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IFP RINGS AND NEAR-IFP RINGS

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Abstract. A ring R is called IFP, due to Bell, if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Huh et al. showed that the IFP condition need not be preserved by polynomial ring extensions. But it is shown that $\sum_{i=0}^{n} Ea_i E$ is a nonzero nilpotent ideal of E whenever R is an IFP ring and $0 \neq f \in F$ is nilpotent, where E is a polynomial ring over R, F is a polynomial ring over E, and a_i's are the coefficients of f. We shall use the term *near-IFP* to denote such a ring as having place near at the IFPness. In the present note the structures of IFP rings and near-IFP rings are observed, extending the classes of them. IFP rings are NI (i.e., nilpotent elements form an ideal). It is shown that the near-IFPness and the NIness are distinct each other, and the relations among them and related conditions are examined.

1. Near-IFP rings

Throughout every ring is associative with identity unless otherwise stated. X denotes a nonempty set of commuting indeterminates over rings. Let R be a ring. The polynomial ring over R with X is denoted by $R[X]$, and if X is a singleton, say $X = \{x\}$, then we write $R[x]$ in place of $R[\{x\}]$. Every polynomial in $R[X]$ is written by $a_0 + \sum_{j=1}^{n} a_j X^{I_j}$ with X^{I_j} a finite product of indeterminates over R, according to the notations in the proof of [10, Theorem 1.1]. The n by n matrix ring over a ring R is denoted by $\mathrm{Mat}_n(R)$, and E_{ij} denotes the n by n matrix with (i, j)-entry 1 and zero elsewhere. The n by n upper and lower triangular matrix rings over R are denoted by $\mathrm{UTM}_n(R)$ and $\mathrm{LTM}_n(R)$, respectively.

An element a of a ring is called *nilpotent* if $a^m = 0$ for some positive integer m. A subset S of a ring is called *nilpotent* if $S^n = 0$ for some positive integer n. A subset T of a ring is called *nil* if each element of T is nilpotent. Given a ring R, $N^*(R)$ and $N(R)$ denote the nilradical (i.e., the sum of all nil ideals) of R and the set of all nilpotent elements in R, respectively. Note $N^*(R) \subseteq N(R)$.

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$r_R(-)$ (resp. $\ell_R(-)$) is used for the right (resp. left) annihilator in a ring R. $a \in R$ is said to be right (resp. left) regular if $r_R(a) = 0$ (resp. $\ell_R(a) = 0$). $a \in R$ is called a left (resp. right) zero-divisor if $r_R(a) \neq 0$ (resp. $\ell_R(a) \neq 0$). A zero-divisor means an element that is neither right nor left regular. A domain means a ring whose nonzero elements are two-sided regular.

A ring R is called *reduced* if $N(R) = 0$. Marks [15] called a ring R NI when $N^*(R) = N(R)$ (equivalently, $N(R)$ forms an ideal in R). Reduced rings are clearly NI and it is obvious that a ring R is NI if and only if $R/N^*(R)$ is reduced. A prime ideal P of a ring R is called *completely prime* if R/P is a domain. Hong et al. showed that a ring R is NI if and only if every minimal strongly prime ideal of R is completely prime [8, Corollary 13].

A well-known property between "commutative" and "NI" is the *insertion-of-factors-property* (simply IFP) due to Bell [1]; a right (or left) ideal I of a ring R is said to have the IFP if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$. So a ring R is called IFP if the zero ideal of R has the IFP. Shin [17] used the term SI for the IFP; while IFP rings are also known as *semicommutative* in Narbonne's paper [16]. IFP rings are NI by [17, Theorem 1.5], and reduced rings are IFP by a simple computation. A ring is called *abelian* if each idempotent is central. IFP rings are abelian by a simple computation.

Huh et al. showed that the IFP condition need not be preserved by polynomial ring extensions [11, Example 2]. But IFP rings have the following useful facts.

Lemma 1.1. (1) *A ring R is* IFP *if and only if $r_R(S)$ is an ideal of R for any $S \subseteq R$ if and only if $\ell_R(S)$ is an ideal of R for any $S \subseteq R$.*

(2) IFP *rings are* NI.

(3) *If R is an* NI *ring and $a_0 + \sum_{j=1}^{n} a_j X^{I_j} \in N(R[X])$ then $\sum_{j=0}^{n} Ra_jR$ is nil.*

(4) *Let R be an* IFP *ring. Then $\sum_{j=0}^{n} Ra_jR$ is nilpotent whenever $a_0 + \sum_{j=1}^{n} a_j X^{I_j} \in R[X]$ is nilpotent.*

Proof. (1) and (2) are proved by [17, Lemma 1.2] and [17, Theorem 1.5], respectively.

(3) Let R be an NI ring and $a_0 + \sum_{j=1}^{n} a_j X^{I_j} \in N(R[X])$. Then $R/N^*(R)$ is reduced with $N^*(R) = N(R)$ by the definition, and so from

$$\frac{R[X]}{N^*(R)[X]} \cong \frac{R}{N^*(R)}[X]$$

we have $N(R[X]) \subseteq N^*(R)[X]$, entailing $a_j \in N^*(R)$ for all j. Then $\sum_{j=0}^{n} Ra_jR$ is nil since $N^*(R)$ is an ideal of R.

(4) Let R be an IFP ring and $a_0 + \sum_{j=1}^{n} a_j X^{I_j} \in N(R[X])$. Then by (2, 3) all a_j's are in $N(R)$. Say $a_j^{k_j} = 0$ for some positive integer k_j, then $(Ra_jR)^{k_j} = 0$

since R is IFP. Thus we obtain

$$(\sum_{j=0}^{n} Ra_jR)^k = 0 \text{ with } k = \sum_{j=0}^{n} k_j.$$

□

Here we consider the following condition that is weaker than the result in Lemma 1.1(4): (∗) $\sum_{i=0}^{n} Ra_iR$ contains a nonzero nilpotent ideal whenever a nonzero polynomial $\sum_{i=0}^{n} a_ix^i$ over a ring R is nilpotent. Then the condition (∗) is placed near at the IFPness by Lemma 1.1(4); hence we call a ring *near-IFP* if it satisfies the condition (∗). However the near-IFPness is distinct from the NIness as we see below. IFP rings are near-IFP by Lemma 1.1(4).

Proposition 1.2. *For a ring R the following conditions are equivalent:*

(1) *R is near-IFP;*
(2) *RaR contains a nonzero nilpotent ideal of R for any $0 \neq a \in N(R)$;*
(3) *$\sum_{j=0}^{n} Ra_jR$ contains a nonzero nilpotent ideal of R whenever $0 \neq a_0 + \sum_{j=1}^{n} a_jX^{I_j} \in R[X]$ is nilpotent.*

Proof. It suffices to obtain (3) from (2). Let $0 \neq f(X) = a_0 + \sum_{j=1}^{n} a_jX^{I_j} \in N(R[X])$ with $I_j < I_{j+1}$ for all $j \geq 1$. Without loss of generality, we can put $a_1 \neq 0$ when $a_0 = 0$. Then by the proof of [10, Theorem 1.1], we get $a_0 \in N(R)$ (when $a_0 \neq 0$) or $a_1 \in N(R)$ (when $a_0 = 0$). By the condition (2), there exists a nonzero nilpotent ideal of R contained in $\sum_{k=0}^{1} Ra_kR \subseteq \sum_{k=0}^{n} Ra_kR$, completing the proof. □

We will use Proposition 1.2 freely. In the following we confirm that there are no containing relations between the classes of near-IFP rings and NI rings, and that NI rings and near-IFP rings need not be IFP.

Example 1.3. (1) Let $R = UTM_2(S)$ over a reduced ring S. Note that $N^*(R) = \left(\begin{smallmatrix} 0 & S \\ 0 & 0 \end{smallmatrix}\right) = N(R)$ (hence R is NI) and $N^*(R)^2 = 0$. Since $N(R) \neq 0$ we can take $0 \neq a_0 \in N(R)$. But $(Ra_0R)^2 = 0$ and thus R is near-IFP. However R is not IFP since R is non-abelian.

(2) There is an NI ring but not near-IFP. Let T be a reduced ring, n be a positive integer and R_n be the 2^n by 2^n upper triangular matrix ring over T. Define a map $\sigma : R_n \to R_{n+1}$ by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$, then R_n can be considered as a subring of R_{n+1} via σ (i.e., $A = \sigma(A)$ for $A \in R_n$). Let R be the direct limit of the direct system (R_n, σ_{ij}), where $\sigma_{ij} = \sigma^{j-i}$. Then R is NI by [12, Proposition 1.1], and semiprime by [7, Corollary 1.3]. Note that $N(R)$ is an infinite subset of R, but RaR cannot contain any nonzero nilpotent ideal for each $0 \neq a \in N(R)$ since R is semiprime. Thus R is not near-IFP.

(3) There is a near-IFP ring but not NI. Let $S = \mathbb{Z}_4$, the ring of integers modulo 4, and $R = \mathrm{Mat}_n(S)$ for $n \geq 2$. Note $(\mathrm{Mat}_n(2\mathbb{Z}_4))^2 = 0$. Let $0 \neq A \in N(R)$. Since $\mathrm{Mat}_n(2\mathbb{Z}_4)$ is the only nonzero proper ideal of R, RAR is either

$\mathrm{Mat}_n(2\mathbb{Z}_4)$ (when $A \in \mathrm{Mat}_n(2\mathbb{Z}_4)$) or R (when $A \in R\backslash\mathrm{Mat}_n(2\mathbb{Z}_4)$). Thus RAR must contain $\mathrm{Mat}_n(2\mathbb{Z}_4)$ and so R is near-IFP. However R is not NI as can be seen by $E_{12} + E_{21} \notin N(R)$.

If given rings are semiprime then near-IFP rings are reduced as follows.

Proposition 1.4. *Let R be a semiprime ring. Then the following conditions are equivalent:*

(1) *R is near*-IFP;
(2) *R is* IFP;
(3) *R is reduced.*

Proof. It suffices to show (1)⇒(3). Let R be near-IFP and $a^2 = 0$ for $a \in R$. If $a \neq 0$ then RaR contains a nonzero nilpotent ideal I of R since R is near-IFP; but R is semiprime by hypothesis and so $I = 0$, a contradiction. Thus R is reduced. □

When R is a semiprime ring we may conjecture that a ring R is NI if and only if R is reduced, based on Proposition 1.4. However there is a semiprime NI ring but not reduced as we see in Example 1.3(2).

The *index of nilpotency* of a subset I of a ring is the supremum of the indices of nilpotency of all nilpotent elements in I. If such a supremum is finite, then I is said to be *of bounded index of nilpotency.*

Proposition 1.5. *Let R be a semiprime ring of bounded index of nilpotency. Then the following conditions are equivalent:*

(1) *R is near*-IFP;
(2) *R is* IFP;
(3) *R is reduced;*
(4) *R is* NI.

Proof. With the help of Proposition 1.4, it suffices to show (4)⇒(3) since reduced rings are clearly NI. Let R be NI and assume $N(R) \neq 0$. Take $0 \neq a \in N(R)$. Then RaR is a nonzero nil ideal of R. Since R is of bounded index of nilpotency, RaR contains a nonzero nilpotent ideal, say J, by Levitzki [6, Lemma 1.1] or Klein [14, Lemma 5]. But R is semiprime and so $J = 0$, a contradiction. Thus $N(R) = 0$. □

The condition "of bounded index of nilpotency" in Proposition 1.5 is not superfluous by Example 1.3(2) (this ring is semiprime but not of bounded index of nilpotency); while, the condition "semiprime" in Proposition 1.5 is also not superfluous by Example 1.3(3) (this ring is of bounded index of nilpotency but not semiprime).

A ring R is called *von Neumann regular* if for each $a \in R$ there exists $x \in R$ such that $a = axa$. A ring is called *right* (resp. *left) duo* if every right (resp. left) ideal is two-sided. Right or left duo rings are IFP by Lemma 1.1(1). Von Neumann regular rings need not be near-IFP in spite of being semiprime. $\mathrm{Mat}_n(R)$ is von Neumann regular by [4, Lemma 1.6] over a von Neumann regular ring R, but it is not near-IFP by Proposition 1.10(2) below when $n \geq 2$. In the following we see some conditions under which von Neumann regular rings can be near-IFP.

Proposition 1.6. *Let R be a von Neumann regular ring. Then the following conditions are equivalent:*

(1) *R is right (left) duo;*
(2) *R is reduced;*
(3) *R is abelian;*
(4) *R is* IFP;
(5) *R is near-*IFP;
(6) *R is* NI.

Proof. The equivalences of the conditions (1), (2), and (3) are proved by [4, Theorem 3.2]. The equivalences of the conditions (2), (4), and (5) are proved by Proposition 1.4 since von Neumann regular rings are semiprime. (2)⇒(6) is obvious.

(6)⇒(2): Let R be NI and assume $N(R) \neq 0$. Take $0 \neq a \in N(R)$. Since R is von Neumann regular, there exists $b \in R$ such that $a = aba$. Then we get $a = aba = ababa = abababa = \cdots$. But $N(R)$ is an ideal of R and so $ab \in N(R)$; hence $(ab)^n = 0$ for some positive integer n. This entails $0 \neq a = aba = \cdots = (ab)^n a = 0$, a contradiction. Thus $N(R) = 0$. □

A ring R is called *strongly regular* if for each $a \in R$ there exists $x \in R$ such that $a = a^2x$. A ring is strongly regular if and only if it is abelian and von Neumann regular [4, Theorem 3.5]. From Proposition 1.6 we obtain a similar result to [4, Theorem 3.5].

Corollary 1.7. *A ring is strongly regular if and only if it is near-*IFP *and von Neumann regular.*

A ring R is called *directly finite* if $ab = 1$ implies $ba = 1$ for $a, b \in R$. NI rings are directly finite by [12, Proposition 2.7(1)]. Abelian rings are also directly finite (hence so are IFP rings). For, if R is an abelian ring and $a, b \in R$ with $ab = 1$, then $baba = ba$ and so $ba = baab = abab = 1$. Thus from Proposition 1.6 one may conjecture that near-IFP rings are directly finite. However the answer is negative by the following.

Example 1.8. Let F be a field and $\mathbb{V}$ be a vector space over F with $\dim_F(\mathbb{V}) = \aleph_0$. Set $S = \mathrm{Hom}_F(\mathbb{V}, \mathbb{V})$. Take $a = \sum_{i=1}^{\infty} E_{i(i+1)}$ and $b = \sum_{j=1}^{\infty} E_{(j+1)j}$ in S. Then $ab = 1$ but $ba \neq 1$ in S. Next let $R = \mathrm{UTM}_n(S)$ for $n \geq 2$. Then R is near-IFP by Proposition 1.10(1) below, but $(aE)(bE) = E$ but $(bE)(aE) \neq E$ in R, where E is the identity matrix in R. Thus R is not directly finite.

A ring R is called *π-regular* if for each $a \in R$ there exist a positive integer n, depending on a, and $b \in R$ such that $a^n = a^n b a^n$. Von Neumann regular rings are obviously π-regular, and so one may ask if a π-regular ring is near-IFP when it is abelian and semiprime, based on Proposition 1.6. However the answer is negative by the following.

Example 1.9. Let S be a division ring. Consider the ring extension of S, that is a subring of $\mathrm{UTM}_{2^n}(S)$,

$$D_n = \{M \in \mathrm{UTM}_{2^n}(S) \mid \text{ the diagonal entries of } M \text{ are equal}\}.$$

Define a map $\sigma : D_n \to D_{n+1}$ by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$, then D_n can be considered as a subring of D_{n+1} via σ (i.e., $A = \sigma(A)$ for $A \in D_n$). Set R be the direct limit of the direct system (D_n, σ_{ij}) with $\sigma_{ij} = \sigma^{j-i}$. Then R is semiprime by [5, Theorem 2.2(2)]. Every D_n is abelian by [9, Lemma 2] such that every idempotent in D_n is of the form

$$\begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix}$$

with $f^2 = f \in S$. Thus R is also abelian. Every element of D_n is either invertible or nilpotent and therefore D_n is π-regular; hence R is also π-regular. However R is not near-IFP by Proposition 1.4.

In the following proposition we see some criteria by which we examine the near-IFPness of given rings.

Proposition 1.10. (1) *For any ring S,* $\mathrm{UTM}_n(S)$ *and* $\mathrm{LTM}_n(S)$ *are near-*IFP *for $n \geq 2$.*

(2) $\mathrm{Mat}_n(S)$ *over a semiprime ring S cannot be near-*IFP *for $n \geq 2$.*

(3) *Suppose that a ring S contains a nonzero nilpotent ideal I such that every element in $S \backslash I$ is invertible. Then* $\mathrm{Mat}_n(S)$ *is near-*IFP *for any n.*

Proof. (1) Let $R = \mathrm{UTM}_n(S)$ for $n \geq 2$ and $0 \neq A = (a_{ij}) \in R$ with $a_{st} \neq 0$. Then $RE_{1s}AE_{tn}R = Sa_{st}SE_{1n}$ is a nonzero nilpotent ideal of R that is contained in RAR. Thus R is near-IFP. The proof of the case $\mathrm{LTM}_n(S)$ is similar.

(2) Let $R = \mathrm{Mat}_n(S)$. Since $n \geq 2$ we have $N(R) \neq 0$. Thus R is not near-IFP by Proposition 1.4.

(3) Let $R = \mathrm{Mat}_n(S)$ and $0 \neq A = (a_{ij}) \in N(R)$. If every a_{ij} is in I then RAR is nilpotent because I is nilpotent. Otherwise we get $RAR = R$ since some a_{ij} is invertible by hypothesis, hence RAR contains the nonzero nilpotent ideal $R\mathrm{Mat}_n(I)R$. □

We can apply Proposition 1.10(3) to Example 1.3(3). $\mathbb{Z}_4$ contains a nonzero nilpotent ideal $2\mathbb{Z}_4$ such that $\mathbb{Z}_4 \backslash 2\mathbb{Z}_4$ is the subset of invertible elements in $\mathbb{Z}_4$. Thus $\mathrm{Mat}_n(\mathbb{Z}_4)$ is near-IFP by Proposition 1.10(3).

2. Structure and examples of (near-)IFP rings

Huh et al. showed that $R[x]$ need not be IFP when R is an IFP ring [11, Example 2]. But $R[X]$ can be near-IFP over an IFP ring R as in the following. The prime radical of a ring R is denoted by $N_*(R)$. Note $N_*(R) \subseteq N^*(R) \subseteq N(R)$ for any ring R.

Proposition 2.1. *Let R be an* IFP *ring. Then $\sum_{i=0}^{n} R[X]f_iR[X]$ is nilpotent whenever $\sum_{i=0}^{n} f_iy^i \in R[X][y]$ is nilpotent, where $f_i \in R[X]$ and y is an indeterminate over $R[X]$; especially $R[X]$ is near-*IFP.

Proof. Let R be an IFP ring. Suppose that $0 \neq g(y) = \sum_{i=0}^{n} f_iy^i \in N(R[X][y])$ with $f_i = a(i)_0 + \sum_{j=1}^{m_i} a(i)_jX(i)^{l_j} \in R[X]$ for $i = 0, 1, \dots, n$, where y is an indeterminate over $R[X]$. We can say $f_0 \neq 0$ after dividing $g(y)$ by powers of y if necessary. Note that there is a finite subset X_0 of X such that $f_i \in R[X_0]$ for all i, say $X_0 = \{x_1, \dots, x_v\}$. Then $g(y)$ is nilpotent in $R[x_1, \dots, x_v, y]$. Write $g(y) = g(x_1, \dots, x_v, y)$.

By [17, Theorem 1.5] we have $N_*(R) = N^*(R) = N(R)$ since R is IFP, and so by [2, Proposition 2.6] we have $N_*(R[X]) = N^*(R[X]) = N(R[X])$. It is well-konwn that $N_*(A[X]) = N_*(A)[X]$ for any ring A. Consequently we get $N(R)[X] = N_*(R[X]) = N^*(R[X]) = N(R[X])$. Applying this result we obtain that every coefficient f_i is nilpotent in $R[x_1, \dots, x_v]$, from $g(x_1, \dots, x_v, y) \in N(R[x_1, \dots, x_v, y])$. Then by Lemma 1.1(4), we obtain that

$$\sum_{i=0}^{n}\sum_{j=0}^{m_i} Ra(i)_jR \text{ is nilpotent.}$$

It then follows that

$$\sum_{i=0}^{n} R[X]f_iR[X] \subseteq \sum_{i=0}^{n} R[X](K(i)_0 + \sum_{j=1}^{m_i} K(i)_jX(i)^{l_j})R[X],$$

where $K(i)_j = Ra(i)_jR$. But $\sum_{i=0}^{n} R[X](K(i)_0 + \sum_{j=1}^{m_i} K(i)_jX(i)^{l_j})R[X]$ is nilpotent and therefore $\sum_{i=0}^{n} R[X]f_iR[X]$ is also nilpotent. It is immediate that $R[X]$ is near-IFP. □

$GF(p^n)$ means the Galois field of order p^n. In the following an infinite direct sum is considered as a ring without identity.

Proposition 2.2. (1) *Every minimal noncommutative near-*IFP *ring is isomorphic to* $\mathrm{UTM}_2(GF(2))$.

(2) *The class of near-*IFP *rings is closed under direct sums and direct products.*

Proof. (1) By [3, Proposition] a finite noncommutative ring R is isomorphic to $\mathrm{UTM}_2(GF(p))$ when the order of R is p^3, p a prime. Next by [3, Theorem] a finite ring R of order m is commutative when m has a cube free factorization. Thus every minimal noncommutative ring is isomorphic to $\mathrm{UTM}_2(GF(2))$. But

$\mathrm{UTM}_2(GF(2))$ is near-IFP by Proposition 1.10(1), and hence every minimal noncommutative near-IFP ring is isomorphic to $\mathrm{UTM}_2(GF(2))$.

(2) Let R_i be a near-IFP ring for $i \in I$ and $R = \oplus_{i\in I} R_i$. Take $0 \neq (a_i) \in N(R)$. There are $j = 1, \ldots, n$ such that $a_{i_j} \neq 0$. Since each R_i is near-IFP, $R_{i_j} a_{i_j} R_{i_j}$ contains a nonzero nilpotent ideal of R_{i_j}, say N_{i_j}. Put $N = \oplus_{i\in I} N_i$ such that $N_i = 0$ for $i \neq i_j$. Then clearly N is nilpotent and is contained in $R(a_i)R$.

Next let $R = \prod_{i\in I} R_i$ and $0 \neq (a_i) \in N(R)$. We can take a finite number of a_i's, say $a_{i_j} \neq 0$ for $j = 1, \ldots, n$. Then as above N is a nilpotent ideal of R contained in $R(a_i)R$. □

From Proposition 2.2(2) and the relation between direct sums and direct products, one may suspect that the class of near-IFP rings may be closed under subrings. However there exists a near-IFP ring whose subrings are non-near-IFP as follows.

Example 2.3. Let S be a semiprime ring and $T = \mathrm{Mat}_n(S)$ for $n \geq 2$. Then T is not near-IFP by Proposition 1.10(2). Next let $R = \mathrm{UTM}_n(T)$ for $n \geq 2$. Then R is near-IFP by Proposition 1.10(1) but the subring T of R is not near-IFP.

Proposition 2.4. *For a ring R suppose that R/I is a near-*IFP *ring for some ideal I of R. If I is nilpotent then R is near-*IFP.

Proof. Let $0 \neq a \in N(R)$. If $a \in I$ then RaR is nilpotent and so we assume $a \notin I$. Write $\bar{R} = R/I$ and $\bar{r} = r + I$ for $r \in R$. Since $\bar{R}$ is near-IFP, $\bar{R}\bar{a}\bar{R}$ contains a nonzero nilpotent ideal of $\bar{R}$, say J/I with $J^k \subseteq I$ for some positive integer k. But since I is nilpotent, say $I^n = 0$ for some positive integer n, we get $J^{kn} = 0$. Take $0 \neq b \in J$. Then there exists $0 \neq c \in RaR$ with $c - b \in I$, and hence $0 \neq c \in J$. Since J is nilpotent, RcR is a nonzero nilpotent ideal of R contained in RaR. Thus R is near-IFP. □

Instead of the condition "I is nilpotent" in Proposition 2.4 we may consider a weaker one "I is nil". However this one cannot guarantee the near-IFPness of R as we see in the following.

Example 2.5. Consider the ring R in Example 1.3(2). Let

$$I = \{M \in R \mid \text{ each diagonal entry of } M \text{ is zero}\}.$$

Then I is a nil ideal of R such that R/I is reduced (hence near-IFP). But R is not near-IFP.

Given rings A and B, suppose that ${}_AU_B$, ${}_BV_A$ are bimodules and

$$\theta : U \otimes_B V \to A, \quad \psi : V \otimes_A U \to B$$

are bimodule homomorphisms (called *pairings*). The array $T = \left(\begin{smallmatrix} A & U \\ V & B \end{smallmatrix}\right)$ can be given the formal operations of 2 by 2 matrices, using θ and ψ in defining multiplication. If θ, ψ satisfy the associativity conditions required to make T

a ring then the collection $(A, B, U, V, \theta, \psi)$ is called a *Morita context*, and T is called the *ring of the Morita context*. If θ, ψ are zero then they are called *zero pairings*.

Proposition 2.6. *Suppose that T is the ring of a Morita context*

$$(A, B, U, V, \theta, \psi)$$

with zero pairings.

(1) *If A and B are near-IFP then so is T.*

(2) *T is NI if and only if so are A and B.*

Proof. (1) Let $I = \left(\begin{smallmatrix} 0 & U \\ V & 0 \end{smallmatrix}\right)$. Then I is a nilpotent ideal of T by the zero pairings. Since $\frac{R}{I} \cong \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \cong A \oplus B$, T is near-IFP by Propositions 2.2(2) and 2.4.

(2) Since $\left(\begin{smallmatrix} 0 & U \\ V & 0 \end{smallmatrix}\right)$ is a nilpotent ideal of T, we have $N(T) = \left(\begin{smallmatrix} N(A) & U \\ V & N(B) \end{smallmatrix}\right)$ because θ, ψ are zero pairings by the hypothesis. Note that $N(T)$ is an ideal of T if and only if $N(A)$ and $N(B)$ are ideals of A and B, respectively. The proof is then complete. □

In Proposition 2.6(2) we have the isomorphisms

$$\frac{T}{N(T)} \cong \begin{pmatrix} \frac{A}{N(A)} & 0 \\ 0 & \frac{B}{N(B)} \end{pmatrix} \cong \frac{A}{N(A)} \oplus \frac{B}{N(B)} \subseteq N(T).$$

Proposition 2.6(1) can be applied to prove Proposition 1.10(1) when $n = 2$, letting $U = S$ and $V = 0$. But the converse of Proposition 2.6(1) need not hold by the following.

Example 2.7. Let $A = B = \mathrm{Mat}_2(\mathbb{Z}_6)$, where $\mathbb{Z}_6$ is the ring of integers modulo 6, and $U = \mathrm{Mat}_2(2\mathbb{Z}_6)$, $V = \mathrm{Mat}_2(3\mathbb{Z}_6)$. Define $\theta(U \otimes_B V) = UV$ and $\psi(V \otimes_A U) = VU$, then they are zero pairings. Let T be the ring of a Morita context $(A, B, U, V, \theta, \psi)$. Take $0 \neq a = \left(\begin{smallmatrix} x & \alpha \\ \beta & y \end{smallmatrix}\right) \in N(T)$. Then $x, y \in N(A)$. If $x = 0 = y$ then $\alpha \neq 0$ or $\beta \neq 0$ and so TaT itself is a nonzero nilpotent ideal of T. Assume $x \neq 0$. If $x \in A \backslash V$, then TaT contains a nonzero nilpotent ideal

$$Ta \begin{pmatrix} 0 & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ 0 & 0 \end{pmatrix} T = \begin{pmatrix} 0 & A(2x)B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}.$$

If $x \in A \backslash U$, then TaT contains a nonzero nilpotent ideal

$$T \begin{pmatrix} 0 & 0 \\ \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} & 0 \end{pmatrix} aT = \begin{pmatrix} 0 & 0 \\ B(3x)A & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix}.$$

The computation for the case of $y \neq 0$ is similar. However A, B are both not near-IFP by Proposition 1.10(2) since $\mathbb{Z}_6$ is semiprime.

Over a reduced ring R the subring of $\mathrm{Mat}_3(R)$

$$\left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

is an IFP ring by [13, Proposition 1.2]. In the following we see an IFP ring of similar structure.

Proposition 2.8. *Let R be a ring and suppose that I, J are ideals of R satisfying $IJ = 0 = JI$. If R is reduced then*

$$\left\{ \begin{pmatrix} a & b & c \\ e & a & d \\ f & g & a \end{pmatrix} \mid a \in R \text{ and } b, c, d \in I, e, f, g \in J \right\}$$

is an IFP *ring.*

Proof. Put

$$S = \left\{ \begin{pmatrix} a & b & c \\ e & a & d \\ f & g & a \end{pmatrix} \mid a \in R \text{ and } b, c, d \in I, e, f, g \in J \right\}.$$

The addition and multiplication of

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ e_1 & a_1 & d_1 \\ f_1 & g_1 & a_1 \end{pmatrix}, \ B = \begin{pmatrix} a_2 & b_2 & c_2 \\ e_2 & a_2 & d_2 \\ f_2 & g_2 & a_2 \end{pmatrix}$$

in S can be rewritten by

$$\begin{aligned} &(a_1; b_1, c_1, d_1; e_1, f_1, g_1) + (a_2; b_2, c_2, d_2; e_2, f_2, g_2) \\ &= (a_1 + a_2; b_1 + b_2, c_1 + c_2, d_1 + d_2; e_1 + e_2, f_1 + f_2, g_1 + g_2) \end{aligned}$$

and

$$\begin{aligned} &(a_1; b_1, c_1, d_1; e_1, f_1, g_1)(a_2; b_2, c_2, d_2; e_2, f_2, g_2) \\ &= (a_1a_2; a_1b_2 + b_1a_2, a_1c_2 + b_1d_2 + c_1a_2, a_1d_2 + d_1a_2; \\ &\quad e_1a_2 + a_1e_2, f_1a_2 + g_1e_2 + a_1f_2, g_1a_2 + a_1g_2) \end{aligned}$$

respectively. Now let $AB = 0$, then

$$(a_1; b_1, c_1, d_1; e_1, f_1, g_1)(a_2; b_2, c_2, d_2; e_2, f_2, g_2) = 0$$

and hence we have the following system of equations:

$$\begin{aligned} a_1a_2 &= 0 \\ a_1b_2 + b_1a_2 = 0, a_1c_2 + b_1d_2 + c_1a_2 = 0, a_1d_2 + d_1a_2 &= 0 \\ e_1a_2 + a_1e_2 = 0, f_1a_2 + g_1e_2 + a_1f_2 = 0, g_1a_2 + a_1g_2 &= 0. \end{aligned}$$

Reduced rings are IFP. So $a_1Ra_2 = 0$ and by the proof of [13, Proposition 1.2], we have

$$a_1Rb_2 = b_1Ra_2 = a_1Rc_2 = b_1Rd_2 = c_1Ra_2 = a_1Rd_2 = d_1Ra_2 = 0.$$

In a similar way to the case of the upper triangular part, we also get

$$e_1Ra_2 = a_1Re_2 = f_1Ra_2 = g_1Re_2 = a_1Rf_2 = g_1Ra_2 = a_1Rg_2 = 0.$$

It is an immediate consequence that for any $(r; s, t, u; x, y, z) \in S$ we have $(a_1; b_1, c_1, d_1; e_1, f_1, g_1)(r; s, t, u; x, y, z)(a_2; b_2, c_2, d_2; e_2, f_2, g_2) = (a_1ra_2, a_1rb_2 + a_1sa_2 + b_1ra_2, a_1rc_2 + a_1sd_2 + b_1rd_2 + a_1ta_2 + b_1ua_2 + c_1ra_2, a_1rd_2 + a_1ua_2 + d_1ra_2) = 0$.

Thus we get

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ e_1 & a_1 & d_1 \\ f_1 & g_1 & a_1 \end{pmatrix} \begin{pmatrix} r & s & t \\ x & r & u \\ y & z & r \end{pmatrix} \begin{pmatrix} a_2 & b_2 & c_2 \\ e_2 & a_2 & d_2 \\ f_2 & g_2 & a_2 \end{pmatrix} = 0$$

for any

$$\begin{pmatrix} r & s & t \\ x & r & u \\ y & z & r \end{pmatrix} \in S;$$

hence S is IFP. □

Let $R = D_1 \oplus D_2$ for domains D_i and $I = D_1 \oplus 0$, $J = 0 \oplus D_2$. Then $IJ = 0 = JI$ and so by Proposition 2.8 we get an IFP ring

$$\left\{ \begin{pmatrix} a & b & c \\ e & a & d \\ f & g & a \end{pmatrix} \mid a \in R \text{ and } b, c, d \in I, e, f, g \in J \right\}.$$

The converse of Proposition 2.8 need not hold as we see below. For that we define a kind of subring of $\mathrm{UTM}_n(S)$

$$D_n(S) = \{M \in \mathrm{UTM}_n(S) \mid \text{ the diagonal entries of } M \text{ are equal}\},$$

where S is a given ring.

Example 2.9. Let S be a commutative domain and

$$R = D_2(S) \oplus S \oplus S.$$

Take the ideals $I = 0 \oplus S \oplus 0$, $J = 0 \oplus 0 \oplus S$ of R, then clearly $IJ = 0 = JI$. Let

$$S = \left\{ \begin{pmatrix} a & b & c \\ e & a & d \\ f & g & a \end{pmatrix} \mid a \in R \text{ and } b, c, d \in I, e, f, g \in J \right\}$$

and $AB = 0$, where

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ e_1 & a_1 & d_1 \\ f_1 & g_1 & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} a_2 & b_2 & c_2 \\ e_2 & a_2 & d_2 \\ f_2 & g_2 & a_2 \end{pmatrix}$$

are in S. Since R is commutative, R is IFP and hence we have $a_1Ra_2 = 0$. Notice that remaining computations are actually done in a reduced $0 \oplus S \oplus S$, from the structure of I and J. Thus we can obtain the same result as in the proof of Proposition 2.8, i.e., $ASB = 0$, concluding that S is IFP. But R is not reduced.

But, in Proposition 2.8, if $(I = R,\ J = 0)$ or $(I = 0,\ J = R)$ then R is reduced by [5, Proposition 2.8] when S is IFP.

For a reduced ring R, $D_3(R)$ is an IFP ring by [13, Proposition 1.2], but $D_n(R)$ $(n \geq 4)$ is not IFP by [13, Example 1.3]. For the near-IFP case we have a different situation as in the following.

Proposition 2.10. *$D_n(S)$ is a near-IFP ring for any ring S when $n \geq 2$.*

Proof. Let $R = D_n(S)$ for $n \geq 2$ and $0 \neq A = (a_{ij}) \in R$ with $a_{st} \neq 0$. When the diagonal of A is nonzero we have the nonzero nilpotent ideal $RAE_{1n}R = Sa_{11}SE_{1n}$ $(\subseteq RAR)$ of R. So we assume that the diagonal of A is zero. Then RAR itself is a nonzero nilpotent ideal of R. Thus R is near-IFP. □

A ring R is called *right Ore* if it has a classical right quotient ring. It is well-known that semiprime right Goldie rings and right Noetherian domains are both right Ore. But not every domain has a classical right quotient ring (e.g., the free algebra in two indeterminates over a field). We denote the set of all regular elements in a ring R by $C(0)$.

Proposition 2.11. *Let R be a right Ore ring and Q be the classical right quotient ring of R.*

(1) *Let R be an IFP ring. If $a_1b_1^{-1}\cdots a_nb_n^{-1} = 0$ for $a_ib_i^{-1} \in Q$ then $a_1\cdots a_n = 0$.*

(2) *Let R be an IFP ring. If $0 \neq ab^{-1} \in N(Q)$ then $Qab^{-1}Q$ contains a nonzero nilpotent ideal of R.*

Proof. (1) Put $a_1b_1^{-1}\cdots a_ib_i^{-1} = 0$ for $a_ib_i^{-1} \in Q$. Since R is right Ore, we have the following computation: There exist $c_1 \in R$, $d_1 \in C(0)$ with $a_2d_1 = b_1c_1$; there exist $c_2 \in R$, $d_2 \in C(0)$ with $a_3d_2 = (b_2d_1)c_2$; inductively there exist $c_{n-1} \in R$, $d_{n-1} \in C(0)$ with $a_nd_{n-1} = (b_{n-1}d_{n-2})c_{n-1}$; consequently we have

$$0 = a_1b_1^{-1}\cdots a_ib_i^{-1} = a_1c_1\cdots c_{n-1}d_{n-1}^{-1}b_n^{-1}$$

and so $a_1c_1\cdots c_{n-1} = 0$. But

$$\begin{aligned} 0 = a_1c_1\cdots c_{n-1} &= a_1b_1c_1\cdots c_{n-1} = a_1a_2d_1\cdots c_{n-1} \\ &= a_1a_2(b_2d_1)c_2\cdots c_{n-1} = a_1a_2a_3d_2\cdots c_{n-1} = \cdots \\ &= a_1a_2\cdots a_{n-1}(b_{n-1}d_{n-2})c_{n-1} = a_1a_2\cdots a_{n-1}a_nd_{n-1}. \end{aligned}$$

Since d_{n-1} is regular, we have $a_1a_2\cdots a_{n-1}a_n = 0$.

(2) Let $0 \neq ab^{-1} \in N(Q)$, say $(ab^{-1})^n = 0$. Then $a^n = 0$ by (1). Since R is IFP, RaR is nilpotent by Lemma 1.1(4) such that

$$RaR \subseteq QaQ = Qab^{-1}bQ \subseteq Qab^{-1}Q.$$

□

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