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OPTIMAL CONTROL PROBLEMS FOR SEMILINEAR EVOLUTION EQUATIONS

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Abstract. This paper deals with the existence of optimal controls and maximal principles for semilinear evolution equations with the nonlinear term satisfying Lipschitz continuity. We also present the necessary conditions of optimality which are described by the adjoint state corresponding to the linear equations without a condition of differentiability for nonlinear term.

1. Introduction

Let H and V be two complex Hilbert spaces such that V is a dense subspace of H. Identifying the antidual of H with H we may consider $V \subset H \subset V^*$.

In this paper we deal with the control problems for the semilinear parabolic type equation in H as follows.

$$\text{(SE)} \qquad \begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + f(t, x(t)) + Bu(t), \\ x(0) = x_0. \end{cases}$$

Let A be the operator associated with a sesquilinear form defined on $V \times V$ satisfying Gårding's inequality. Let Y be a complex Banach space and B be a bounded linear operator from Y to H.

The optimal control problems of linear systems have been so extensively studied by [3, 6, 9] and the references cited there. In [7], Papageorgiou gives the existence of the optimal control for a broad class of nonlinear evolution control systems and in [8], the author obtained necessary conditions for optimality using the penalty method first introduced in [2] for optimal control problems governed by nonlinear evolution equations with nonmonotone nonlinearities in the state on condition of the Gateaux differentiability of the nonlinear terms.

In this paper, combining techniques for the linear control problems and the properties of solutions of semilinear systems in [5], we can obtain the existence

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of optimal controls and maximal principles for (SE) where the nonlinear term is given by

$$f(t,x) = \int_0^t k(t-s)g(s,x(s))ds,$$

where k belongs to $L^2(0,T)$ for $T > 0$.

In the proofs of the main theorems, we need some compactness hypothesis. We make the natural assumption that the embedding $D(A) \subset V$ is compact. Then the embedding $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V)$ is compact in view of Aubin's result ([1]), and we show that the mapping which maps u to the mild solution of (SE) is a compact operator from $L^2(0,T;Y)$ to $L^2(0,T;H)$.

Two applications of the optimal control problems for cost functions are given; one is the averaging observation control and the other is the observation of terminal value. We give the maximal principle for given cost functions and present the necessary conditions of optimality which are described by the adjoint state corresponding to the linear equation without a condition of differentiability of nonlinear term.

The unique problem of the optimal control for the terminal value cost function is an open problem. One of the difficulties is that we can not obtain the convexity property of nonlinear term.

2. Fundamental properties

Let H and V be Hilbert spaces whose norms will be denoted by $|\cdot|$ and $||\cdot||$, respectively. Let A be the operator associated with a sesquilinear form $b(u,v)$ which is defined Gårding's inequality

$$\operatorname{Re} b(u,u) \geq c_0||u||^2 - c_1|u|^2, \quad c_0 > 0,\ c_1 \geq 0 \quad \text{for } u \in V,$$

that is,

$$(v, Au) = -b(u,v), \quad u,\ v \in V,$$

where $(\cdot, \cdot)$ denotes also the duality pairing between V and V^*.
Then A is a bounded linear operator from V to V^* and its realization in H which is the restriction of A to

$$D(A) = \{u \in V; Au \in H\}$$

is also denoted by A. Here, we note that $D(A)$ is dense in V. Hence, it is also dense in H. We endow the domain $D(A)$ of A with graph norm, that is, for $u \in D(A)$, we define $||u||_{D(A)} = |u| + |Au|$. So, for the brevity, we may regard that $|u| \leq ||u|| \leq ||u||_{D(A)}$ for all $u \in V$.

It is known that A generates an analytic semigroup $S(t)(t \geq 0)$ in both H and V^*.

From the following inequalities

$$\begin{aligned} c_0||u||^2 &\leq \operatorname{Re} b(u,u) + c_1|u|^2 \leq C|Au|\,|u| + c_1|u|^2 \\ &\leq (C|Au| + c_1|u|)|u| \leq \max\{C, c_1\}||u||_{D(A)}|u|, \end{aligned}$$

it follows that there exists a constant $C_0 > 0$ such that

$$||u|| \le C_0 ||u||_{D(A)}^{1/2} |u|^{1/2}.$$

We assume that the embedding $D(A) \subset V$ is compact.

The control space will be modeled by a Banach space Y. Let the controller B is a bounded linear operator from Y to H and $g : [0,T] \times V \longrightarrow H$ be a nonlinear mapping such that $t \mapsto g(t,x)$ is measurable and

$$\text{(F)} \qquad |g(t,x) - g(t,y)| \le L||x-y||$$

for a positive constant L. For the sake of simplicity we assume that $g(t,0) = 0$.

For $x \in L^2(0,T;V)$ we set

$$\text{(2.1)} \qquad f(t,x) = \int_0^t k(t-s) g(s,x(s)) ds,$$

where k belongs to $L^2(0,T)$.

Lemma 2.1. *Let $x \in L^2(0,T;V)$, $T > 0$. Then $f(\cdot,x) \in L^2(0,T;H)$ and*

$$||f(\cdot,x)||_{L^2(0,T;H)} \le L||k||_{L^2(0,T)} \sqrt{T} ||x||_{L^2(0,T;V)}.$$

Moreover if $x_1,\ x_2 \in L^2(0,T;V)$, then

$$||f(\cdot,x_1) - f(\cdot,x_2)||_{L^2(0,T;H)} \le L||k||_{L^2(0,T)} \sqrt{T} ||x_1 - x_2||_{L^2(0,T;V)}.$$

Proof. From (F) and using the Hölder inequality it is easily seen that

$$\begin{aligned} ||f(\cdot,x)||^2_{L^2(0,T;H)} &\le \int_0^T |\int_0^t k(t-s) g(s,x(s)) ds|^2 dt \\ &\le ||k||^2_{L^2(0,T)} \int_0^T \int_0^t L^2 ||x(s)||^2 ds dt \\ &\le T L^2 ||k||^2_{L^2(0,T)} ||x||^2_{L^2(0,T;V)}. \end{aligned}$$

The proof of the second paragraph is similar. □

We denote by $W^{m,p}(0,T;V^*)$ Sobolev space of V^*-valued functions on $(0,T)$ whose distributional derivatives up to m belong to $L^p(0,T:V^*)$.

Using the maximal regularity for more general retarded parabolic system in [4, 6], we establish the following results on the solvability of (SE)(Cf. Theorem 3.1 of [5]).

Proposition 2.2. *Suppose that the assumption (F) holds. Then the following properties hold:*

1) *Let the assumption (F) be satisfied and $x_0 \in (D(A),H)_{\frac{1}{2},2}$ where $(D(A), H)_{\frac{1}{2},2}$ is the real interpolation space between $D(A)$ and H. Then there exists a unique solution x of (SE) belonging to*

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset C([0,T];(D(A),H)_{\frac{1}{2},2})$$

and satisfying

$$||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \le c_1(||x_0||_{(D(A),H)_{\frac{1}{2},2}} + ||u||_{L^2(0,T;Y)}),$$

where c_1 *is a constant depending on* T.

2) *Let* $x_0 \in H$ *and the assumption* (F) *be satisfied. Then there exists a unique solution* x *of equation* (SE) *such that*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).$$

Moreover, there exists a constant c_1 *such that*

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le c_1(|x_0| + ||u||_{L^2(0,T;Y)}).$$

Choose a bounded subset U of Y and call it a control set. Suppose that an admissible control $u \in L^2(0,T;Y)$ is strongly measurable function satisfying $u(t) \in U$ for almost all t, and let $x(t;f,u)$ be a solution of (SE) associated with the nonlinear term f and a control u at time t. The solution $x(t;f,u)$ of (SE) for each admissible control u is called a trajectory corresponding to u. Then, it is represented by

$$x(t;f,u) = S(t)x_0 + \int_0^t S(t-s)\{f(s,x(s)) + Bu(s)\}ds.$$

Let x_0 and x_1 be two different elements of H. Let us assume that there exists an admissible control u satisfying $x(t;f,u) = x_1$ and $x(0) = x_0$ for some $t > 0$. We can define the transition time that is the first time t_u such that $x(t_u;f,u) = x_1$.

The lower limit t_0 of t_u for which there exists an admissible control satisfying $x(t_u;f,u) = x_1$ is called the optimal time and we will show the existence of the admissible control satisfying $x(t_0;f,u_0) = x_1$ that is the time optimal control u_0 with respect to $\{x_0,x_1\}$.

Lemma 2.3. *Let* x_u *be the solution of* (SE) *corresponding to* u. *Then the mapping* $u \mapsto x_u$ *is compact from* $L^2(0,T;Y)$ *to* $L^2(0,T;V)$.

Proof. We define the solution mapping Q from $L^2(0,T;Y)$ to $L^2(0,T;V)$ by

$$(Qu)(t) = x_u(t), \quad u \in L^2(0,T;Y).$$

In virtue of 1) of Proposition 2.2

$$\begin{aligned}||Qu||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} &= ||x_u||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)}\\ &\le c_1\{||x_0||_{(D(A),H)_{\frac{1}{2},2}} + ||u||_{L^2(0,T;Y)}\}.\end{aligned}$$

Hence if u is bounded in $L^2(0,T;Y)$, then so is x_u in

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H).$$

Noting that $D(A)$ is compactly embedded in V by assumption, the embedding $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V))$ is also compact in view of Theorem 2 of Aubin [1]. Hence, the mapping $u \mapsto Qu = x_u$ is compact from $L^2(0,T;Y)$ to $L^2(0,T;V)$. □

Theorem 2.4. *Suppose that the assumption (F) holds. Then there exists a time optimal control with respect to* $\{x_0, x_1\}$

Proof. Put $t_0 = \inf\{t : x(t; f, u) = x_1$ where u is an admissible $\}$. Let $\{u_n, x_n\}$ be a minimizing sequence such that

$$x_n(t_n; f, u_n) = S(t_n)x_0 + \int_0^{t_n} S(t_n - s)\{f(s, x_n(s)) + Bu_n(s)\}ds.$$

Note that $x_n(t_n; f, u_n) \in C([0,T]; H)$ by 2) of Proposition 2.2. Since U is bounded, $\{x_n(t_n; f, u_n)\}$ is also bounded and hence weakly sequentially compact. Hence, we set

$$\begin{aligned} &u_n \to u_0 \text{ weakly in } L^2(0,T;Y)\\ &x(t_n; f, u_n) \to \hat{x} \text{ weakly in } H\\ &t_n \downarrow t_0. \end{aligned}$$

Let $x^* \in V^*$. Then, the duality pairing between V and V^* by $(\cdot,\cdot)$ is extension of the scalar product of H,

$$\begin{aligned} &(x_n(t_n; f, u_n), x^*)\\ &= (S(t_n)x_0, x^*) + \int_0^{t_0} (S(t_n - s)\{f(s, x_n(s)) + Bu_n(s)\}, x^*)ds \\ &+ \int_{t_0}^{t_n} (S(t_n - s)\{f(s, x_n(s)) + Bu_n(s)\}, x^*)ds. \end{aligned} \tag{2.2}$$

From the assumptions (F) and Proposition 2.2 it follows that

$$\begin{aligned} &\left|\int_{t_0}^{t_n} (S(t_n - s)\{f(s, x_n(s)) + Bu_n(s)\}, x^*)ds\right|\\ &\le (\sup_{t\in[0,T]} ||S(t)||)\{L||k||_{L^2(0,T)}||x_n||_{L^2(0,T;V)}\\ &\quad + ||B||||u_n||_{L^2(0,T;Y)}\}|x^*|(t_n - t_0)^{1/2}\\ &\le (\sup_{t\in[0,T]} ||S(t)||)\{L||k||_{L^2(0,T)}c_1(|x_0| + ||u_n||_{L^2(0,T;Y)})\\ &\quad + ||B||||u_n||_{L^2(0,T;Y)}\}|x^*|(t_n - t_0)^{1/2}. \end{aligned}$$

Thus, the first and third on the right hand side of (2.2) converge strongly to $S(t_0)x_0$ and 0, respectively. Let $\mathbb{F}$ be the Nemitsky operator corresponding to the map f, which is defined by

$$(\mathbb{F}u)(\cdot) = f(\cdot, x_u).$$

From (F) and Lemma 2.3 we see that $\mathbb{F}$ is a compact operator from $L^2(0,T;Y)$ to $L^2(0,T;H)$ and hence, it holds $\mathbb{F}u_n \to \mathbb{F}u_0$ strongly in $L^2(0,T;H)$. Therefore, by tending $n \to \infty$, it follows that

$$(\hat{x}, x^*) = (S(t_0)x_0, x^*) + \int_0^{t_0} (S(t_0 - s)\{(\mathbb{F}u_0)(s) + Bu_0(s)\}, x^*)ds.$$

Since x^* is arbitrary chosen

$$\hat{x} = S(t_0)x_0 + \int_0^{t_0} S(t_0 - s)\{(\mathbb{F}u_0)(s) + Bu_0(s)\}ds,$$

and hence, u_0 is the time optimal control. □

3. Averaging observation control

Let Z be a real Hilbert space and let $C(t)$ be bounded from H to Z for each t and be continuous in $t \in [0, T]$. Let $y \in L^2(0, T; Z)$. Suppose that there exists no admissible control which satisfies $C(t)x(t; f, u) = y(t)$ for almost all t. So we consider a cost functional given by

$$J(u) = \frac{1}{2} \int_0^T |C(t)x(t; f, u) - y(t)|^2 dt. \tag{3.1}$$

Let $u \in L^1(0, T; Y)$. Then it is well known that

$$\lim_{h \to 0} h^{-1} \int_0^h ||u(t + s) - u(t)||_Y ds = 0 \tag{3.2}$$

for almost all point of $t \in (0, T)$.

Definition 3.1. The point t which permits (3.2) to hold is called the Lebesgue point of u.

Theorem 3.2. *Let U be a bounded closed convex subset of Y. Then, there exists an optimal control for the cost functional* (3.1).

Proof. Let $\{u_n\}$ be a minimizing sequence of J such that

$$\inf_{u \in U} J(u) = \lim_{n \to \infty} J(u_n).$$

Since U is bounded and weakly closed, there exist a subsequence, which we write again by $\{u_n\}$, of $\{u_n\}$ and a $\hat{u} \in U$ such that

$$u_n \to \hat{u} \quad \text{weakly in } L^2(0, T; Y).$$

Now we show that $\hat{u}$ is admissible as follows. Since U is a closed convex set of Y, by Mazur theorem as an important consequence of the Hahn-Banach theorem, there exists an $f_0 \in Y^*$ and $c \in (-\infty, \infty)$ be such that $f_0(u) \le c$ for all $u \in U$. Let s be a Lebesgue point of $\hat{u}$ and put

$$w_{\epsilon,n} = \frac{1}{\epsilon} \int_s^{s+\epsilon} u_n(t)dt$$

for each $\epsilon > 0$ and n. Then, $f_0(w_{\epsilon,n}) \le c$ and we have

$$w_{\epsilon,n \to} w_\epsilon = \frac{1}{\epsilon} \int_s^{s+\epsilon} \hat{u}(t)dt \quad \text{weakly as } n \to \infty.$$

By letting $\epsilon \to 0$, it holds that $w_\epsilon \to \hat{u}(s)$ and $f_0(\hat{u}) \le c$, so that $\hat{u}(s) \in U$.

From Proposition 2.2 it follows that $\{x(t; f, u_n)\}$ is also bounded and hence weakly sequentially compact. Hence, as seen in the prove of Theorem 2.4, we have

$$x(t; f, u_n) \to x(t; f, \hat{u}) \text{ weakly in } H.$$

Therefore, we have

$$\inf J(u) \leq J(\hat{u}) \leq \liminf J(u_n) = \inf J(u).$$

Thus, this $\hat{u}$ is an optimal control. □

For the sake of simplicity we assume that $S(t)$ is uniformly bounded: then

$$|S(t)| \leq M (t \geq 0)$$

for some $M > 0$ (e.g. [6]).

The optimality condition J is often used to derive the uniqueness of optimal control. So, we give the conditions for the uniqueness of optimal control as follows.

Theorem 3.3. *Let $\mathbb{F}$ and $\mathbb{B}$ defined by $(\mathbb{F}u)(\cdot) = f(\cdot, x_u)$ and $(\mathbb{B}u)(\cdot) = Bu(\cdot)$, respectively. Let $\mathbb{F} + \mathbb{B}$ and $C(t)(t \geq 0)$ be one to one mappings. Then the optimal control for the cost function* (3.1) *is unique.*

Proof. Let $\hat{u}$ be an optimal control and $\hat{x}(t) = x(t; f, \hat{u})$. Let t_0 be a Lebesgue point of $\hat{u}, v$ and $\mathbb{F}(v - \hat{u})$. For $t_0 < t_0 + \epsilon < T$, put

$$u(t) = \begin{cases} v & \text{if } t_0 < t < t_0 + \epsilon \\ \hat{u}(t) & \text{otherwise.} \end{cases} \tag{3.3}$$

Then u is an admissible control. Let $x(t) = x(t; f, u)$. Then, $x(t) - \hat{x}(t) = 0$ for $0 \leq t \leq t_0$ and

$$x(t) - \hat{x}(t) = \int_{t_0}^{t_0+\epsilon} S(t-s)\{f(s, x(s)) - f(s, \hat{x}(s)) + B(v - \hat{u}(s))\} ds \tag{3.4}$$

for $t_0 < t < t_0 + \epsilon$. Using the Hölder inequality it is easily seen that

$$\begin{aligned} &||f(\cdot, x) - f(\cdot, \hat{x})||^2_{L^2(t_0, t_0+\epsilon; H)} \\ &= \int_{t_0}^{t_0+\epsilon} |\int_0^s k(s-\tau)(g(\tau, x(\tau)) - g(\tau, \hat{x}(\tau))) d\tau|^2 ds \\ &\leq ||k||^2_{L^2(0,T)} \int_{t_0}^{t_0+\epsilon} \int_0^s L^2 ||x(\tau) - \hat{x}(\tau)||^2 d\tau ds \\ &\leq \epsilon L^2 ||k||^2_{L^2(0,T)} ||x - \hat{x}||^2_{L^2(t_0, t_0+\epsilon; V)} \end{aligned}$$

and hence, with the aid of suitable change of variables and Hölder inequality

$$\begin{aligned} &|\int_{t_0}^{t_0+\epsilon} S(t-s)(f(s, x(s)) - f(s, \hat{x}(s))) ds| \\ &\leq \epsilon M L ||k||_{L^2(0,T)} ||x - \hat{x}||_{L^2(t_0, t_0+\epsilon; V)} \end{aligned} \tag{3.5}$$

for $t_0 < t < t_0 + \epsilon$. Noting that $v - \hat{u}$ is admissible and t_0 is Lebesgue point of $v - \hat{u}$, there exists a constant $c > 0$ such that

$$||v - \hat{u}(t)||_Y \leq c \quad \text{for } t_0 \leq t \leq t_0 + \epsilon.$$

Thus, we obtain

$$\begin{aligned}|x(t) - \hat{x}(t)| &\leq |\int_{t_0}^{t_0+\epsilon} S(t-s)\{f(s, x(s)) - f(s, \hat{x}(s)) + B(v - \hat{u}(s))\}ds| \\ &\leq \epsilon ML||k||_{L^2(0,T)}||x - \hat{x}||_{L^2(t_0, t_0+\epsilon; V)} + \epsilon c M||B||.\end{aligned}$$

Hence, since the control set U is bounded, in virtue of Proposition 2.2, there exists a constant c such that

$$|x(t) - \hat{x}(t)| \leq c\epsilon \tag{3.6}$$

holds for any $0 \leq t \leq T$. Since $\hat{u}$ is optimal, we have

$$\begin{aligned}0 &\leq \frac{1}{\epsilon}(J(u) - J(\hat{u})) \\ &= \frac{1}{\epsilon}\int_0^T (C(t)(x(t) - \hat{x}(t)), C(t)\hat{x}(t) - y(t))dt \\ &\quad + \frac{1}{2\epsilon}\int_0^T |C(t)(x(t) - \hat{x}(t))|^2 dt \\ &= I + II.\end{aligned} \tag{3.7}$$

From (3.6) it follows that

$$\lim_{\epsilon \to 0} II = 0. \tag{3.8}$$

The first term of (3.7) can be represented as

$$\begin{aligned}I &= \frac{1}{\epsilon}\int_{t_0}^T (C(t)(x(t) - \hat{x}(t)), C(t)\hat{x}(t) - y(t))dt \\ &= \frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon} + \frac{1}{\epsilon}\int_{t_0+\epsilon}^T = I_1 + I_2.\end{aligned}$$

On account of (3.6), it holds that

$$\lim_{\epsilon \to 0} I_1 = 0. \tag{3.9}$$

Let $t > t_0$ and $\epsilon \to 0$. Then, we obtain

$$\begin{aligned}&\lim_{\epsilon \to 0} \frac{1}{\epsilon}(x(t) - \hat{x}(t)) \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon} S(t-s)(\mathbb{F} + \mathbb{B})(v - \hat{u})(s)ds \\ &= S(t - t_0)(\mathbb{F} + \mathbb{B})(v - \hat{u})(t_0).\end{aligned}$$

Hence,

$$
\begin{aligned}
\lim_{\epsilon\to 0} I_2 &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{t_0+\epsilon}^{T}(C(t)(x(t)-\hat{x}(t)), C(t)\hat{x}(t)-y(t))dt \\
&= \int_{t_0}^{T}(C(t)S(t-t_0)(\mathbb{F}+\mathbb{B})(v-\hat{u})(t_0), C(t)\hat{x}(t)-y(t))dt.
\end{aligned} \tag{3.10}
$$

By (3.7)-(3.10), the inequality

$$\int_s^T (C(t)S(t-s)(\mathbb{F}+\mathbb{B})(v-\hat{u})(s), C(t)\hat{x}(t)-y(t))dt \geq 0$$

holds for every $v \in U$ and for all Lebesgue points s of $\hat{u}$. Let us denote two optimal controls by u_1 and u_2 and their corresponding by x_1 and x_2. Then, by the similar procedure mentioned above, the inequalities

$$\int_s^T (C(t)S(t-s)(\mathbb{F}+\mathbb{B})(u_2-u_1)(s), C(t)x_1(t)-y(t))dt \geq 0$$

and

$$\int_s^T (C(t)S(t-s)(\mathbb{F}+\mathbb{B})(u_1-u_2)(s), C(t)x_2(t)-y(t)dt \geq 0$$

hold. Add both inequalities and integrate the resultant inequality from 0 to T with respect to s. Then, since

$$x_2(t)-x_1(t) = \int_0^t S(t-s)(\mathbb{F}+\mathbb{B})(u_1-u_2)(s)ds,$$

it holds

$$\int_0^T |C(t)(x_2(t)-x_1(t))|^2 \leq 0.$$

Since $C(t)$ is one to one, we have that $x_2(t)-x_1(t)\equiv 0$. Hence, by the property of semigroup $S(t)$, it holds that $(\mathbb{F}+\mathbb{B})(u_1-u_2)(t)=0$ almost everywhere. From that $\mathbb{F}+\mathbb{B}$ is one to one, $u_1(t)=u_2(t)$ holds for almost all t. □

In order to derive necessary optimality conditions for the optimal control for J, we will establish the maximum principle, which is derived from the optimal condition as follows.

Theorem 3.4. *Let $\hat{u}$ be an optimal control. Then the inequality*

$$\max_{v\in U}(v, B^*z(s)) = (\hat{u}(s), B^*z(s))$$

holds, where

$$z(s) = \int_s^T S^*(t-s)C^*(t)(y(t)-C(t)\hat{x}(t))dt.$$

Here, $z(s)$ satisfies the following transposed system:

$$
\begin{cases}
\frac{d}{ds}z(s) = & -A^*z(s) + C^*(s)(y(s)-C(s)\hat{y}(s)), \\
z(T) \;\;= & 0
\end{cases} \tag{As}
$$

in the weak sense.

Proof. Let u be an admissible control defined by (3.3) and let t_0 be a Lebesgue point of $\hat{u}, v \in U$. Putting that $\hat{x}(t) = x(t; f, \hat{u})$, $x(t) = x(t; f, u)$, and noting

$$|f(s, x(s)) - f(s, \hat{x}(s))| \leq L||k||_{C[0,T]} \int_0^s |x(\tau) - \hat{x}(\tau)| d\tau,$$

then, by similar to the proof of (3.5), we obtain that the inequality

$$\begin{aligned} &| \int_{t_0}^{t_0+\epsilon} S(t-s)(f(s,x(s)) - f(s,\hat{x}(s)))ds| \\ &\leq ML||k||_{C[0,T]} \int_{t_0}^{t_0+\epsilon} \int_0^s |x(\tau) - \hat{x}(\tau)| d\tau ds \\ &= ML||k||_{C[0,T]} \int_{t_0}^{t_0+\epsilon} \int_{\tau}^{t_0+\epsilon} ||x(\tau) - \hat{x}(\tau)|| ds d\tau \\ &\leq \epsilon^2 ML||k||_{C[0,T]} ||x - \hat{x}||_{L^2(t_0, t_0+\epsilon; V)} \end{aligned} \tag{3.11}$$

holds.

Hence, it follows from Proposition 2.2 and the boundedness of U that there exists a constant c' such that

$$| \int_{t_0}^{t_0+\epsilon} S(t-s)(f(s,x(s)) - f(s,\hat{x}(s)))ds| \leq c'\epsilon^2. \tag{3.12}$$

Hence, noting that for $t_0 \leq t \leq t_0 + \epsilon$

$$x(t) - \hat{x}(t) = \int_{t_0}^{t_0+\epsilon} S(t-s)\{f(s,x(s)) - f(s,\hat{x}(s)) + B(v - \hat{u}(s))\}ds,$$

we obtain

$$\begin{aligned} &\lim_{\epsilon \to 0} \frac{1}{\epsilon}(x(t) - \hat{x}(t)) \\ = &\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} S(t-s)\{f(s,x(s)) - f(s,\hat{x}(s)) + B(v - \hat{u})(s)\}ds \\ = &\; S(t - t_0)B(v - \hat{u})(t_0). \end{aligned}$$

Thus, as in (3.10), we have

$$\lim_{\epsilon \to 0} I_2 = \int_{t_0}^{T} (C(t)(S(t - t_0)B(v - \hat{u})(t_0), C(t)\hat{x}(t) - y(t))dt,$$

that is, from (3.7) it follows

$$\int_s^T (C(t)S(t-s)B(v - \hat{u})(s), C(t)\hat{x}(t) - y(t))dt \geq 0$$

holds for every $v \in U$ and for all Lebesgue points s of $\hat{u}$. Hence, we have

$$(v - \hat{u}(s), B^*z(s)) \leq 0,$$

where

$$z(s) = \int_s^T S^*(t-s)C^*(t)(y(t) - C(t)\hat{x}(t))dt.$$

Here, $z(s)$ is a solution in some sense of the equation (AS). □

4. Observation of terminal value

Let y be an element of H and suppose there exists no admissible control which satisfies $x(T; f, u) = y$. We assume a cost functional given by

$$J_1 = \frac{1}{2}|x(T; f, u) - y|. \tag{4.1}$$

Theorem 4.1. *Let (F) be satisfied. Then there exists an optimal control for the cost functional* (4.1). *Moreover, if $\hat{u}$ is an optimal control for* (4.1), *then*

$$\max_{v \in U}(v, B^* z(t)) = (\hat{u}(t), B^* z(t)) \tag{4.2}$$

almost everywhere in $0 \le t \le T$, where $z(t) = S^*(T-t)(y - x(T; f, \hat{u}))$ satisfies the initial value problem

$$\begin{cases} \frac{d}{dt}z(t) = & -A^* z(t), \\ z(T) \;\; = & y - x(T; f, \hat{u}) \end{cases}$$

in the weak sense.

Proof. Let $v \in U$. Let u be an admissible control defined by (3.3) and t_0 be a Lebesgue point of $\hat{u}, v \in U$. Put $x(t) = x(t; f, u)$ and $\hat{x}(t) = x(t; f, \hat{u})$. Then

$$x(T) - \hat{x}(T) = \int_{t_0}^{t_0+\epsilon} S(T-s)\{f(s, x(s)) - f(s, \hat{x}(s)) + B(v - \hat{u}(s))\}ds.$$

Since $\hat{u}$ is an optimal control, we have

$$\begin{aligned} 0 &\le \frac{1}{\epsilon}(J_1(u)^2 - J_1(\hat{u})^2) \\ &= \frac{1}{2\epsilon}(x(T) - \hat{x}(T), \hat{x}(T) - y) + \frac{1}{4\epsilon}|x(T) - \hat{x}(T)|^2 \\ &= I + II. \end{aligned} \tag{4.3}$$

By (3.11), we have $II \to 0$ as $\epsilon \to 0$. From (3.12) it follows that

$$\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon} S(T-s)(f(s, x(s)) - f(s, \hat{x}(s)))ds \to 0,$$

thus,

$$\begin{aligned} I &= (\frac{1}{2\epsilon}\int_{t_0}^{t_0+\epsilon} S(T-s)\{f(s, x(s)) - f(s, \hat{x}(s)) + B(v - \hat{u})(s)\}ds, \hat{x}(T) - y) \\ &\to (\frac{1}{2}S(T - t_0)B(v - \hat{u})(t_0), \hat{x}(T) - y), \end{aligned}$$

as $\epsilon \to 0$. Therefore, from (4.3) we have

$$0 \leq ((v - \hat{u})(t_0), B^* S^*(T - t_0)(\hat{x}(T) - y)),$$

which implies that (4.2) holds at each Lebesgue point $\hat{u}$. □

Definition 4.2. Let $z(t) = S(T-t)^* z_0$ be a solution of the equation

$$\text{(AS-1)} \qquad \begin{cases} \frac{d}{dt} z(t) = & -A^* z(t), \\ z(T) \;\; = & z_0. \end{cases}$$

We say the adjoint system (AS-1) is weakly regular if $z_0 = 0$ follows from the existence of a set $E \subset [0,T]$ such that the measure of E is positive and $z(t) = S(T-t)^* z_0 = 0$ for all $t \in E$.

The examples for which the system (AS-1) is weakly regular are given in [9, section 7.3].

Theorem 4.3. *Let the cost J_1 be given as (4.1). Assume that the adjoint system $(AS-1)$ is weakly regular and B^* is one to one, then the optimal control $\hat{u}(t)$ is the bang-bang control, i.e., $\hat{u}(t)$ satisfies*

$$\text{(4.4)} \qquad \hat{u}(t) \in \partial U \quad \textit{for almost everywhere } t \in [0,T].$$

Proof. For the cost function J_1, the maximal principle is written by

$$\max_{v \in U} (v, B^* z(t)) = (\hat{u}(t), B^* z(t)) \quad \text{a.e. } t \in [0,T],$$

where $z(t) = S(T-t)^* z_0$. It is sufficient to show (4.4) that $B^* z(t) \neq 0$ a.e. $t \in [0,T]$. Suppose the contrary that there exists a set E such that the measure of E is positive and $B^* z(t) = 0$ $t \in E$. Since B^* is one to one and (AS-1) is weakly regular, we have that $z_0 = 0$, which is a contraction. □

The unique problem of the optimal control for the terminal value cost function J_1 is an open problem. One of the difficulties is that we can not obtain the convexity property of nonlinear term.

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