J. Korean Math. Soc. **45** (2008), No. 3, pp. 741–756

ON SB-RINGS

HUANYIN CHEN

ABSTRACT. In this paper, we introduce a new class of rings, SB-rings. We establish various properties of this concept. These shows that, in several respects, SB-rings behave like rings satisfying unit 1-stable range. We will give necessary and sufficient conditions under which a semilocal ring is a SB-ring. Furthermore, we extend these results to exchange rings with all primitive factors artinian. For such rings, we observe that the concept of the SB-ring coincides with Goodearl–Menal condition. These also generalize the results of Huh et al., Yu and the author on rings generated by their units.

1. Introduction

A ring R satisfies unit 1-stable range if $aR + bR = R$ with $a, b \in R$ implies that there exists a $u \in U(R)$ such that $a + bu \in U(R)$. If R satisfies unit 1-stable range, then $K_1(R) = U(R)/V(R)$, where $V(R) = \{(1+ab)(1+ba)^{-1} \mid a, b \in R, 1+ab \in U(R)\}$. A ring R is said to satisfies Goodearl–Menal condition provided that for any $x, y \in R$, there exists a $u \in U(R)$ such that $x-u, y-u^{-1} \in U(R)$. In [7], Goodearl and Menal provided many classes of rings satisfying such condition. If R satisfies Goodearl–Menal condition, then $K_1(R) \cong U(R)^{ab}$ (cf. [7, Theorem 1.4]), i.e., $V(R) = \big[U(R), U(R)\big]$. Obviously, every ring satisfying Goodearl–Menal condition satisfies unit 1-stable range, but the converse is not true, e.g., $\mathbb{Z}/3\mathbb{Z}$. These two conditions play important roles in algebraic K-theory. In this paper, we introduce a new class of rings: SB-rings. We say that a ring R is a SB-ring provided that $aR + bR = R$ with $a, b \in R$ implies that there exists $u \in U(R)$ such that $a \pm bu \in U(R)$. We show that, in several respects, SB-rings behave like rings satisfying unit 1-stable range. On the other hand, we see that in many large classes of rings the concept of SB-ring coincides with Goodearl–Menal condition.

Let $J(R)$ denote the Jacobson radical of a ring R. A ring R is said to be a semilocal ring provided that $R/J(R)$ is semisimple artinian (cf. [2], [5]). We will give necessary and sufficient conditions under which a semilocal ring is a SB-ring. Recall that a ring R is an exchange ring if for every right R-module A

Received September 20, 2006.
2000 *Mathematics Subject Classification.* 16E50, 19U99, 15A33.
Key words and phrases. SB-ring, semilocal ring, exchange ring.

and any two decompositions $A = M \oplus N = \bigoplus_{i \in I} A_i$, where $M_R \cong R_R$ and I is a finite index set, there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus \big(\bigoplus_{i \in I} A_i'\big)$. It is well known that a ring R is an exchange ring if and only if for any $x \in R$ there exists an idempotent $e \in xR$ such that $1 - e \in (1 - x)R$. Regular rings, π-regular rings, unit C^*-algebras of real rank zero, semiperfect rings, left or right continuous rings and clean rings are all exchange rings (cf. [1] and [10-13]). Furthermore, we prove that if R is an exchange ring with all primitive factors artinian then $M_n(R)$ is a SB-ring for all $n \geq 2$. These also generalize the results of Huh et al., Yu and the author on rings generalized by units (cf. [3] and [9]).

Throughout, all rings are associative with identity 1_R, $U(R)$ denotes the set of all units of $R, GL_n(R)$ denotes the n-dimensional general linear group of R. We always use D^* to stand for the set of all non-zero elements of a division ring D.

2. Equivalent characterizations

In this section, we assemble a few elementary properties of SB-rings. It is shown that the concept of SB-ring is right and left symmetric.

Theorem 2.1. *Let R be a ring. Then the following are equivalent:*

(1) *R is a SB-ring.*

(2) *For any $x, y \in R$, there exists $u \in U(R)$ such that $1 + x(y \pm u) \in U(R)$.*

Proof. (1) $\Rightarrow$ (2) For any $x, y \in R$, it follows from $(1 + xy)R + (-x)R = R$ that there is a $u \in U(R)$ such that $(1+xy)+(-x)u \in U(R)$, i.e., $1+x(y \pm u) \in U(R)$.

(2) $\Rightarrow$ (1) Given $aR + bR = R$ with $a, b \in R$, then there exist $x, y \in R$ such that $ax+by = 1$. By hypothesis, we have a $u \in U(R)$ such that $1+(-a)(x-u) \in U(R)$; hence, $au + by \in U(R)$. Thus, $auv + byv = 1$ for a $v \in U(R)$. By hypothesis again, there exists $w \in U(R)$ such that $1 + (-b)(yv \pm w) \in U(R)$, i.e., $auv \mp bw \in U(R)$. Therefore $a \pm bwv^{-1}u^{-1} \in U(R)$, as required. □

Corollary 2.2. *A ring R is a SB-ring if and only if so is the opposite ring R^{op}.*

Proof. Let R be a SB-ring. For any $x^{op}, y^{op} \in R^{op}$, we have $x, y \in R$. In view of Theorem 2.1, we have a $u \in U(R)$ such that $1 + x(y \pm u) \in U(R)$. It is well known that $1 + ab \in U(R)$ is and only if $1 + ba \in U(R)$ for any $a, b \in R$. Thus, $1 + (y \pm u)x \in U(R)$, i.e., $1^{op} + x^{op}\big(y^{op} \pm u^{op}\big) \in U\big(R^{op}\big)$. Therefore R^{op} is a SB-ring from Theorem 2.1. The converse is symmetric. □

Corollary 2.2 shows that the concept of SB-ring is right and left symmetric.

Lemma 2.3. *Let R be a ring. Then the following are equivalent:*

(1) *R is a SB-ring.*

(2) *$ax + b = 1$ with $a, x, b \in R$ implies that there exists $u \in U(R)$ such that $a \pm bu \in U(R)$.*

(3) *$ax+b=1$ with $a,x,b\in R$ implies that there exists $u\in U(R)$ such that $x\pm ub\in U(R)$.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) For any $x,y\in R$, we see that $(-x)y+(1+xy)=1$. By hypothesis, there exists $u\in U(R)$ such that $(-x)\pm(1+xy)u\in U(R)$, i.e., $1+x(y\pm u^{-1})\in U(R)$. According to Theorem 2.1, R is a SB-ring.

(1) $\Leftrightarrow$ (3) Applying "(1) $\Leftrightarrow$ (2)" to the opposite ring R^{op}, we complete the proof. □

Let I be an ideal of a SB-ring R. As an immediate consequence of Lemma 2.3, R/I is a SB-ring. Let $[\alpha,\beta]=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$, $B_{12}(a)=\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}$ and $B_{21}(a)=\begin{pmatrix}1 & 0\\ a & 1\end{pmatrix}$. Now we derive a new characterization of SB-ring.

Theorem 2.4. *Let R be a ring. Then the following are equivalent:*

(1) *R is a SB-ring.*

(2) *For any $A\in GL_2(R)$, there exists $u\in U(R)$ such that $A=[*,*]B_{21}(*)B_{12}(*)B_{21}(\pm u)$.*

Proof. (1) $\Rightarrow$ (2) Given any $A=(a_{ij})\in GL_2(R)$, we have $a_{11}R+a_{12}R=R$. Since R is a SB-ring, there exists a $u\in U(R)$ such that $v_1:=a_{11}+a_{12}u\in U(R)$ and $v_2:=a_{11}-a_{12}u\in U(R)$. This implies that

$$AB_{21}(u)=\begin{pmatrix}v_1 & a_{12}\\ a_{21}+a_{22}u & a_{22}\end{pmatrix}.$$

It is easy to verify that

$$B_{21}(*)\begin{pmatrix}v_1 & a_{12}\\ a_{21}+a_{22}u & a_{22}\end{pmatrix}B_{12}(*)=[v_1,w_1]$$

for some $w_1\in U(R)$. Thus,

$$\begin{pmatrix}v_1 & a_{12}\\ a_{21}+a_{22}u & a_{22}\end{pmatrix}=[*,*]B_{21}(*)B_{12}(*).$$

Therefore $A=[*,*]B_{21}(*)B_{12}(*)B_{21}(u)$. Likewise, we have

$$AB_{21}(-u)=\begin{pmatrix}v_2 & a_{12}\\ a_{21}-a_{22}u & a_{22}\end{pmatrix}.$$

Thus, $A=[*,*]B_{21}(*)B_{12}(*)B_{21}(-u)$, as required.

(2) $\Rightarrow$ (1) Given $ax+b=1$ with $a,x,b\in R$, then

$$\begin{pmatrix}a & b\\ -1 & x\end{pmatrix}=\begin{pmatrix}a & b\\ -1 & x\end{pmatrix}^{-1}\in GL_2(R).$$

By assumption, there exists $u\in U(R)$ such that

$$\begin{pmatrix}a & b\\ -1 & x\end{pmatrix}=[*,*]B_{21}(*)B_{12}(*)B_{21}(\pm u).$$

Thus, we can find $\alpha_i, \beta_i \in U(R), c_i, d_i \in R (i = 1, 2)$ such that

$$\begin{pmatrix} a & b \\ -1 & x \end{pmatrix} B_{21}(-u) = [\alpha_1, \beta_1] B_{21}(c_1) B_{12}(d_1)$$

and

$$\begin{pmatrix} a & b \\ -1 & x \end{pmatrix} B_{21}(u) = [\alpha_2, \beta_2] B_{21}(c_2) B_{12}(d_2).$$

It follows that $a - bu = \alpha_1 \in U(R)$ and $a + bu = \alpha_2 \in U(R)$. Therefore we complete the proof by Lemma 2.3. □

Corollary 2.5. *Let R be a ring. Then the following are equivalent:*

(1) *R is a SB-ring.*

(2) *For any $A \in GL_2(R)$, there exists $u \in U(R)$ such that $A = [*, *]B_{12}(*) B_{21}(*)B_{12}(\pm u)$.*

Proof. (1) ⇒ (2) For any $A \in GL_2(R)$, we have $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(R)$. According to Theorem 2.4, there is a $u \in U(R)$ such that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = [*, *]B_{21}(*)B_{12}(*)B_{21}(\pm u).$$

As a result, we deduce that

$$\begin{aligned} A = & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} [*, *] \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B_{21}(*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B_{12}(*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B_{21}(\pm u) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

This implies that $A = [*, *]B_{12}(*)B_{21}(*)B_{12}(\pm u)$.

(2) ⇒ (1) Given $ax + b = 1$ with $a, x, b \in R$, then

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ -1 & x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(R).$$

By assumption, there exists $u \in U(R)$ such that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ -1 & x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = [*, *]B_{12}(*)B_{21}(*)B_{12}(\pm u).$$

As in the proof above, we see that

$$\begin{pmatrix} a & b \\ -1 & x \end{pmatrix} = [*, *]B_{21}(*)B_{12}(*)B_{21}(\pm u).$$

Similarly to the consideration in Theorem 2.4, we show that $a \pm bu \in U(R)$, and therefore the proof is true. □

3. Semilocal rings

Semilocal rings have been studied by many authors (cf. [2], [5]). In this section, we investigate semilocal SB-rings. These give new matrix decompositions over semilocal rings as well.

Lemma 3.1. *Let D be a division ring, and let $A \in M_n(D)(n \geq 2)$. Then there exists a $U \in GL_n(D)$ such that $A \pm U \in GL_n(D)$.*

Proof. Since D is a division ring, there exist $U, V \in GL_n(D)$ such that

$$UAV = \begin{pmatrix} 1_D & & & & & & \\ & 1_D & & & & & \\ & & \ddots & & & & \\ & & & 1_D & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}_{n\times n}.$$

Assume that n is an even number. Let

$$W = \begin{pmatrix} 1_D & & & & & & 1_D \\ 1_D & 0 & & & & & \\ & 1_D & \ddots & & & & \\ & & \ddots & 0 & & & \\ & & & 1_D & 0 & & \\ & & & & \ddots & \ddots & \\ & & & & & 1_D & 0 \end{pmatrix}_{n\times n}.$$

Then $W \in GL_n(D)$. Clearly,

$$\begin{aligned} UAV - W &= \begin{pmatrix} 0 & & & & & & -1_D \\ -1_D & 1_D & & & & & \\ & -1_D & \ddots & & & & \\ & & \ddots & 1_D & & & \\ & & & -1_D & 0 & & \\ & & & & \ddots & \ddots & \\ & & & & & -1_D & 0 \end{pmatrix}_{n\times n} \\ &\in GL_n(D). \end{aligned}$$

On the other hand,

$$UAV+W=\begin{pmatrix} 2\times 1_D & & & & & & 1_D \\ 1_D & 1_D & & & & & \\ & 1_D & \ddots & & & & \\ & & \ddots & 1_D & & & \\ & & & 1_D & 0 & & \\ & & & & \ddots & \ddots & \\ & & & & & 1_D & 0 \end{pmatrix}_{n\times n}$$

or

$$UAV+W=\begin{pmatrix} 2\times 1_D & & & & & & 1_D \\ 1_D & 1_D & & & & & \\ & 1_D & 1_D & & & & \\ & & 1_D & \ddots & & & \\ & & & 1_D & \ddots & & \\ & & & & \ddots & \ddots & \\ & & & & & 1_D & 1_D \end{pmatrix}_{n\times n}.$$

As n is an even number, one easily checks that $UAV+W\in GL_n(D)$.

Assume that n is an odd number. Let

$$W=\begin{pmatrix} 1_D & & & & & & 1_D \\ -1_D & 0 & & & & & \\ & 1_D & \ddots & & & & \\ & & \ddots & 0 & & & \\ & & & 1_D & 0 & & \\ & & & & \ddots & \ddots & \\ & & & & & 1_D & 0 \end{pmatrix}_{n\times n}.$$

Then

$$\begin{aligned} UAV-W &= \begin{pmatrix} 0 & & & & & & -1_D \\ 1_D & 1_D & & & & & \\ & -1_D & \ddots & & & & \\ & & \ddots & 1_D & & & \\ & & & -1_D & 0 & & \\ & & & & \ddots & \ddots & \\ & & & & & -1_D & 0 \end{pmatrix}_{n\times n} \\ &\in GL_n(D). \end{aligned}$$

On the other hand,

$$UAV + W = \begin{pmatrix} 2\times 1_D & & & & & & & 1_D \\ -1_D & 1_D & & & & & & \\ & 1_D & \ddots & & & & & \\ & & \ddots & 1_D & & & & \\ & & & 1_D & 0 & & & \\ & & & & \ddots & \ddots & & \\ & & & & & & 1_D & 0 \end{pmatrix}_{n\times n}$$

or

$$UAV + W = \begin{pmatrix} 2\times 1_D & & & & & & & 1_D \\ -1_D & 1_D & & & & & & \\ & 1_D & 1_D & & & & & \\ & & 1_D & \ddots & & & & \\ & & & 1_D & \ddots & & & \\ & & & & \ddots & \ddots & & \\ & & & & & & 1_D & 1_D \end{pmatrix}_{n\times n}.$$

As n is an odd number, one easily checks that $UAV + W \in GL_n(D)$. In any case, we can find a $W \in GL_n(D)$ such that $UAV \pm W \in GL_n(D)$, and so $A \pm U^{-1}WV^{-1} \in GL_n(D)$. □

Lemma 3.2. *Let D be a division ring. Then the following hold:*

(1) *D is a SB-ring if and only if $|D| \geq 4$.*

(2) *$M_n(D)$ is a SB-ring for any $n \geq 2$.*

Proof. (1) Assume that D is a SB-ring. Since $1_D \times 0 + 1_D = 1_D$, we can find a $u \in D^*$ such that $1_D \pm u \in D^*$. Thus, $\{0, 1_D, u, 1_D + u\}$ is a subset of D, and so $|D| \geq 4$. Conversely, assume that $|D| \geq 4$. Suppose that $ax + b = 1_D$ with $a, x, b \in D$. If $x = 0$, then $b = 1_D$. As $|D| \geq 4$, we can choose a $u \notin \{0, a, -a\}$. Thus, $a \pm bu \in D^*$. If $x \neq 0$ and $b = 0$, then $a \pm b \times 1_D \in D^*$. If $x \neq 0$ and $b \neq 0$, we choose a $u \notin \{0, xb^{-1}a, -xb^{-1}\}$. Then $u \neq \pm xb^{-1}a$, and so $a \pm bx^{-1}u \in D^*$. In any case, we can find a $v \in D^*$ such that $a \pm bv \in D^*$. Therefore D is a SB-ring.

(2) Assume that $AX + B = I_n$ in $M_n(D)$. Since D is a division ring, there exist $U, V \in GL_n(D)$ such that $UAV = diag(I_r, 0)$ for some $0 \leq r \leq n$. Let $X' = V^{-1}XU^{-1}$ and $B' = UBU^{-1}$. Then $diag(I_r, 0)X' + B' = (UAV)(V^{-1}XU^{-1}) + UBU^{-1} = I_n$. Assume that $X' = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ and $B' = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, where $X_{11}, B_{11} \in M_{r\times r}(D), X_{12}, B_{12} \in M_{r\times(n-r)}(D)$, $X_{21}, B_{21} \in M_{(n-r)\times r}(D), X_{22}, B_{22} \in M_{(n-r)\times(n-r)}(D)$. One simply checks

that

$$X_{11} + B_{11} = I_r,$$
$$X_{12} = -B_{12}, B_{21} = 0,$$
$$B_{22} = I_{n-r}.$$

If $r = 0$, then $A = 0$; hence, $B = I_n$. Thus, $A \pm B \times I_n \in GL_n(D)$.

If $r = 1$ and $B_{11} = 0$, then

$$\begin{pmatrix} 1_D & 0 \\ 0 & 0_{n-1} \end{pmatrix} \pm \begin{pmatrix} 0 & B_{12} \\ 0 & I_{n-1} \end{pmatrix}\begin{pmatrix} 1_D & 0 \\ 0 & I_{n-1} \end{pmatrix} = \begin{pmatrix} 1_D & * \\ 0 & \pm I_{n-1} \end{pmatrix} \in GL_n(D).$$

That is, $UAV \pm UBU^{-1} \in GL_n(D)$. Let $\Lambda = U^{-1}V^{-1}$. Then $A \pm B\Lambda, \Lambda \in GL_n(D)$.

If $r = 1$ and $B_{11} \neq 0$, then $B_{11} \in D^*$. This implies that $B' \in GL_n(D)$. In view of Lemma 3.1, there exists some $U \in GL_n(D)$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0_{n-1} \end{pmatrix} \pm U \in GL_n(D).$$

Choose $Y = (B')^{-1}U$. Then $Y \in GL_n(D)$ and

$$\begin{pmatrix} 1 & 0 \\ 0 & 0_{n-1} \end{pmatrix} \pm B'Y \in GL_n(D).$$

That is, $UAV \pm UBU^{-1}Y \in GL_n(D)$. Let $\Lambda = U^{-1}YV^{-1}$. Then $A \pm B\Lambda, \Lambda \in GL_n(D)$.

If $r \geq 2$, by Lemma 3.1, we can find some $W \in GL_r(D)$ such that $B_{11} \pm W \in GL_r(D)$. Hence $I_r \pm B_{11}W^{-1} \in GL_r(D)$. Thus, we see that

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \pm \begin{pmatrix} B_{11} & B_{12} \\ 0 & I_{n-r} \end{pmatrix}\begin{pmatrix} W^{-1} & 0 \\ 0 & I_{n-r} \end{pmatrix} \in GL_n(D).$$

That is,

$$UAV \pm UBU^{-1}\begin{pmatrix} W^{-1} & 0 \\ 0 & I_{n-r} \end{pmatrix} \in GL_n(D).$$

Let $\Lambda = U^{-1}\begin{pmatrix} W^{-1} & 0 \\ 0 & I_{n-r} \end{pmatrix}V^{-1}$. Then $A \pm B\Lambda, \Lambda \in GL_n(D)$.

Therefore we conclude that $M_n(D)(n \geq 2)$ is a SB-ring. □

Lemma 3.3. *$R/J(R)$ is a SB-ring if and only if so is R.*

Proof. Assume that $R/J(R)$ is a SB-ring. Given $ax + b = 1$ in R, then $\overline{ax+b} = \overline{1}$ in $R/J(R)$. Thus, we can find a $\overline{u} \in U(R/J(R))$ such that $\overline{a \pm bu} \in U(R/J(R))$. As $\overline{u} \in U(R/J(R))$, we have a $\overline{v} \in R/J(R)$ such that $\overline{uv} = \overline{1} = \overline{vu}$; hence, $1-uv \in J(R)$. This implies that $uv = 1-(1-uv) \in U(R)$, and then $u \in R$ is right invertible. Likewise, we see that $u \in R$ is left invertible. As a result, $u \in U(R)$. Similarly, $a \pm bu \in U(R)$. Thus, R is a SB-ring. The converse is clear. □

Recall that a ring R is a homogeneous semilocal ring provided that $R/J(R)$ is simple artinian. By virtue of Lemma 3.2 and Lemma 3.3, we show that every homogeneous semilocal ring with $cd(R) \geq 2$ is a SB-ring (cf. [5, Lemma 3.1]). In [4, Theorem 4.1], the author proved that if R satisfies Goodearl-Menal condition, then so does $M_n(R)$ for any $n \in \mathbb{N}$. Now we observe that this condition coincides with the concept of SB-ring for a semilocal ring.

Theorem 3.4. *Let R be a semilocal ring. Then the following are equivalent:*

(1) *R is a SB-ring.*
(2) *There exists $u \in U(R)$ such that $1_R \pm u \in U(R)$.*
(3) *For any $x, y \in R$, there exists $u \in U(R)$ such that $x-u, y-u^{-1} \in U(R)$.*
(4) *R has no homomorphic images $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$.*

Proof. (1) $\Rightarrow$ (4) Let I be an ideal of R. Since R is a SB-ring, so is R/I. But neither $\mathbb{Z}/2\mathbb{Z}$ nor $\mathbb{Z}/3\mathbb{Z}$ is a SB-ring. Hence, $R/I \not\cong \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$, as desired.

(4) $\Rightarrow$ (3) Since R is a semilocal ring, we can find division rings $D_1, \dots, D_s$ such that $R/J(R) \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_s}(D_s)$. Clearly, each $D_i \cong R/M_i$ for a maximal ideal M_i of R. Hence $|D_i| \geq 4$. For any $x, y \in D_i^*$, we choose $u \notin \{0, x, y^{-1}\}$. For $x \neq 0, y = 0$, we choose $u \notin \{0, x\}$. For $x = 0, y \neq 0$, we choose $u \notin \{0, y^{-1}\}$. For $x = y = 0$, we choose $u = 1$. In any case, we have that $x - u, y - u^{-1} \in D_i^*$. For any $X, Y \in M_{n_i}(D_i)$, it follows from [4, Theorem 4.1] that there exists $U \in GL_{n_i}(D_i)$ such that $X - U, Y - U^{-1} \in GL_{n_i}(D_i)$. So for any $x, y \in R$, there exists a $u \in U(R)$ such that $x - u, y - u^{-1} \in U(R)$.

(3) $\Rightarrow$ (2) By hypothesis, there is a $u \in U(R)$ such that $1 - u, -1 - u^{-1} \in U(R)$. As a result, $1 \pm u \in U(R)$.

(2) $\Rightarrow$ (1) Since R is a semilocal ring, we can find division rings $D_1, \dots, D_s$ such that $R/J(R) \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_s}(D_s)$. If $n_i \geq 2$, then $M_{n_i}(D_i)$ is a SB-ring by Lemma 3.2. If $n_i = 1$, then D_i is isomorphic to a homomorphic image of R. Thus, we can find a $u_i \in D_i^*$ such that $1_{D_i} \pm u_i \in D_i^*$. This implies that $\{0, 1_{D_i}, u_i, 1_{D_i} + u_i\}$ is a subset of D_i, and so $|D_i| \geq 4$. According to Lemma 3.2, D_i is a SB-ring. Therefore $R/J(R)$ is a SB-ring, and so the result follows by Lemma 3.3. □

Let R be a semilocal ring. We claim that $M_n(R)$ is a SB-ring for any $n \geq 2$. Since there exist division ring $D_1, \dots, D_s$ such that

$$M_n(R)/J(M_n(R)) \cong M_n(R/J(R)) \cong M_{nn_1}(D_1) \oplus \cdots \oplus M_{nn_s}(D_1)$$

for some natural numbers $n_1, \dots, n_s$, and we are done by Lemma 3.2 and Lemma 3.3.

Let $R = \{\frac{m}{n} \in \mathbb{Q} \mid (m, n) = 1, (n, pq) = 1\}$, where $p, q (\neq 2, 3)$ are distinct prime numbers. Then we claim that R is a SB-ring. As $\text{Max}(R) = \{pR, qR\}$, R is a semilocal ring. It is easy to verify that $R/J(R) \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$. As $p, q \neq 2, 3$, $1_R \pm u \in U(R)$ for some $u \in U(R)$. According to Theorem 3.4, we are through.

Corollary 3.5. *Let R be a semilocal ring. If $2, 3 \in U(R)$, then for any $A \in M_n(R)$, there exist $U, V \in GL_n(R)$ such that $A = U + V$ with $U - V \in GL_n(R)$.*

Proof. Let $S = M_n(R)$. Then S is semilocal. Clearly, $1_S + 2^{-1} = 3 \times 2^{-1} \in U(S)$ and $1_S - 2^{-1} = 2^{-1} \in U(S)$. According to Theorem 3.4, $M_n(R)$ is a SB-ring for any $n \geq 1$. For any $A \in M_n(R)$, we have $AM_n(R) + M_n(R) = M_n(R)$. Thus, we can find $U', V', W' \in GL_n(R)$ such that $A + W' = U'$ and $A - W' = V'$. As a result, $A = \frac{1}{2}U' + \frac{1}{2}V'$ and $W' = \frac{1}{2}U' - \frac{1}{2}V'$. Let $U = \frac{1}{2}U'$ and $V = \frac{1}{2}V'$. Then $A = U + V$ with $U - V \in GL_n(R)$, as required. □

Corollary 3.6. *Let A be an artinian right R-module. If $2, 3 \in U(R)$, then for any $\alpha \in \mathrm{End}_R(A)$, there exist $\beta, \gamma \in \mathrm{Aut}_R(A)$ such that $\alpha = \beta + \gamma$ with $\beta - \gamma \in \mathrm{Aut}_R(A)$.*

Proof. Let $\sigma : A \to A$ given by $\sigma(a) = a \cdot 2$ for any $a \in A$ and $\tau : A \to A$ given by $\tau(a) = a \cdot \frac{1}{2}$ for any $a \in A$. It is easy to verify that $\sigma\tau = 1_A = \tau\sigma$. This means that $2 \times 1_A \in U\big(\mathrm{End}_R(A)\big)$. Likewise, $3 \times 1_A \in U\big(\mathrm{End}_R(A)\big)$. In view of [2, Corollary 6], $\mathrm{End}_R(A)$ is a semilocal ring. Therefore we complete the proof by Corollary 3.5. □

4. Exchange rings with primitive factors artinian

Many authors investigated exchange rings with primitive factors artinian (cf. [3], [9], and [12]). In [12, Theorem 3], Yu proved that every exchange ring of bounded index has artinian primitive factors. In this section, we extend Theorem 3.4 to such exchange rings. This also shows that Goodearl-Menal condition coincides with the concept of the SB-ring for an exchange ring with primitive factors artinian.

Theorem 4.1. *Let R be an exchange ring with primitive factors artinian. Then the following are equivalent:*

(1) *R is a SB-ring.*
(2) *There exists $u \in U(R)$ such that $1_R \pm u \in U(R)$.*
(3) *For any $x, y \in R$, there exists $u \in U(R)$ such that $x - u, y - u^{-1} \in U(R)$.*
(4) *R has no homomorphic images $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$.*

Proof. (1) ⇒ (4) and (3) ⇒ (2) are proved as in Theorem 3.4.

(2) ⇒ (1) Suppose that $ax + b = 1$ in R and $a + bu \notin U(R)$ or $a - bu \notin U(R)$ for any $u \in U(R)$. Let $\Omega = \{P \text{ is an ideal of } R \mid \overline{a + bu} \notin U(R/P) \text{ or } \overline{a - bu} \notin U(R/P) \text{ for any } \overline{u} \in U(R/P)\}$. Then $\Omega \neq \emptyset$. Given $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n \subseteq \cdots$ in Ω, we claim that $\bigcup\limits_{i=1}^{\infty} P_i \in \Omega$. If not, we have a $\overline{u} \in R/\bigcup\limits_{i=1}^{\infty} P_i$ such that $\overline{a + bu}, \overline{a - bu} \in U\big(R/\bigcup\limits_{i=1}^{\infty} P_i\big)$. Then there are some $v, s, t \in R$ such that

$$\overline{uv} = \overline{1} = \overline{vu}, \overline{(a + bu)s} = \overline{1} = \overline{s(a + bu)}, \overline{(a - bu)t} = \overline{1} = \overline{t(a - bu)}$$

in $R/\bigcup_{i=1}^{\infty} P_i$. So we have $i,j,k,l,m,n \in \mathbb{N}$ such that

$$\overline{uv} = \overline{1} \text{ in } R/P_i, \overline{vu} = \overline{1} \text{ in } R/P_j, \overline{(a+bu)s} = \overline{1} \text{ in } R/P_k,$$
$$\overline{s(a+bu)} = \overline{1} \text{ in } R/P_l, \overline{(a-bu)t} = \overline{1} \text{ in } R/P_m, \overline{t(a-bu)} = \overline{1} \text{ in } R/P_n.$$

Choose $q = \max\{i,j,k,l,m,n\}$. Then $\overline{a \pm bu}, \overline{u} \in U(R/P_q)$, and so $P_q \notin \Omega$. This gives a contradiction. Thus, Ω is inductive. By Zorn's Lemma, there exists an ideal Q which is maximal in Ω. If $R/Q/J(R/Q)$ is a decomposable ring, we have two ideals K and L of R such that $R/Q/J(R/Q) = K/Q/J(R/Q) \oplus L/Q/J(R/Q)$ with $K/Q, L/Q \subsetneqq R/Q$. Since $Q \subsetneqq K, L$, we can find $\overline{v} \in R/K, \overline{w} \in R/L$ such that $\overline{a} \pm \overline{bv} \in U(R/K), \overline{a} \pm \overline{bw} \in U(R/L)$. Clearly, we have

$$L/Q/J(R/Q) \cong R/Q/J(R/Q)/K/Q/J(R/Q) \cong R/K$$

and

$$K/Q/J(R/Q) \cong R/Q/J(R/Q)/L/Q/J(R/Q) \cong R/L.$$

This implies that $\overline{(a+Q)} \pm \overline{(bu+Q)} \in U\big(R/Q/J(R/Q)\big)$ for some $\overline{(u+Q)} \in U\big(R/Q/J(R/Q)\big)$. Since every unit lifts modulo the Jacobson radical, we deduce that $Q \notin \Omega$, a contradiction. This shows that $R/Q/J(R/Q)$ is an indecomposable ring. By virtue of [13, Lemma 3.7], $R/Q/J(R/Q)$ is a simple artinian ring, and so $R/Q/J(R/Q) \cong M_n(D)$ for some $n \in \mathbb{N}$, where D is a division ring. If $n = 1$, by assumption, we can find a $\overline{v} \in U\big(R/Q/J(R/Q)\big)$ such that $\overline{1} \pm \overline{v} \in U\big(R/Q/J(R/Q)\big)$. Since $R/Q/J(R/Q)$ is a division ring, as in the proof of Lemma 3.2, we see that $|R/Q/J(R/Q)| \geq 4$. It follows by Lemma 3.2 that $\overline{a} \pm \overline{bw} \in U(R/Q)$ for some $\overline{w} \in U(R/Q)$, which gives a contradiction. If $n \geq 2$, it follows by Lemma 3.2 that $\overline{a} \pm \overline{bv} \in U(R/Q)$ for some $\overline{v} \in U(R/Q)$, which gives a contradiction. Therefore we have a $u \in U(R)$ such that $a \pm bu \in U(R)$, as required.

(4) $\Rightarrow$ (3) Assume that there exist some $x, y \in R$ such that for any $u \in U(R)$, we have $x - u \notin U(R)$ or $y - u^{-1} \notin U(R)$. Let $\Omega = \{P \text{ is an ideal of } R \mid \overline{x} - \overline{u} \notin U(R/P) \text{ or } \overline{y} - \overline{u}^{-1} \notin U(R/P) \text{ for any } \overline{u} \in U(R/P)\}$. It is easy to check that Ω is a nonempty inductive set. By using Zorn's Lemma, there exists an ideal Q which is maximal in Ω. As in the proof in "(2) $\Rightarrow$ (1)", $R/Q/J(R/Q) \cong M_n(D)$ for some $n \in \mathbb{N}$, where D is a division ring. By assumption, $M_n(D)$ has no homomorphic images $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$. In view of Theorem 3.4, $M_n(D)$ satisfies Goodearl-Menal condition, Since every unit lifts modulo the Jacobson, we show that $x - v, y - v^{-1} \in U(R/Q)$ for a $v \in U(R/Q)$. This contradicts the choice of Q. Therefore for any $x, y \in R$, there is a $u \in U(R)$ such that $a \pm bu \in U(R)$, as desired. □

Corollary 4.2. *Let R be an exchange ring with primitive factors artinian. Then $M_n(R)$ is a SB-ring for all $n \geq 2$.*

Proof. Since R is an exchange ring with primitive factors artinian, so is $M_n(R)$. Assume that n is an even number. Choose

$$U=\begin{pmatrix} 1_R & & & & & & 1_R \\ 1_R & 0 & & & & & \\ & 1_R & \ddots & & & & \\ & & \ddots & 0 & & & \\ & & & 1_R & 0 & & \\ & & & & \ddots & \ddots & \\ & & & & & 1_R & 0 \end{pmatrix}_{n\times n}.$$

Then $U\in GL_n(R)$. Clearly,

$$I_n-U=\begin{pmatrix} 0 & & & & & & -1_R \\ -1_R & 1_R & & & & & \\ & -1_R & \ddots & & & & \\ & & \ddots & 1_R & & & \\ & & & -1_R & 1_R & & \\ & & & & \ddots & \ddots & \\ & & & & & -1_R & 1_R \end{pmatrix}\in GL_n(R).$$

Furthermore,

$$I_n+U=\begin{pmatrix} 2\times 1_R & & & & & & 1_R \\ 1_R & 1_R & & & & & \\ & 1_R & 1_R & & & & \\ & & 1_R & \ddots & & & \\ & & & 1_R & \ddots & & \\ & & & & \ddots & \ddots & \\ & & & & & 1_R & 1_R \end{pmatrix}_{n\times n}.$$

As n is an even number, one easily checks that $I_n+U\in GL_n(R)$.

Assume that n is an odd number. Let

$$U=\begin{pmatrix} 1_R & & & & & & 1_R \\ -1_R & 0 & & & & & \\ & 1_R & \ddots & & & & \\ & & \ddots & 0 & & & \\ & & & 1_R & 0 & & \\ & & & & \ddots & \ddots & \\ & & & & & 1_R & 0 \end{pmatrix}_{n\times n}.$$

Then

$$I_n - U = \begin{pmatrix} 0 & & & & & & -1_R \\ 1_R & 1_R & & & & & \\ & -1_R & \ddots & & & & \\ & & \ddots & 1_R & & & \\ & & & -1_R & 0 & & \\ & & & & \ddots & \ddots & \\ & & & & & -1_R & 0 \end{pmatrix} \in GL_n(R).$$

On the other hand,

$$I_n + U = \begin{pmatrix} 2\times 1_R & & & & & & 1_R \\ -1_R & 1_R & & & & & \\ & 1_R & 1_R & & & & \\ & & 1_R & \ddots & & & \\ & & & 1_R & \ddots & & \\ & & & & \ddots & \ddots & \\ & & & & & 1_R & 1_R \end{pmatrix}_{n\times n}.$$

As n is an odd number, one easily checks that $I_n + U \in GL_n(R)$. In any case, we have a $U \in GL_n(R)$ such that $I_n \pm U \in GL_n(R)$. According to Theorem 4.1, we complete the proof. □

Let R be an exchange ring with primitive factors artinian. We claim that $M_n(R)(n \geq 2)$ satisfies Goodearl-Menal condition. Clearly, $M_n(R)$ is an exchange ring with primitive factors artinian as well. In view of Corollary 4.2, $M_n(R)(n \geq 2)$ is a SB-ring, and therefore we are done by Theorem 4.1. Let R be a semilocal ring. By a similar route, we prove that $M_n(R)(n \geq 2)$ satisfies Goodearl-Menal condition. In these two cases, $K_1(R) \cong K_1\big(M_2(R)\big) \cong GL_2(R)^{ab}$.

Let D be a division ring, and let $R = \{x_1, x_2, \ldots, x_n, y, y, \ldots\} \mid x_i \in M_i(D)$, $n \in \mathbb{N}, y \in D\}$, where y is treated as a scalar matrix of proper size under multiplied with x_i. In view of [13, Example 2.3], R is an exchange ring with primitive factors artinian. By virtue of Corollary 4.2, $M_n(R)(n \geq 2)$ is a SB-ring.

In view of [13, Example 3.10], $\begin{pmatrix} \mathbb{Z}/5\mathbb{Z} & \mathbb{Z}/5\mathbb{Z} \\ 0 & \mathbb{Z}/5\mathbb{Z} \end{pmatrix}$ is a quasi-duo exchange ring; hence, it has artinain primitive factors. One simply checks that

$$\begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix} + \begin{pmatrix} \overline{2} & \overline{0} \\ \overline{0} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{3} & \overline{0} \\ \overline{0} & \overline{3} \end{pmatrix},$$
$$\begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix} - \begin{pmatrix} \overline{2} & \overline{0} \\ \overline{0} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{4} & \overline{0} \\ \overline{0} & \overline{4} \end{pmatrix}.$$

It follows from Theorem 4.1 that $\begin{pmatrix} \mathbb{Z}/5\mathbb{Z} & \mathbb{Z}/5\mathbb{Z} \\ 0 & \mathbb{Z}/5\mathbb{Z} \end{pmatrix}$ is a SB-ring.

Theorem 4.3. *Let R be an exchange ring with primitive factors artinian. If $2,3 \in U(R)$, then for any $A \in M_n(R)$, there exist $U, V \in GL_n(R)$ such that $A = U + V$ with $U - V \in GL_n(R)$.*

Proof. Let $S = M_n(R)$. Then S is an exchange ring with primitive factors artinian. Clearly, $1_S + 2^{-1} = 3 \times 2^{-1} \in U(S)$ and $1_S - 2^{-1} = 2^{-1} \in U(S)$. By virtue of Theorem 4.1 and Corollary 4.2, $M_n(R)$ is a SB-ring for all $n \in \mathbb{N}$. For any $A \in M_n(R)$, it follows from $AM_n(R) + M_n(R) = M_n(R)$ that $A + W' = U'$ and $A - W' = V'$ for some $U', V', W' \in GL_n(R)$. Let $U = \frac{1}{2}U'$ and $V = \frac{1}{2}V'$. As in the proof of Corollary 3.5, we have that $A = U + V$ with $U - V \in GL_n(R)$, as asserted. □

Since every commutative exchange ring and every exchange PI-ring have artinian primitive factors, Theorem 4.3 holds for such exchange rings. Recall that a ring R is of bounded index if there is some $n \in \mathbb{N}$ such that $x^n = 0$ for any nilpotent x of R. In addition, we can derive the following.

Corollary 4.4. *Let R be an exchange ring of bounded index. If $2,3 \in U(R)$, then for any $A \in M_n(R)$, there exist $U, V \in GL_n(R)$ such that $A = U + V$ with $U - V \in GL_n(R)$.*

Proof. Since R is an exchange ring of bounded index, it follows by [12, Theorem 3] that R is an exchange ring with primitive factors artinian. Therefore we complete the proof by Theorem 4.3. □

5. Exchange rings with all idempotents central

The main purpose of this section is to investigate exchange rings with all idempotents cental. Let R be an exchange ring with all idempotents central. In view of Theorem 4.1, we deduce that R is a SB-ring if and only if R has no homomorphic images $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$ if and only if for any $x, y \in R$ there exists $u \in U(R)$ such that $x - u, y - u^{-1} \in U(R)$. Furthermore, we can derive the following.

Theorem 5.1. *Let R be an exchange ring with all idempotents central. Then the following are equivalent:*

(1) *R is a SB-ring.*
(2) *$\mid R/M \mid \geq 4$ for all maximal ideal M of R.*

Proof. (1) ⇒ (2) Since R is a SB-ring, we can find a $u \in U(R)$ such that $1 \pm u \in U(R)$. Let M be a maximal ideal of R. Then $\overline{1} \pm \overline{u} \in U(R/M)$. This implies that $\{\overline{0}, \overline{1}, \overline{u}, \overline{1} + \overline{u}\} \subseteq R/M$. Therefore $\mid R/M \mid \geq 4$, as required.

(2) ⇒ (1) Suppose that $ax + b = 1$ in R and $a + bu \notin U(R)$ or $a - bu \notin U(R)$ for any $u \in U(R)$. Let $\Omega = \{P$ is an ideal of $R \mid \overline{a + bu} \notin U(R/P)$ or $\overline{a - bu} \notin U(R/P)$ for any $\overline{u} \in U(R/P)\}$. As in the proof of Theorem 4.1, there

exists an ideal Q which is maximal in Ω. In addition, $R/Q/J(R/Q)$ is an indecomposable ring. Since every idempotent in R is central, it follows from [11, Theorem 29.5] that $R/Q/J(R/Q)$ is a local ring; hence, it is a division ring. Assume that $J(R/Q) = M/J(R)$ for an ideal M of R. As $R/M \cong R/Q/M/Q \cong R/Q/J(R/Q)$, M is a maximal ideal of R. This implies that $\mid R/M \mid \geq 4$, i.e., $\mid R/Q/J(R/Q) \mid \geq 4$. In view of Lemma 3.2, $R/Q/J(R/Q)$ is a SB-ring, and so is R/Q. Thus, $\overline{a} \pm \overline{bw} \in U(R/Q)$ for some $\overline{w} \in U(R/Q)$, a contradiction. Therefore R is a SB-ring. □

Recall that a ring R is an abelian regular ring provided that R is a regular ring with all idempotents central.

Corollary 5.2. *Let R be an abelian regular ring. Then the following are equivalent:*

(1) *R is a SB-ring.*
(2) *$R/M \not\cong \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$ for all maximal ideals M of R.*

Proof. (2) $\Rightarrow$ (1) Let M be a maximal ideal of R. By [11, Proposition 7.4], R/M is a division ring. If $|R/M| = 2$, then $R/M \cong \mathbb{Z}/2\mathbb{Z}$. If $|R/M| = 3$, then $R/M = \{\overline{0}, \overline{1}, \overline{x}\}$. Clearly, $\overline{1} + \overline{x} = \overline{0}$, and so $\overline{x} = -\overline{1}$. This means that $R/M \cong \mathbb{Z}/3\mathbb{Z}$. Therefore $\mid R/M \mid \geq 4$. According to Theorem 5.1, R is a SB-ring.

(1) $\Rightarrow$ (2) Clearly, R is an exchange ring with all idempotent central. Let M be a maximal ideal of R. In view of Theorem 5.1, $\mid R/M \mid \geq 4$. This implies that $R/M \not\cong \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$, as asserted. □

Now we extend Theorem 5.1 to semilocal rings as follows.

Theorem 5.3. *Let R be a semilocal ring. Then the following are equivalent:*

(1) *R is a SB-ring.*
(2) *$\mid R/M \mid \geq 4$ for all maximal ideal M of R.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) Since R is a semilocal ring, we have division rings $D_1, \dots, D_s$ such that $R/J(R) \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_s}(D_s)$. For each D_i, there exists a maximal ideal M_i of R such that $D_i \cong R/M_i$. By hypothesis, each $\mid D_i \mid \geq 4$; hence, $M_{n_i}(D_i)$ is a SB-ring from Lemma 3.2. Thus, $R/J(R)$ is a SB-ring, and then so is R. Therefore the proof is true. □

Example 5.4. Let $m \in \mathbb{N}$. If $2, 3 \nmid m$, then $\mathbb{Z}/m\mathbb{Z}$ is a SB-ring.

Proof. Since $m \in \mathbb{N}$, there exist prime $p_1, \dots, p_s \in \mathbb{N}$ such that $m = p_1^{r_1} \cdots p_s^{r_s}$. Thus

$$\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/(p_1^{r_1}) \oplus \cdots \oplus \mathbb{Z}/(p_s^{r_s}).$$

Clearly, each $\mathbb{Z}/(p_1^{r_1})$ is a local ring with the maximal ideal $(p_i)/(p_i^{r_i})$. It is easy to verify that

$$\mid \mathbb{Z}/(p_s^{r_s})/(p_i)/(p_i^{r_i}) \mid = \mid \mathbb{Z}/(p_i) \mid \geq 4.$$

In view of Theorem 5.3, $\mathbb{Z}/(p_s^{r_s})$ is a SB-ring. Therefore $\mathbb{Z}/m\mathbb{Z}$ is a SB-ring. □

References

[1] P. Ara, K. R. Goodearl, K. C. O'Meara, and E. Pardo, *Separative cancellation for projective modules over exchange rings*, Israel J. Math. **105** (1998), 105-137.
[2] R. Camps and W. Dicks, *On semilocal rings*, Israel J. Math. **81** (1993), no. 1-2, 203–211.
[3] H. Chen, *Exchange rings with Artinian primitive factors*, Algebr. Represent. Theory **2** (1999), no. 2, 201–207.
[4] ———, *Units, idempotents, and stable range conditions*, Comm. Algebra **29** (2001), no. 2, 703–717.
[5] R. Corisello and A. Facchini, *Homogeneous semilocal rings*, Comm. Algebra **29** (2001), no. 4, 1807–1819.
[6] J. W. Fisher and R. L. Snider, *Rings generated by their units*, J. Algebra **42** (1976), no. 2, 363–368.
[7] K. R. Goodearl and P. Menal, *Stable range one for rings with many units*, J. Pure Appl. Algebra **54** (1988), no. 2-3, 261–287.
[8] M. Henriksen, *Two classes of rings generated by their units*, J. Algebra **31** (1974), 182–193.
[9] C. Huh, N. K. Kim, and Y. Lee, *On exchange rings with primitive factor rings Artinian*, Comm. Algebra **28** (2000), no. 10, 4989–4993.
[10] T. Y. Lam, *A crash course on stable range, cancellation, substitution and exchange*, J. Algebra Appl. **3** (2004), no. 3, 301–343.
[11] A. Tuganbaev, *Rings Close to Regular*, Mathematics and its Applications, 545. Kluwer Academic Publishers, Dordrecht, 2002.
[12] H. P. Yu, *Stable range one for exchange rings*, J. Pure Appl. Algebra **98** (1995), no. 1, 105–109.
[13] ———, *On the structure of exchange rings*, Comm. Algebra **25** (1997), no. 2, 661–670.

Department of Mathematics
Hunan Normal University
Changsha, 410006, P. R. China
E-mail address: chyzxl@hunnu.edu.cn