathcal{HBD}_{\mathcal{A}}(E, \partial E)$ and $g' \in \mathcal{BD}_{s,0}(E)$ such that $f = h' + g'$. Then since $g' - g \in \mathcal{BD}_{s,0}(E)$, $h - h' \in \mathcal{BD}_{s,0}(E)$. Hence there exists a sequence $\{\phi_n\}$ of compactly supported continuous functions converging to $h - h'$ in $\mathcal{BD}_s(E)$ with $\phi_n = 0$ on ∂E. On the other hand, since $h, h' \in \mathcal{HBD}_{\mathcal{A}}(E, \partial E)$,

$$\int_E \langle \mathcal{A}(\nabla h), \nabla(h - h') \rangle = 0 \ \text{ and } \ \int_E \langle \mathcal{A}(\nabla h'), \nabla(h - h') \rangle = 0,$$

hence

$$\begin{aligned} &\int_E \langle \mathcal{A}(\nabla h) - \mathcal{A}(\nabla h'), \nabla h - \nabla h' \rangle \\ = &\int_E \langle \mathcal{A}(\nabla h), \nabla(h - h') \rangle - \int_E \langle \mathcal{A}(\nabla h'), \nabla(h - h') \rangle = 0. \end{aligned}$$

By $(A6)$, this implies that $h - h' \equiv C$ on E for some constant C. Since $h - h'$ is continuous and $h - h' \equiv 0$ on ∂E, $C \equiv 0$. Thus we have $h \equiv h'$ on E, hence also $g \equiv g'$ on E. □

We give a characterization of the s-parabolicity in terms of s-harmonic boundary as follows:

Theorem 3.5. *Let E be an end of a complete Riemannian manifold. Then E is s-parabolic if and only if s-harmonic boundary Δ_E is empty.*

Proof. Suppose that there exists a point $\mathbf{x} \in \Delta_E$. Let $\{u_r\}_{r \geq r_1}$ be a sequence of continuous functions on E such that

$$\begin{cases} \mathcal{A} u_r = 0 & \text{in } E \cap B_r(o); \\ u_r = 0 & \text{on } \partial E; \\ u_r = 1 & \text{on } E \setminus B_r(o). \end{cases}$$

Obviously, each u_r is a function in $\mathcal{BD}_s(E)$ with $u_r(\mathbf{x}) = 1$. Since the sequence $\{u_r\}$ is decreasing, it converges to a limit function u in $\mathcal{BD}_s(E)$ such that

$$\begin{cases} \mathcal{A} u = 0 & \text{in } E; \\ u = 0 & \text{on } \partial E; \\ 0 \leq u \leq 1 & \text{on } E. \end{cases}$$

Since $\{u_{r_1} - u_r\}_{r > r_1}$ is a sequence of compactly supported functions in $\mathcal{BD}_s(E)$, there exists a function $g \in \mathcal{BD}_{s,0}(E)$ such that $u_{r_1} = u + g$, hence by Lemma 3.3, $u(\mathbf{x}) = 1$. This implies that u is an $\mathcal{A}$-harmonic measure of E, which means that E is s-nonparabolic.

Conversely, let E be an s-nonparabolic end and u_E be an $\mathcal{A}$-harmonic measure of E. Suppose that Δ_E is empty. By Lemma 3.1, there exists a function $w \in \mathcal{BD}_{s,0}(E)$ such that $w > 1$ on $\tilde{E}$. Since $\mathcal{BD}_{s,0}(E)$ is an ideal of $\mathcal{BD}_s(E)$, $1 = (1/w)w \in \mathcal{BD}_{s,0}(E)$. Thus there exists a sequence $\{\phi_n\}$ of compactly supported functions converging to 1 in $\mathcal{BD}_s(E)$. Clearly, $\{u_E \phi_n\}$ converges to u_E in $\mathcal{BD}_s(E)$. Therefore, we have

$$\mathbf{J}(u_E, E) = \int_M \langle \mathcal{A}(\nabla u_E), \nabla(u_E) \rangle = \lim_{n \to \infty} \int_M \langle \mathcal{A}(\nabla u_E), \nabla(u_E \phi_n) \rangle = 0.$$

This implies that u_E is constant on E, which is a contradiction. □

Now, we solve the Dirichlet problem on the s-harmonic boundary of each end, and we describe the set of all bounded energy finite $\mathcal{A}$-harmonic functions on each end in terms of s-harmonic boundary as follows:

Theorem 3.6. *Let E be an end of a complete Riemannian manifold and $\mathcal{A}$ be an elliptic operator on E satisfying $(A1), (A2), (A7)$, and $(A8)$. Then for any continuous function f on Δ_E, there exists an $\mathcal{A}$-harmonic function h on E, which is a limit of a sequence of functions in $\mathcal{HBD}_{\mathcal{A}}(E, \partial E)$ in the sense of supremum norm, such that for any $\mathbf{x} \in \Delta_E$,*

$$h(\mathbf{x}) = f(\mathbf{x}).$$

In particular, in the case that Δ_E consists of finitely many points $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_l$, for given real numbers $a_1, a_2, \ldots, a_l \in \mathbf{R}$, there exists an $\mathcal{A}$-harmonic function $h \in \mathcal{HBD}_{\mathcal{A}}(E, \partial E)$ such that

$$h(\mathbf{x}_i) = a_i \quad \text{for } i = 1, 2, \ldots, l. \tag{3}$$

Conversely, each $\mathcal{A}$-harmonic function $h \in \mathcal{HBD}_{\mathcal{A}}(E, \partial E)$ is uniquely determined by the values in (3).

Proof. For any continuous function f on Δ_E, f can be extended to a continuous function, denoted again by f, on $\tilde{E}$. Since $\mathcal{BD}_s(E)$ is dense in the set of continuous functions on $\tilde{E}$ with respect to the supremum norm, there exists a sequence $\{f_n\}$ converging to f in $\mathcal{BD}_s(E)$. By Proposition 3.4, there exist $h_n \in \mathcal{HBD}_\mathcal{A}(E, \partial E)$ and $g_n \in \mathcal{BD}_{s,0}(E)$ such that $f_n = h_n + g_n$. Since $g_n = 0$ on $\tilde{\Delta}_E$, we have $h_n(\mathbf{x}) = f_n(\mathbf{x})$ for each $\mathbf{x} \in \Delta_E$. Since the sequence $\{h_n\}$ is uniformly bounded, it is equicontinuous, and by Ascoli's theorem there exists a subsequence $\{h_{n_k}\}$ converging uniformly to a limit function h on any compact subset of E. Then h is an $\mathcal{A}$-harmonic function by the result of [3]. Since for any $\mathbf{x} \in \Delta_E$ and $\epsilon > 0$,

$$|h(\mathbf{x}) - f(\mathbf{x})| \le |h(\mathbf{x}) - h_{n_k}(\mathbf{x})| + |f_{n_k}(\mathbf{x}) - f(\mathbf{x})| < 2\epsilon,$$

we have

$$h(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Delta_E.$$

Since $\mathcal{BD}_s(E)$ is dense in the set of continuous functions on $\tilde{E}$ with respect to the supremum norm, we can find $f_1, f_2, \ldots, f_l \in \mathcal{BD}_s(E)$ such that $f_i(\mathbf{x}_j) = \delta_{ij}$ for $i, j = 1, 2, \ldots, l$, where δ_{ij} denotes Kronecker's delta. Let $f = \sum_{i=1}^{l} a_i f_i$. Then $f(\mathbf{x}_i) = a_i$ for each $i = 1, 2, \ldots, l$. By Proposition 3.4, there exist $h \in \mathcal{HBD}_\mathcal{A}(E, \partial E)$ and $g \in \mathcal{BD}_{s,0}(E)$ such that $f = h + g$. Since $g = 0$ on $\tilde{\Delta}_E$ by Lemma 3.3, we have $h(\mathbf{x}_i) = a_i$ for each $i = 1, 2, \ldots, l$.

Suppose that $\tilde{h}$ is another $\mathcal{A}$-harmonic function in $\mathcal{HBD}_\mathcal{A}(E, \partial E)$ satisfying (3). By Lemma 3.3, $h - \tilde{h} \in \mathcal{BD}_{s,0}(E)$. Since there exists a sequence of compactly supported continuous functions converging to $h - \tilde{h}$ in $\mathcal{BD}_s(E)$,

$$\int_E \langle \mathcal{A}(\nabla h), \nabla(h - \tilde{h})\rangle = 0 \quad \text{and} \quad \int_E \langle \mathcal{A}(\nabla \tilde{h}), \nabla(h - \tilde{h})\rangle = 0.$$

Arguing similarly as Proposition 3.4, we have $h - \tilde{h} \equiv C$ on E for some constant C. However, since E is s-nonparabolic, $C \equiv 0$. Thus we have $h \equiv \tilde{h}$ on E. □

Slightly modifying the proof of Theorem 3.6, one can prove Theorem 1.1.

4. Proof of the main theorem

In this section, we give a precise relation between the totality of bounded energy finite $\mathcal{A}$-harmonic functions on the whole manifold and those on its ends. We first prove that the set of all s-harmonic boundary points of the whole manifold coincides the union of the set of all s-harmonic boundary points of each end.

Theorem 4.1. *Let M be a complete Riemannian manifold with s-nonparabolic ends $E_1, E_2, \ldots, E_l$, $l \ge 1$. Then $\Delta_M \cap \hat{E}_i = \Delta_{E_i}$ for each $i = 1, 2, \ldots, l$,*

$$\Delta_M = \bigcup_{i=1}^{l} \Delta_{E_i}$$

and $\sharp\Delta_M = \sum_{i=1}^{l} \sharp\Delta_{E_i}$, where $\sharp X$ denotes the cardinality of the set X.

Proof. Let $\mathbf{x}$ be a point in $\Delta_M \cap \hat{E}_i$ and f be a function in $\mathcal{BD}_{s,0}(E_i)$. There exists a sequence $\{f_n\}$ of compactly supported functions converging to f in $\mathcal{BD}_s(E_i)$. Choose a smooth function η defined on E_i in such a way that $\eta = 1$ in $E_i \setminus B_{r_2}(o)$ for some $r_2 > r_1$, $\eta = 0$ on ∂E_i, and $|\nabla \eta|$ is bounded. For each n, let us extend ηf and ηf_n outside E_i with zero, respectively. Then $\{\eta f_n\}$is a sequence of compactly supported functions converging to ηf in $\mathcal{BD}_s(M)$ hence ηf is a function in $\mathcal{BD}_{s,0}(M)$. By definition of ηf,

$$\lim_{x \to \mathbf{x}, x \in E_i} (f - \eta f)(x) = 0.$$

Since $\eta f \in \mathcal{BD}_{s,0}(M)$ and $\mathbf{x} \in \Delta_M$, $\eta f(\mathbf{x}) = 0$, hence $f(\mathbf{x}) = 0$. Therefore, by Lemma 3.3, $\mathbf{x} \in \Delta_{E_i}$.

Conversely, let $\mathbf{x}$ be a point in Δ_{E_i} and f be a function in $\mathcal{BD}_{s,0}(M)$. There exists a sequence $\{f_n\}$ of compactly supported functions converging to f in $\mathcal{BD}_s(M)$. Then $\{f_n|_{E_i}\}$ is a sequence of compactly supported functions converging to $f|_{E_i}$ in $\mathcal{BD}_s(E_i)$, hence $f|_{E_i} \in \mathcal{BD}_{s,0}(E)$. Since $f|_{E_i}(\mathbf{x}) = 0$, we conclude that $f(\mathbf{x}) = 0$. Hence by Lemma 2.2, $\mathbf{x} \in \Delta_M \cap \hat{E}_i$.

If E is an s-parabolic end, then by Theorem 3.5, we have $\Delta_E = \emptyset$. Combining this together with the above arguments, we have the rest equalities. □

We are ready to prove our main theorem:

Proof of Theorem 1.2. Let h be a function in $\mathcal{HBD}_{\mathcal{A}}(M)$. Combining Theorem 1.1, Theorem 4.1, and Theorem 3.6, we can construct a unique function h_i in $\mathcal{HBD}_{\mathcal{A}}(E_i, \partial E_i)$ in such a way that

$$h(\mathbf{x}) = h_i(\mathbf{x}), \quad \mathbf{x} \in \Delta_{E_i}$$

for each $i = 1, 2, \ldots, l$. Let us define $\Phi : \mathcal{HBD}_{\mathcal{A}}(M) \to \prod_{i=1}^{l} \mathcal{HBD}_{\mathcal{A}}(E_i, \partial E_i)$ by

$$\Phi(h) = (h_1, h_2, \ldots, h_l).$$

Then by the uniqueness of the functions $h_1, h_2, \ldots, h_l$, the map Φ is well defined.

Suppose that $\Phi(h) = \Phi(h')$ for some functions h and h' in $\mathcal{HBD}_{\mathcal{A}}(M)$. Then by definition of Φ, we have $h - h' \in \mathcal{BD}_{s,0}(M)$. Since there exists a sequence of compactly supported continuous functions converging to $h - h'$ in $\mathcal{BD}_s(M)$,

$$\int_M \langle \mathcal{A}(\nabla h), \nabla(h - h') \rangle = 0 \quad \text{and} \quad \int_M \langle \mathcal{A}(\nabla h'), \nabla(h - h') \rangle = 0.$$

Arguing similarly as Proposition 3.4, we have $h - h' \equiv C$ on M for some constant C. However, since M is s-nonparabolic, $C \equiv 0$. Thus we have $h \equiv h'$ on M. Hence the map Φ is injective.

Let $(h_1, h_2, \ldots, h_l) \in \prod_{i=1}^{l} \mathcal{HBD}_{\mathcal{A}}(E_i, \partial E_i)$. Extend h_i outside E_i with zero for each $i = 1, 2, \ldots, l$. Then the function $f = \sum_{i=1}^{l} h_i$ belongs to $\mathcal{BD}_s(M)$. By Proposition 3.4, there exist $h \in \mathcal{HBD}_{\mathcal{A}}(M)$ and $g \in \mathcal{BD}_{s,0}(M)$ such that

$f = h+g$. Since $g = 0$ on Δ_M and $f(\mathbf{x}) = h_i(\mathbf{x})$, $\mathbf{x} \in \Delta_{E_i}$ for each $i = 1, 2, \ldots, l$, we have

$$h(\mathbf{x}) = h_i(\mathbf{x}), \quad \mathbf{x} \in \Delta_{E_i}$$

for each $i = 1, 2, \ldots, l$. Hence $\Phi(h) = (h_1, h_2, \ldots, h_l)$, i.e., the map Φ is surjective. □

Applying our argument to the case of harmonic functions, we have the following linear isomorphism between the space of bounded harmonic functions with finite Dirichlet integral on a complete Riemannian manifold and the Cartesian product of those on its ends:

Corollary 4.2. *Let M be a complete Riemannian manifold with nonparabolic ends $E_1, E_2, \ldots, E_l$, $l \geq 1$. Then there exists an isomorphism*

$$\Phi : \mathcal{HBD}(M) \to \prod_{i=1}^{l} \mathcal{HBD}(E_i, \partial E_i),$$

where $\mathcal{HBD}(X)$ and $\mathcal{HBD}(X, \partial X)$ denote the space of bounded harmonic functions with finite Dirichlet integral on X and the subspace of elements of $\mathcal{HBD}(X)$ vanishing at ∂X, respectively.

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Seok Woo Kim
Department of Mathematics Education
Konkuk University
Seoul 143-701, Korea
E-mail address: swkim@konkuk.ac.kr

Yong Hah Lee
Department of Mathematics Education
Ewha Womans University
Seoul 120-750, Korea
E-mail address: yonghah@ewha.ac.kr