

## WEYL SPECTRUM OF THE PRODUCTS OF OPERATORS

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ABSTRACT. Let  $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$  be a  $2 \times 2$  upper triangular operator matrix acting on the Hilbert space  $H \oplus K$  and let  $\sigma_w(\cdot)$  denote the Weyl spectrum. We give the necessary and sufficient conditions for operators  $A$  and  $B$  which  $\sigma_w \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \sigma_w \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$  or  $\sigma_w \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \sigma_w(A) \cup \sigma_w(B)$  holds for every  $C \in B(K, H)$ . We also study the Weyl's theorem for operator matrices.

### 1. Introduction

Throughout this note, let  $H$  and  $K$  be complex, separable, infinite dimensional Hilbert spaces, and let  $B(H, K)$  denote the set of bounded linear operators from  $H$  to  $K$ , abbreviate  $B(H, H)$  to  $B(H)$ . If  $A \in B(H)$ , write  $N(A)$  and  $R(A)$  for the null space and the range of  $A$ ;  $\sigma(A)$  for the spectrum of  $A$ . An operator  $A \in B(H)$  is called upper semi-Fredholm if it has closed range with finite dimensional null space and if  $R(A)$  has finite co-dimension,  $A \in B(H)$  is called a lower semi-Fredholm operator. We call  $A \in B(H)$  Fredholm if it has closed range with finite dimensional null space and its range of finite co-dimension. If  $A$  is semi-Fredholm (that is upper semi-Fredholm or lower semi-Fredholm), let  $n(A) = \dim N(A)$  and  $d(A) = \dim H/R(A)$ , then the index of  $A$ ,  $\text{ind}(A)$ , is defined to be  $\text{ind}(A) = n(A) - d(A)$ . The operator  $A$  is Weyl if it is Fredholm of index zero, and  $A$  is said to be Browder if it is Fredholm "of finite ascent and descent". The essential spectrum  $\sigma_e(A)$ , the Weyl spectrum  $\sigma_w(A)$ , the Browder spectrum  $\sigma_b(A)$ , the upper (lower) semi-Fredholm spectrum  $\sigma_{SF_+}(A)$  ( $\sigma_{SF_-}(A)$ ) and the essential approximate point spectrum  $\sigma_{ea}(A)$  of  $A$  are defined by:

$$\sigma_e(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm}\},$$

$$\sigma_w(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Weyl}\},$$

$$\sigma_b(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Browder}\},$$

$$\sigma_{SF_+}(A)(\sigma_{SF_-}(A)) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not upper (lower) semi-Fredholm}\},$$

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$$\sigma_{ea}(A) = \{\lambda \in \mathbb{C} : \lambda \in \sigma_{SF_+}(A) \text{ or } \text{ind}(A - \lambda I) > 0\}.$$

When  $A \in B(H)$  and  $B \in B(K)$  are given, we denote by  $M_C$  an operator on  $H \oplus K$  of the form  $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$  and let  $M_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ . For operators  $A$ ,  $B$  and  $C$ , the equality  $\sigma_w(M_C) = \sigma_w(M_0)$  or  $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$  was studied by number authors. The aim of this note is the investigation of the set of operators  $A$  and  $B$  for which  $\sigma_w(M_C) = \sigma_w(M_0)$  or  $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$  holds for every  $C \in B(K, H)$ .

## 2. CW operator and Weyl spectrum for operator matrix

For  $A \in B(H)$ , we say that  $A$  is consistent in Weyl or, briefly, a CW operator if, for each  $B \in B(H)$ ,  $AB$  and  $BA$  are Weyl or not Weyl together. Remember ([5],  $P_{38}$ ) that if  $\lambda \neq 0$ , then

$$\begin{pmatrix} AB - \lambda I & 0 \\ 0 & I \end{pmatrix} = F(\lambda) \begin{pmatrix} BA - \lambda I & 0 \\ 0 & I \end{pmatrix} E(\lambda),$$

where  $F(\lambda)$  and  $E(\lambda)$  are both invertible for each  $\lambda \neq 0$ , it follows that if  $\lambda \neq 0$ ,

$$N(AB - \lambda I) \cong N(BA - \lambda I), \quad [R(AB - \lambda I)]^\perp \cong [R(BA - \lambda I)]^\perp,$$

which means that the nonzero elements of  $\sigma_w(AB)$  and  $\sigma_w(BA)$  are the same. Thus  $A$  is a CW operator if and only if  $\sigma_w(AB) = \sigma_w(BA)$  for every  $B \in B(H)$ . In this section, first we invest which elements in  $B(H)$  are CW operator, and we will give a complete characterization of  $A$  in  $B(H)$  which satisfies the Weyl spectral condition  $\sigma_w(AB) = \sigma_w(BA)$  for every  $B \in B(H)$ .

$A \in B(H)$  must be included in one of the following five cases, and in each of them the problem is definitely answered.

**Case 1.** If  $A$  is Fredholm, then  $A$  is a CW operator.

*Proof.* For every  $B \in B(H)$ , if  $AB$  ( $BA$ ) is Weyl, then  $B$  is Fredholm and  $\text{ind}(A) + \text{ind}(B) = \text{ind}(AB) = 0$ . Thus  $BA$  ( $AB$ ) is Fredholm with  $\text{ind}(BA) = \text{ind}(A) + \text{ind}(B) = 0$ , which means that  $BA$  ( $AB$ ) is Weyl.  $\square$

**Case 2.** If  $R(A)$  is not closed, then  $A$  is a CW operator.

In this case, we know for every  $B \in B(H)$ , both  $AB$  and  $BA$  are not Weyl.

**Case 3.** If  $A$  is upper semi-Fredholm with  $d(A) = \infty$ , then  $A^*A$  is Weyl but  $AA^*$  is not Weyl, and so  $A$  is not a CW operator.

*Proof.* Since  $R(A)$  is closed if and only if  $R(A^*A)$  is closed (see [1]), and since  $A^*A$  is normal operator, we know that  $H = N(A^*A) \oplus R(A^*A)$  and  $N(A^*A) = N(A)$ . Then  $A^*A$  is Weyl. But since  $n(AA^*) = n(A^*) = d(A) = \infty$ , it follows that  $AA^*$  is not Weyl.  $\square$

**Case 4.** If  $A$  is lower semi-Fredholm operator with  $n(A) = \infty$ , similar to the proof of case 3, we know  $A^*A$  is not Weyl but  $AA^*$  is Weyl, thus  $A$  is not a CW operator.

**Case 5.** If  $R(A)$  is closed and  $n(A) = d(A) = \infty$ , then  $A$  is CW operator. In fact, if  $n(A) = d(A) = \infty$ , we know for every  $B \in B(H)$ , both  $A$  and  $B$  are not Weyl.

From the results just proved above we can conclude:

**Theorem 2.1.**  $A \in B(H)$  is a CW operator if and only if one of the following three mutually disjoint cases occurs:

- (1)  $A$  is Fredholm;
- (2)  $R(A)$  is not closed;
- (3)  $R(A)$  is closed and  $\dim N(A) = \text{comdim} R(A) = \infty$ .

**Corollary 2.2.** (1)  $A \in B(H)$  is CW operator if and only if  $\sigma_w(A^*A) = \sigma_w(AA^*)$  if and only if  $\sigma_e(AB) = \sigma_e(BA)$  for every  $B \in B(H)$ ;

(2)  $A \in B(H)$  is not a CW operator if and only if  $A$  is semi-Fredholm with  $\text{ind}(A) = \infty$ ;

(3) If  $A \in B(H)$  is a CW operator, then  $\sigma(AB) = \sigma(BA)$  and  $\sigma_b(AB) = \sigma_b(BA)$  for every  $B \in B(H)$ .

*Proof.* From the statements before Theorem 2.1, we know the result (1) and (2) are true. For (3), Since the nonzero elements of  $\sigma(AB)$  and  $\sigma(BA)$  are the same, we only need to prove that  $AB$  and  $BA$  are invertible or noninvertible together. If  $AB$  is invertible, using the fact that  $A$  is CW operator,  $BA$  is Weyl and there exists  $\epsilon > 0$  such that  $AB - \lambda I$  is invertible if  $0 < |\lambda| < \epsilon$ . Then  $BA - \lambda I$  is invertible if  $0 < |\lambda| < \epsilon$ . Thus  $BA$  is Browder. This implies that  $A$  has finite ascent, then  $n(A) \leq d(A) = 0$ , which means that  $A$  is invertible. Therefore  $B$  is invertible and  $BA$  is invertible. If  $BA$  is invertible, using the same way, we can prove that  $AB$  is invertible. For the case of Browder spectrum, there is a similar proof.  $\square$

In [8] and [9], W. Y. Lee gave the sufficient conditions for  $A$  and  $B$  which the equivalent  $\sigma_w(M_C) = \sigma_w(M_0)$  holds for every  $C \in B(K, H)$ . The following theorem is an investigation for the set of operators  $A \in B(H)$  and  $B \in B(K)$  for which  $\sigma_w(M_C) = \sigma_w(M_0)$  holds for every  $C \in B(K, H)$ .

**Theorem 2.3.** Let  $A \in B(H)$  and  $B \in B(K)$ . Then the following statements are equivalent:

- (1)  $\sigma_w \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \sigma_w \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$  for every  $C \in B(K, H)$ ;
- (2)  $\{\lambda \in \mathbb{C}, A - \lambda I \text{ or } B - \lambda I \text{ is CW operator}\} \cup \{\lambda \in \mathbb{C}, A - \lambda I \text{ is lower semi-Fredholm with } n(A - \lambda I) = \infty\} \cup \{\lambda \in \mathbb{C}, B - \lambda I \text{ is upper semi-Fredholm with } d(B - \lambda I) = \infty\} = \mathbb{C}$ ;
- (3)  $\{\lambda \in \mathbb{C}, A - \lambda I \text{ is upper semi-Fredholm with } d(A - \lambda I) = \infty\} \cap \{\lambda \in \mathbb{C}, B - \lambda I \text{ is lower semi-Fredholm with } n(B - \lambda I) = \infty\} = \emptyset$ ;
- (4)  $\sigma(A) \cap \sigma(B) \subseteq \{\lambda \in \mathbb{C}, A - \lambda I \text{ or } B - \lambda I \text{ is CW operator}\} \cup \{\lambda \in \mathbb{C}, A - \lambda I \text{ is lower semi-Fredholm with } n(A - \lambda I) = \infty\} \cup \{\lambda \in \mathbb{C}, B - \lambda I \text{ is upper semi-Fredholm with } d(B - \lambda I) = \infty\}$ ;

$$(5) \quad \sigma_e \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \sigma_e \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \text{ for every } C \in B(K, H).$$

*Proof.* (1)  $\implies$  (2). Let  $\lambda_0 \notin \{\lambda \in \mathbb{C}, A - \lambda I \text{ or } B - \lambda I \text{ is CW operator}\} \cup \{\lambda \in \mathbb{C}, A - \lambda I \text{ is lower semi-Fredholm with } n(A - \lambda I) = \infty\} \cup \{\lambda \in \mathbb{C}, B - \lambda I \text{ is upper semi-Fredholm with } d(B - \lambda I) = \infty\}$ . Then  $A - \lambda_0 I$  is upper semi-Fredholm with  $d(A - \lambda_0 I) = \infty$  and  $B - \lambda_0 I$  is lower semi-Fredholm with  $n(B - \lambda_0 I) = \infty$ .

Clearly,  $\lambda_0 \in \sigma_w \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ . Suppose  $A_1 = A - \lambda_0 I$  and  $B_1 = B - \lambda_0 I$ .

Suppose  $n(A_1) = N$  and let  $\{e_1, e_2, \dots, e_N\}$  be an orthonormal set in  $N(B_1)^\perp$  and let  $M = \text{span}\{e_1, e_2, \dots, e_N\}$ . Since  $\dim(N(B_1) + M) = \dim R(A_1)^\perp = \infty$ , there exists an isometrical isomorphism  $J : N(B_1) + M \longrightarrow R(A_1)^\perp$ . Define an operator  $C : K \longrightarrow H$  by

$$C = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} N(B_1) + M \\ (N(B_1) + M)^\perp \end{pmatrix} \longrightarrow \begin{pmatrix} R(A_1)^\perp \\ R(A_1) \end{pmatrix}.$$

Then  $M_C - \lambda_0 I$  is Weyl (See the proof in Theorem 2.1 in [2]). This implies that  $\lambda_0 \notin \sigma_w \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ , it is a contradiction.

(2)  $\implies$  (3). From the Corollary 2.2 (2), the result is true.

(3)  $\implies$  (4). Clear.

(4)  $\implies$  (5). For every  $C \in B(K, H)$ , the inclusion  $\sigma_e(M_C) \subseteq \sigma_e(M_0)$  is clear. For the converse inclusion, let  $M_C - \lambda_0 I$  is Fredholm. Then  $A - \lambda_0 I$  is upper semi-Fredholm,  $B - \lambda_0 I$  is lower semi-Fredholm and  $A - \lambda_0 I$  is Fredholm if and only if  $B - \lambda_0 I$  is Fredholm. Without loss of generality, we suppose  $\lambda_0 \in \sigma(A) \cap \sigma(B)$ , then there must have that  $A - \lambda_0 I$  or  $B - \lambda_0 I$  is CW operator. Thus from Theorem 2.1,  $A - \lambda_0 I$  or  $B - \lambda_0 I$  is Fredholm, then both  $A - \lambda_0 I$  and  $B - \lambda_0 I$  are Fredholm, which means that  $\lambda_0 \notin \sigma_e(M_0)$ .

(5)  $\implies$  (1). There is inclusion  $\sigma_w(M_C) \subseteq \sigma_w(M_0)$  for every  $C \in B(K, H)$ . Conversely, if  $M_C - \lambda_0 I$  is Weyl, then  $M_0 - \lambda_0 I$  is Fredholm, which means that both  $A - \lambda_0 I$  and  $B - \lambda_0 I$  are Fredholm. Then  $M_0 - \lambda_0 I$  is Fredholm with  $\text{ind}(M_0 - \lambda_0 I) = \text{ind}(A - \lambda_0 I) + \text{ind}(B - \lambda_0 I) = \text{ind}(M_C - \lambda_0 I) = 0$ , it tells us that  $M_0 - \lambda_0 I$  is Weyl.  $\square$

**Example 1.** Let  $H = K = \ell_2$  and let  $A \in B(H)$  is defined by

$$A(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots).$$

Then for every  $\lambda \in \mathbb{C}$ ,  $A - \lambda I$  is CW operator. From Theorem 2.3, for every  $B \in B(K)$  and  $C \in B(K, H)$ ,  $\sigma_w(M_C) = \sigma_w(M_0)$ .

Recall that the spectral picture of an operator  $A \in B(H)$ , denoted by  $SP(A)$ , is the structure consisting of the set  $\sigma_e(A)$ , the collection of holes and pseudoholes in  $\sigma_e(A)$ , and the indices associated with these holes and pseudoholes, where a hole in  $\sigma_e(A)$  is a nonempty bounded component of  $\mathbb{C} \setminus \sigma_e(A)$  and a pseudohole in  $\sigma_e(A)$  is a nonempty component of  $\sigma_e(A) \setminus \sigma_{SF_+}(A)$  or of  $\sigma_e(A) \setminus \sigma_{SF_-}(A)$ . If  $SP(A)$  (or  $SP(B)$ ) has no pseudoholes, then for every

$\lambda \in \mathbb{C}$ ,  $A - \lambda I$  is semi-Fredholm implies  $A - \lambda I$  is Fredholm. Thus if either  $SP(A)$  or  $SP(B)$  has no pseudoholes, the fact (2) in Theorem 2.4 hold, then for every  $C \in B(K, H)$ ,  $\sigma_w(M_C) = \sigma_w(M_0)$ .

Using Theorem 2.1 in [2], similar to the proof of Theorem 2.3, for essential approximate point spectrum, we have:

**Theorem 2.4.** *Let  $A \in B(H)$  and  $B \in B(K)$ . Then the following statements are equivalent:*

- (1)  $\sigma_{ea} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \sigma_{ea} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$  for every  $C \in B(K, H)$ ;
- (2)  $\{\lambda \in \mathbb{C}, A - \lambda I \text{ is CW operator}\} \cup \{\lambda \in \mathbb{C}, A - \lambda I \text{ is lower semi-Fredholm with } n(A - \lambda I) = \infty\} \cup \{\lambda \in \mathbb{C}, n(B - \lambda I) < \infty\} = \mathbb{C}$ .

If  $SP(A)$  has no pseudoholes, then  $\{\lambda \in \mathbb{C}, A - \lambda I \text{ is CW operator}\} \cup \{\lambda \in \mathbb{C}, A - \lambda I \text{ is lower semi-Fredholm with } n(A - \lambda I) = \infty\} = \mathbb{C}$ . Thus  $\sigma_{ea}(M_C) = \sigma_{ea}(M_0)$  for every  $C \in B(K, H)$ .

Weyl's theorem for an operator says that the complement in the spectrum of the Weyl spectrum coincides with the isolated points of the spectrum which are eigenvalues of finite multiplicity. H. Weyl ([10]) discovered that this property holds for hermitian operators and it has been extended to many other operators. In recent years, a number of researchers have considered the Weyl's theorem for operators and operator matrices (such as [2], [3], [4], [7], [9], et.) In the following, we consider Weyl type theorem for  $2 \times 2$  operator matrices.

We say that the Weyl's theorem holds for  $A \in B(H)$  if there is equality

$$\sigma(A) \setminus \sigma_w(A) = \pi_{00}(A),$$

where  $\pi_{00}(A)$  for the isolated points of  $\sigma(A)$  which are eigenvalues of finite multiplicity. Harte and Lee ([7]) have discussed a variant of Weyl's theorem: "the Browder's theorem holds" for  $A$  if

$$\sigma(A) = \sigma_w(A) \cup \pi_{00}(A).$$

What is missing is the disjointness between the Weyl spectrum and the isolated eigenvalues of finite multiplicity: equivalently

$$\sigma_w(A) = \sigma_b(A).$$

Rakočević has looked at variants of "Weyl's theorem" and "Browder's theorem" in which the spectrum is replaced by the approximate point spectrum: "the a-Weyl's theorem holds" for  $A$  if

$$\sigma_a(A) \setminus \sigma_{ea}(A) = \pi_{00}^a(A),$$

where we write  $\sigma_a(A)$  for the approximate point spectrum of  $A$ ,  $\pi_{00}^a(A)$  for the set of all  $\lambda \in \mathbb{C}$  such that  $\lambda$  is isolated point in  $\sigma_a(A)$  and  $0 < \dim N(A - \lambda I) < \infty$ . Finally "the a-Browder's theorem holds" for  $A$  if

$$\sigma_{ea}(A) = \sigma_{ab}(A),$$

where  $\sigma_{ab}(A) = \bigcap \{\sigma_a(A + K) : K \in B(H) \text{ is compact and } AK = KA\}$ .

**Theorem 2.5.** Suppose  $A \in B(H)$  and  $B \in B(K)$ . If  $A - \lambda I$  is CW operator for every  $\lambda \in \mathbb{C}$  (or  $B$  has this property), then for every  $C \in B(K, H)$ ,

- (1) Browder's theorem holds for  $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$   
 $\implies$  Browder's theorem holds for  $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ ;
- (2) a-Browder's theorem holds for  $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$   
 $\implies$  a-Browder's theorem holds for  $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ .

*Proof.* If  $A - \lambda I$  is CW operator for every  $\lambda \in \mathbb{C}$ , then for every  $C \in B(K, H)$ ,  $\sigma_w(M_C) = \sigma_w(M_0)$ . Let  $M_C - \lambda_0 I$  be Weyl. Then  $M_0 - \lambda_0 I$  Weyl. Since Browder's theorem holds for  $M_0$ , it follows that both  $A - \lambda_0 I$  and  $B - \lambda_0 I$  are Browder. Then  $M_C - \lambda_0 I$  is Browder. For the case of a-Browder's theorem, using Theorem 2.4 and similar to the proof of the case of Browder's theorem, we get the result (2).  $\square$

**Example 2.** Let  $A \in B(\ell_2)$  be defined as Example 1 and let  $B \in B(\ell_2)$  be defined by

$$B(x_1, x_2, x_3, \dots) = (0, x_1, 0, x_2, 0, x_3, \dots).$$

Then  $\sigma_{ea}(M_0) = \sigma_{ab}(M_0) = \partial D$ , which means that a-Browder's theorem holds for  $M_0$ . Then for every  $C \in B(\ell_2, \ell_2)$ , a-Browder's theorem holds for  $M_C$ .

*Remark.* Theorem 2.5 may fail for Weyl's theorem even with the additional assumption that Weyl's theorem holds for  $A$  and  $B$ . To see this let the operator  $A, B$  and  $C$  on  $\ell_2$  are defined by

$$A(x_1, x_2, x_3, \dots) = (0, x_1, 0, \frac{1}{2}x_2, 0, \frac{1}{3}x_3, 0, \frac{1}{4}x_4, \dots);$$

$$B(x_1, x_2, x_3, \dots) = (0, x_2, 0x_4, 0, x_6, 0, x_8, \dots);$$

$$C(x_1, x_2, x_3, \dots) = (0, 0, x_2, 0, x_3, 0x_4, 0, \dots).$$

Then  $\sigma(A) = \sigma_w(A) = \{0\}$  and  $R(A)$  is not closed, which means that  $A - \lambda I$  is CW operator for every  $\lambda \in \mathbb{C}$ . We can find that Weyl's theorem holds for  $A$  and  $B$  and also Weyl's theorem holds for  $M_0$ . But since  $\sigma(M_C) = \sigma_w(M_C) = \{0, 1\}$  and  $\pi_{00}(M_C) = \{0\}$ , which implies that Weyl's theorem fails for  $M_C$ .

An operator  $A \in B(H)$  is called isoloid if every isolated point of  $\sigma(A)$  is an eigenvalue of  $A$ , and we call  $A$  is reguloid if each isolated point of its spectrum has closed range.

**Theorem 2.6.** Suppose  $A \in B(H)$  and  $B \in B(K)$  and suppose that  $A - \lambda I$  is CW operator for every  $\lambda \in \mathbb{C}$ . If  $A$  is reguloid (or  $A$  is isoloid and Weyl's theorem holds for  $A$ ), then for every  $C \in B(K, H)$ ,

Weyl's theorem holds for  $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \implies$  Weyl's theorem holds for  $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ .

*Proof.* Suppose  $A$  is reguloid. Theorem 2.5 tells us that  $\sigma(M_C) \setminus \sigma_w(M_C) \subseteq \pi_{00}(M_C)$ . For the reverse inclusion, let  $\lambda_0 \in \pi_{00}(M_C)$ . Then there exists  $\epsilon > 0$  such that  $M_C - \lambda I$  is invertible if  $0 < |\lambda - \lambda_0| < \epsilon$ . This induces that  $A - \lambda I$  is bounded from below and  $B - \lambda I$  is surjective. Since  $A - \lambda I$  is CW operator, it follows that  $\sigma_w(M_C) = \sigma_w(M_0)$ . Then  $\sigma_w(M_0) - \lambda I$  is Weyl. The fact that Weyl's theorem holds for  $M_0$  induces that both  $A - \lambda I$  and  $B - \lambda I$  are Browder. Thus both  $A - \lambda I$  and  $B - \lambda I$  are invertible. This means that  $\lambda_0 \in \text{iso}\sigma(M_0)$ . Without loss of generality, we suppose that  $\lambda_0 \in \sigma(A)$ . Since  $A$  is reguloid and  $n(A - \lambda_0 I) \leq n(M_C - \lambda_0 I) < \infty$ , we know that  $A - \lambda_0 I$  is upper semi-Fredholm and hence  $A - \lambda_0 I$  has topological uniform descent which defined in [6]. Then Theorem 4.7 in [6] tells us that  $A - \lambda_0 I$  is Browder. Similar to the proof in Theorem 2.4 in [9], we can get that  $\lambda_0 \in \pi_{00}(M_0)$ . Since Weyl's theorem holds for  $M_0$ ,  $M_0 - \lambda_0 I$  is Weyl. Then  $M_C - \lambda_0 I$  is Weyl. Thus  $\sigma(M_C) \setminus \sigma_w(M_C) = \pi_{00}(M_C)$ , which means that Weyl's theorem holds for  $M_C$  for every  $C \in B(K, H)$ .  $\square$

Let  $A, B \in B(\ell_2)$  be defined as Example 2, a straightforward calculation shows that:

- (1)  $\sigma(M_0) = \sigma_w(M_0)$  and  $\pi_{00}(M_0) = \emptyset$ , which means that Weyl's theorem holds for  $M_0$ ;
- (2)  $A$  is isoloid and Weyl's theorem holds for  $A$ .

Then for every  $C \in B(\ell_2, \ell_2)$ , Weyl's theorem holds for  $M_C$ .

The "reguloid" condition is essential in Theorem 2.6. For example, let  $A, B, C \in B(\ell_2)$  be defined as Remark, the operator  $A$  is not reguloid.

### 3. CFI operator and Weyl spectrum for operator matrix

In this section we invest the set of operators  $A$  and  $B$  which  $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$  holds for every  $C \in B(K, H)$ . We begin with a definition:

**Definition 3.1.** We say  $A \in B(H)$  is consistent in Fredholm and index (abbrev. a CFI operator), if for each  $B \in B(H)$ , one of the case occurs:

- (1)  $AB$  and  $BA$  are Fredholm together and  $\text{ind}(AB) = \text{ind}(BA) = \text{ind}(B)$ ;
- (2) Both  $AB$  and  $BA$  are not Fredholm.

Clearly, CFI operators are CW operators. But the converse is not true. Similar to section 2, we need to study which elements in  $B(H)$  are CFI operator.

Also  $A \in B(H)$  must be included in one of the following five cases, and in each of them the problem is definitely answered.

Case 1. If  $A$  is Weyl, then  $A$  is a CFI operator.

Case 2.  $A$  is Fredholm of  $\text{ind}(A) \neq 0$ ,  $A$  is not a CFI operator.

In fact, we know that both  $A^*A$  and  $AA^*$  are Weyl. But since  $\text{ind}(A^*) \neq 0$ , it follows that  $\text{ind}(A^*A) = \text{ind}(AA^*) \neq \text{ind}(A^*)$ .

Case 3. If  $A$  is semi-Fredholm of  $\text{ind}(A) = \infty$ , form Theorem 2.1,  $A$  is not CFI operator.

Case 4. If  $R(A)$  is not closed,  $A$  is a CFI operator.

Case 5. If  $R(A)$  is closed and  $n(A) = d(A) = \infty$ , then  $A$  is CFI operator. Then we conclude that:

**Theorem 3.2.**  $A \in B(H)$  is a CFI operator if and only if one of the following three mutually disjoint cases occurs:

- (1)  $A$  is Weyl;
- (2)  $R(A)$  is not closed;
- (3)  $R(A)$  is closed and  $\dim N(A) = \text{comdim}R(A) = \infty$ .

**Corollary 3.3.**  $A \in B(H)$  is not a CFI operator if and only if  $A$  is semi-Fredholm of  $\text{ind}(A) \neq 0$ .

**Theorem 3.4.** Let  $A \in B(H)$  and  $B \in B(K)$ . Then the following statements are equivalent:

- (1)  $\sigma_w \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \sigma_w(A) \cup \sigma_w(B)$  for every  $C \in B(K, H)$ ;
- (2)  $\{\lambda \in \mathbb{C}, A - \lambda I \text{ or } B - \lambda I \text{ is CFI operator}\} \cup \{\lambda \in \mathbb{C}, A - \lambda I \text{ is lower semi-Fredholm with } n(A - \lambda I) = \infty\} \cup \{\lambda \in \mathbb{C}, B - \lambda I \text{ is upper semi-Fredholm with } d(B - \lambda I) = \infty\} \cup \{\lambda \in \mathbb{C}, A - \lambda I \text{ and } B - \lambda I \text{ is semi-Fredholm and } n(A - \lambda I) + n(B - \lambda I) \neq d(A - \lambda I) + d(B - \lambda I)\} = \mathbb{C}$ .

*Proof.* (1)  $\implies$  (2). If there exists  $\lambda_0 \in \mathbb{C}$  such that  $\lambda_0$  is not in the left set in (2), then  $A - \lambda_0 I$  and  $B - \lambda_0 I$  is semi-Fredholm and  $\text{ind}(A - \lambda_0 I) \neq 0$ ,  $\text{ind}(B - \lambda_0 I) \neq 0$ ,  $n(A - \lambda_0 I) + n(B - \lambda_0 I) = d(A - \lambda_0 I) + d(B - \lambda_0 I)$ . There are three cases to consider:

Case 1. Both  $A - \lambda_0 I$  and  $B - \lambda_0 I$  are upper (lower) semi-Fredholm.

In this case, both  $A - \lambda_0 I$  and  $B - \lambda_0 I$  are Fredholm. Using Theorem 2.4 in [2], there exists  $C_0 \in B(K, H)$  such that  $M_{C_0} - \lambda_0 I$  is Weyl.

Case 2.  $A - \lambda_0 I$  is upper semi-Fredholm and  $B - \lambda_0 I$  is lower semi-Fredholm.

In this case, one of the cases exists:

- (a)  $d(A - \lambda_0 I) < \infty$ ,  $n(B - \lambda_0 I) < \infty$  and  $n(A - \lambda_0 I) + n(B - \lambda_0 I) = d(A - \lambda_0 I) + d(B - \lambda_0 I)$ ;
- (b)  $d(A - \lambda_0 I) = n(B - \lambda_0 I) = \infty$ .

Then there exists  $C_0 \in B(K, H)$  such that  $M_{C_0} - \lambda_0 I$  is Weyl (Theorem 2.4, [2]).

Case 3.  $A - \lambda_0 I$  is lower semi-Fredholm and  $B - \lambda_0 I$  is upper semi-Fredholm.

Since  $\lambda_0 \notin \{\lambda \in \mathbb{C}, A - \lambda I \text{ is lower semi-Fredholm with } n(A - \lambda I) = \infty\} \cup \{\lambda \in \mathbb{C}, B - \lambda I \text{ is upper semi-Fredholm with } d(B - \lambda I) = \infty\}$ , then both  $A - \lambda_0 I$  and  $B - \lambda_0 I$  is Fredholm. Using Theorem 2.4 in [2] again, there exists  $C_0 \in B(K, H)$  such that  $M_{C_0} - \lambda_0 I$  is Weyl.

From the preceding proof, we know if  $\lambda_0$  is not in the left set in (2), there must have  $C \in B(K, H)$  such that  $\lambda_0 \notin \sigma_w(M_C)$ . But we know that  $\lambda_0 \in \sigma_w(A) \cup \sigma_w(B)$ , it is a contradiction.

(2)  $\implies$  (1). For each  $C \in B(K, H)$ , the inclusion  $\sigma_w(M_C) \subseteq \sigma_w(A) \cup \sigma_w(B)$  is clear. For the converse, let  $M_C - \lambda_0 I$  is Weyl, then  $A - \lambda_0 I$  is upper semi-Fredholm,  $B - \lambda_0 I$  is lower semi-Fredholm and  $A - \lambda_0 I$  is Fredholm if and



only if  $B - \lambda_0 I$  is Fredholm. Also there is equality  $n(A - \lambda_0 I) + n(B - \lambda_0 I) = d(A - \lambda_0 I) + d(B - \lambda_0 I)$ . Thus  $\lambda_0 \in \{\lambda \in \mathbb{C}, A - \lambda I \text{ or } B - \lambda I \text{ is CFI operator}\}$ . From Theorem 3.2, we know that  $A - \lambda_0 I$  or  $B - \lambda_0 I$  is Weyl. Then both  $A - \lambda_0 I$  and  $B - \lambda_0 I$  is Weyl, which means that  $\lambda_0 \notin \sigma_w(A) \cup \sigma_w(B)$ . Thus  $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$  for every  $C \in B(K, H)$ .  $\square$

If  $\sigma_w(A) \cap \sigma_w(B)$  has no interior points, we can prove that the equality (2) in Theorem 3.4 holds. In fact, if there exists  $\lambda_0 \in \mathbb{C}$  such that  $\lambda_0$  is not in the left set of equality (2) in Theorem 3.4, both  $A - \lambda_0 I$  and  $B - \lambda_0 I$  are semi-Fredholm and  $\lambda_0 \in \sigma_w(A) \cap \sigma_w(B)$ . Using perturbation neighborhood theorem,  $\text{ind}(A - \lambda I) = \text{ind}(A - \lambda_0 I)$  and  $\text{ind}(B - \lambda I) = \text{ind}(B - \lambda_0 I)$  if  $0 < |\lambda - \lambda_0|$  is sufficiently small. But since  $\sigma_w(A) \cap \sigma_w(B)$  has no interior points, this means that for any  $\epsilon > 0$ , there exists  $\lambda \in \mathbb{C}$  such that  $0 < |\lambda - \lambda_0| < \epsilon$  and  $A - \lambda I$  or  $B - \lambda I$  is Weyl. Then  $A - \lambda_0 I$  or  $B - \lambda_0 I$  is Weyl, it is in contradiction to the fact that  $\lambda_0 \in \sigma_w(A) \cap \sigma_w(B)$ . From Theorem 3.4, we have that:

**Corollary 3.5.** *If  $\sigma_w(A) \cap \sigma_w(B)$  has no interior points, then for every  $C \in B(K, H)$ ,*

$$\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B).$$

We apply the property of CFI operators to the Weyl type theorem:

**Theorem 3.6.** *Suppose  $A \in B(H)$  and  $B \in B(K)$  and suppose that  $A - \lambda I$  is CFI operator for every  $\lambda \in \mathbb{C}$ . If  $A$  is reguloid (or  $A$  is isoloid and Weyl's theorem holds for  $A$ ), then for every  $C \in B(K, H)$ ,*

- (1) *Weyl's theorem holds for  $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$*   
 $\implies$  *Weyl's theorem holds for  $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ ;*
- (2) *a-Weyl's theorem holds for  $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$*   
 $\implies$  *a-Weyl's theorem holds for  $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ .*

*Proof.* Since CFI operators are CW operator, using Theorem 2.5 and Theorem 2.6, we only need to prove that  $\pi_{00}^a(M_C) \subseteq \sigma_a(M_C) \setminus \sigma_{ea}(M_C)$  for every  $C \in B(K, H)$ . Let  $\lambda_0 \in \pi_{00}^a(M_C)$ . Then  $0 < n(M_C - \lambda_0 I) < \infty$  and there exists  $\epsilon > 0$  such that  $M_C - \lambda I$  is bounded from below if  $0 < |\lambda - \lambda_0| < \epsilon$ . Thus  $A - \lambda I$  is Bounded from below if  $0 < |\lambda - \lambda_0| < \epsilon$ . Since  $A - \lambda I$  is CFI operator, it follows from Theorem 3.2 that  $A - \lambda I$  is invertible. Then  $B - \lambda I$  is bounded from below and  $\lambda_0 \in \text{iso}\sigma(A) \cup \rho(A)$ , where  $\rho(A) = \mathbb{C} \setminus \sigma(A)$ . Thus  $\lambda_0 \in \text{iso}\sigma_a(M_0)$ . Similar to the proof of Theorem 2.6, we can get that  $\lambda_0 \in \pi_{00}^a(M_0)$ , then  $M_0 - \lambda_0$  is upper semi-Fredholm with  $\text{ind}(M_0 - \lambda_0 I) \leq 0$  and  $A - \lambda_0 I$  is Browder. Thus  $B - \lambda_0 I$  is upper semi-Fredholm of  $\text{ind}(B - \lambda_0 I) \leq 0$ . This can induces that  $\lambda_0 \in \sigma_a(M_C) \setminus \sigma_{ea}(M_C)$ . Thus  $\sigma_a(M_C) \setminus \sigma_{ea}(M_C) = \pi_{00}^a(M_C)$ , which means that a-Weyl's theorem holds for  $M_C$  for every  $C \in B(K, H)$ .  $\square$

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