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GRÖBNER-SHIRSHOV BASES FOR IRREDUCIBLE $\mathfrak{sp}_4$-MODULES

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ABSTRACT. We give an explicit construction of Gröbner–Shirshov pairs and monomial bases for finite-dimensional irreducible representations of the simple Lie algebra $\mathfrak{sp}_4$. We also identify the monomial basis consisting of the reduced monomials with a set of semistandard tableaux of a given shape, on which we give a colored oriented graph structure.

1. Introduction

Since Buchberger's algorithm [5] for computing Gröbner bases was introduced and Shirshov's Composition Lemma [15] was proved, the *Gröbner–Shirshov basis theory* has been a useful tool for understanding the structure of associative algebras and their representations.

For finite-dimensional simple Lie algebras, the Gröbner–Shirshov bases were completely determined by Bokut and Klein [2, 3, 4]. For Lie superalgebras and their universal enveloping algebras, Bokut, *et al.* [1] developed the theory and gave an explicit construction of Gröbner–Shirshov bases.

In [12], Gröbner–Shirshov basis theory for representations of associative algebras was developed by introducing the notion of the *Gröbner–Shirshov pair*, which yields an explicit monomial basis of a finite-dimensional irreducible module. The monomial basis is useful, for example, in that we can easily compute the weight of each element in the basis. Moreover, we can give a colored oriented graph structure on the monomial basis, which is called the *Gröbner–Shirshov graph*.

Gröbner–Shirshov basis theory for representations of associative algebras is so general that it can be applied to the representation theory of various interesting algebras such as finite-dimensional simple Lie (super)algebras, Kac-Moody (super)algebras, (affine) Hecke algebras, and so on. The works on the

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representation theory of Hecke algebras (of A-type) and Ariki–Koike algebras were given in [10] and [11], respectively.

Gröbner–Shirshov pairs for finite-dimensional irreducible modules over A-type simple Lie algebras were determined in [13]. But there have been no results for modules over finite-dimensional simple Lie algebras of the other types. In this paper, the case of type C_2 has been constructed. First, we review the Gröbner–Shirshov basis theory for representations in the next section, and then give an explicit construction of Gröbner–Shirshov pairs and monomial bases for finite-dimensional irreducible $\mathfrak{sp}_4$-modules. We also show that the monomial basis can be realized as a set of semistandard tableaux of a given shape and its colored oriented graph. For illustrations, we give several examples of monomial bases for some representations.

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2. Gröbner–Shirshov pair

We give a brief summary of the Gröbner–Shirshov basis theory for representations developed by Kang and Lee, which is a fundamental tool for the main result in this paper. For details of the theory, one may refer to [9, 12, 13].

Let X be a set and let X^* be the free monoid of associative words on X. We denote the empty word by 1 and the *length* (or *degree*) of a word u by $l(u)$. A well-ordering $<$ on X^* is called a *monomial order* if $x < y$ implies $axb < ayb$ for all $a, b \in X^*$.

Fix a monomial order $<$ on X^* and let $\mathcal{A}_X$ be the free associative algebra generated by X over a field $\mathbb{F}$. Given a nonzero element $p \in \mathcal{A}_X$, we denote by $\overline{p}$ the maximal monomial (called the *leading monomial*) appearing in p under the ordering $<$. Thus $p = \alpha\overline{p} + \sum \beta_i w_i$ with $\alpha, \beta_i \in \mathbb{F}$, $w_i \in X^*$, $\alpha \neq 0$ and $w_i < \overline{p}$. If $\alpha = 1$, p is said to be *monic*.

Let (S, T) be a pair of subsets of monic elements in $\mathcal{A}_X$, let J be the two-sided ideal of $\mathcal{A}_X$ generated by S, and let I be the left ideal of the algebra $A = \mathcal{A}_X/J$ generated by (the image of) T. Then we say that the algebra $A = \mathcal{A}_X/J$ is *defined by* S and that the left A-module $M = A/I$ is *defined by the pair* (S, T). The images of $p \in \mathcal{A}_X$ in A and in M under the canonical quotient map will also be denoted by p.

Definition 2.1. Given a pair (S, T) of subsets of monic elements in $\mathcal{A}_X$, a monomial $u \in X^*$ is said to be *(S,T)-standard* (or *(S,T)-reduced*) if $u \neq a\overline{s}b$ and $u \neq c\overline{t}$ for any $s \in S$, $t \in T$ and $a, b, c \in X^*$. Otherwise, the monomial u is said to be *(S,T)-reducible*.

Lemma 2.2 ([12, 13]). *Every $p \in \mathcal{A}_X$ can be expressed as*

$$p = \sum \alpha_i a_i s_i b_i + \sum \beta_j c_j t_j + \sum \gamma_k u_k, \tag{2.1}$$

where $\alpha_i, \beta_j, \gamma_k \in \mathbb{F}$, $a_i, b_i, c_j, u_k \in X^*$, $s_i \in S$, $t_j \in T$, $a_i \overline{s_i} b_i \leq \overline{p}$, $c_j \overline{t_j} \leq \overline{p}$, $u_k \leq \overline{p}$ and u_k are (S,T)-standard.

Remark. The term $\sum \gamma_k u_k$ in the expression (2.1) is called a *normal form* (or a *remainder*) of p with respect to the pair (S,T) (and with respect to the monomial order $<$). In general, a normal form is not unique.

As an immediate corollary of Lemma 2.2, we obtain:

Proposition 2.3 ([12, 13]). *The set of (S,T)-standard monomials spans the left A-module $M = A/I$ defined by the pair (S,T).*

Let p and q be monic elements in $\mathcal{A}_X$ with leading monomials $\overline{p}$ and $\overline{q}$. We define the *composition* of p and q as follows.

Definition 2.4. (a) If there exist a and b in X^* such that $\overline{p}a = b\overline{q} = w$ with $l(\overline{p}) > l(b)$, then the *composition of intersection* is defined to be $(p,q)_w = pa - bq$. Furthermore, if $a = 1$, the composition $(p,q)_w$ is called *right-justified.*

(b) If there exist a and b in X^* such that $a \neq 1$, $a\overline{p}b = \overline{q} = w$, then the *composition of inclusion* is defined to be $(p,q)_{a,b} = apb - q$.

Remark. The composition of inclusion has an ambiguity if we denote it by $(p,q)_w$ where $w = a\overline{p}b = \overline{q}$. For example, if $p = x_2 + x_3$ and $q = x_1 x_2^2 x_3$, then $(p,q)_w$ may be $x_1 p x_2 x_3 - q$ or $x_1 x_2 p x_3 - q$. So we should specify the monomials a and b.

Let $p, q \in \mathcal{A}_X$ and $w \in X^*$. We define the *congruence relation* on $\mathcal{A}_X$ as follows: $p \equiv q \mod (S,T;w)$ if and only if $p - q = \sum \alpha_i a_i s_i b_i + \sum \beta_j c_j t_j$, where $\alpha_i, \beta_j \in \mathbb{F}$, $a_i, b_i, c_j \in X^*$, $s_i \in S$, $t_j \in T$, $a_i \overline{s_i} b_i < w$, and $c_j \overline{t_j} < w$. When $T = \emptyset$, we simply write $p \equiv q \mod (S;w)$.

Definition 2.5. A pair (S,T) of subsets of monic elements in $\mathcal{A}_X$ is said to be *closed under composition* if

(i) $(p,q)_w \equiv 0 \mod (S;w)$ and $(p,q)_{a,b} \equiv 0 \mod (S;w)$ for all $p, q \in S$, $a, b \in X^*$ whenever the compositions $(p,q)_w$ and $(p,q)_{a,b}$ are defined,

(ii) $(p,q)_w \equiv 0 \mod (S,T;w)$ for all $p, q \in T$, $w \in X^*$ whenever the right-justified composition $(p,q)_w$ is defined,

(iii) $(p,q)_w \equiv 0 \mod (S,T;w)$ and $(p,q)_{a,b} \equiv 0 \mod (S,T;w)$ for all $p \in S$, $q \in T$, $a, b \in X^*$ whenever the compositions $(p,q)_w$ and $(p,q)_{a,b}$ are defined.

Theorem 2.6 ([12]). *Let (S,T) be a pair of subsets of monic elements in the free associative algebra $\mathcal{A}_X$ generated by X, let $A = \mathcal{A}_X/J$ be the associative algebra defined by S, and let $M = A/I$ be the left A-module defined by (S,T). If (S,T) is closed under composition and the image of $p \in \mathcal{A}_X$ is trivial in M, then the word $\overline{p}$ is (S,T)-reducible.*

As a corollary, we obtain:

Proposition 2.7 ([13]). *Let (S,T) be a pair of subsets of monic elements in $\mathcal{A}_X$. Then the following conditions are equivalent:*

(a) *(S,T) is closed under composition.*
(b) *For each $p \in \mathcal{A}_X$, the normal form of p is unique.*
(c) *The set of (S,T)-standard monomials forms a linear basis of the left A-module $M = A/I$ defined by the pair (S,T).*

Definition 2.8. A pair (S,T) of subsets of monic elements in $\mathcal{A}_X$ is a *Gröbner–Shirshov pair* if (S,T) satisfies one of the equivalent conditions in Proposition 2.7. In this case, we say that (S,T) is a *Gröbner–Shirshov pair* for the module M defined by (S,T). If a pair $(S,\emptyset)$ is a Gröbner–Shirshov pair, then we say that S is a *Gröbner–Shirshov basis* for the algebra $A = \mathcal{A}_X/J$ defined by S.

3. Irreducible modules over $\mathfrak{sp}_4$

The base field will be the complex field $\mathbb{C}$, which is an algebraically closed field of characteristic 0. And our monomial order $<$ will be the degree-lexicographic order throughout this paper.

It is well known that there is a 1-1 correspondence between the set of finite-dimensional irreducible representations of a simple Lie algebra and the set of dominant integral weights. For each dominant integral highest weight λ, we can use *Weyl's character formula* to compute the dimension of the finite-dimensional irreducible module $V(\lambda)$. (cf. [7, Chapter VI])

We consider the simplest case, that is, we apply the part (c) of Proposition 2.7 to computing Gröbner–Shirshov pairs for the finite-dimensional irreducible representations of the rank 2 classical simple Lie algebras. The results of $\mathfrak{sl}_3$-modules have been calculated in [12].

Recall that the symplectic Lie algebra $\mathfrak{sp}_4$ is the Kac-Moody algebra associated with the Cartan matrix $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$. Hence the algebra U_- is the associative algebra defined by the set $S_- = \{[[f_1f_2]f_2], [f_1f_1f_1f_2]\}$ of the Serre relations in $\mathcal{A}_F$, where $F = \{f_1, f_2\}$ and $[xy] = xy - yx$. On the contrary to the case of A_2 [9, 12, 13], we set $f_2 < f_1$ to consider a *left* U_--module $V(\lambda)$. The notation $[z_1 \cdots z_r]$ will mean $[z_1[z_2 \cdots z_r]]$.

Theorem 3.1 ([2, 14]). *Let*

$$\mathcal{S} = \{\ [[f_1f_2]f_2],\ [f_1f_1f_1f_2],\ [[f_1f_1f_2][f_1f_2]]\ \}.$$

Then $\mathcal{S}$ is a Gröbner–Shirshov basis for the algebra U_-.

Corollary 3.2. *The algebra U_- has a monomial basis consisting of $\mathcal{S}$-standard monomials $f_2^a(f_1f_2)^b(f_1^2f_2)^cf_1^d$ in F^* $(a,b,c,d \geq 0)$.*

Lemma 3.3. *The following relations hold in U_-. In other words, they belong to the two-sided ideal generated by S_- in $\mathcal{A}_F$:*

(a) $f_1 f_2^k - k f_1^{k-1}[f_1 f_2] - f_2^k f_1 = 0 \quad (k \geq 1)$,
(b) $f_1 [f_1 f_2]^k - k [f_1 f_2]^{k-1} [f_1 f_1 f_2] - [f_1 f_2]^k f_1 = 0 \quad (k \geq 1)$.

Proof. Since $[f_1 f_2] f_2 = f_2 [f_1 f_2]$ and $[f_1 f_1 f_2][f_1 f_2] = [f_1 f_2][f_1 f_1 f_2]$, the relations follow by the same induction on k as shown in [12, Lemma 5.1]. □

Let $\lambda = m\Lambda_1 + n\Lambda_2$ be a dominant integral weight for $\mathfrak{sp}_4$, where Λ_i are the fundamental weights ($i = 1, 2$), and let $V(\lambda)$ be the irreducible highest weight module over $\mathfrak{sp}_4$ with highest weight λ. Then, as a left U_--module, $V(\lambda)$ is defined by the pair (S_-, T_λ), where $T_\lambda = \{f_1^{m+1}, f_2^{n+1}\}$. From now on, we will say that *a relation $R = 0$ holds in $V(\lambda)$* whenever R is contained in the left ideal of U_- generated by T_λ.

Lemma 3.4 ([12]). *The following relations hold in $V(\lambda)$:*

(a) $f_2^{n+d+1} f_1^d = 0 \quad (d \geq 0)$,
(b) $f_2^{n+d}[f_1 f_2] f_1^d + \dfrac{1}{n+d+1} f_2^{n+d+1} f_1^{d+1} = 0 \quad (d \geq 0)$.

Lemma 3.5. *The following relations hold in $V(\lambda)$:*

(b2) $f_2^{n+d-1}[f_1 f_2]^2 f_1^d + \frac{f_2^{n+d}}{n+d} \left([f_1 f_1 f_2] + 2[f_1 f_2] f_1\right) f_1^d + \frac{f_2^{n+d+1} f_1^{d+2}}{(n+d+1)(n+d)} = 0$
$(n + d \geq 1)$,

(b3) $f_2^{n+d-2}[f_1 f_2]^3 f_1^d + \frac{f_2^{n+d-1}}{n+d-1} \left(3[f_1 f_2][f_1 f_1 f_2] + 3[f_1 f_2]^2 f_1\right) f_1^d$
$+ \frac{f_2^{n+d}}{(n+d)(n+d-1)} \left(3[f_1 f_1 f_2] f_1 + 3[f_1 f_2] f_1^2\right) f_1^d + \frac{f_2^{n+d+1} f_1^{d+3}}{(n+d+1)(n+d)(n+d-1)} = 0$
$(n + d \geq 2)$,

(b4) $f_2^{n+d-3}[f_1 f_2]^4 f_1^d + \frac{f_2^{n+d-2}}{n+d-2} \left(6[f_1 f_2]^2 [f_1 f_1 f_2] + 4[f_1 f_2]^3 f_1\right) f_1^d$
$+ \frac{f_2^{n+d-1}}{(n+d-1)(n+d-2)} \left(3[f_1 f_1 f_2]^2 + 12[f_1 f_2][f_1 f_1 f_2] f_1 + 6[f_1 f_2]^2 f_1^2\right) f_1^d$
$+ \frac{f_2^{n+d}}{(n+d)(n+d-1)(n+d-2)} \left(6[f_1 f_1 f_2] f_1^2 + 4[f_1 f_2] f_1^3\right) f_1^d$
$+ \frac{f_2^{n+d+1} f_1^{d+4}}{(n+d+1)(n+d)(n+d-1)(n+d-2)} = 0 \qquad (n + d \geq 3)$.

Proof. Multiplying the relation (b) by f_1 from the left and using Lemma 3.3, we get the relation (b2). In the same way, (b3) and (b4) follow. Note that the coefficients appear in a symmetric manner. □

In general, the relations

$$
\begin{aligned}
& f_2^{n-b+d+1}[f_1 f_2]^b f_2^d \\
& + \frac{f_2^{n-b+d+2}}{n-b+d+2} \left(\frac{b(b-1)}{2} [f_1 f_2]^{b-2} [f_1 f_1 f_2] + b [f_1 f_2]_1^{b-1} \right) f_1^d \\
& + \frac{f_2^{n-b+d+3}}{(n-b+d+3)(n-b+d+2)} \left(\frac{b(b-1)(b-2)(b-3)}{2 \cdot 4} [f_1 f_2]^{b-4} [f_1 f_1 f_2]^2 \right.
\end{aligned}
$$

$$+ \frac{b(b-1)(b-2)}{2}[f_1f_2]^{b-3}[f_1f_1f_2]f_1 + \frac{b(b-1)}{2}[f_1f_2]^{b-2}f_1^2\Big)f_1^d$$
$$+ \frac{f_2^{n-b+d+4}}{(n-b+d+4)(n-b+d+3)(n-b+d+2)}$$
$$\times \Big(\frac{b\cdots(b-5)}{2\cdot 4\cdot 6}[f_1f_2]^{b-6}[f_1f_1f_2]^3$$
$$+ \frac{b(b-1)\cdots(b-4)}{2\cdot 4}[f_1f_2]^{b-5}[f_1f_1f_2]^2f_1 + \frac{b\cdots(b-3)}{4}[f_1f_2]^{b-4}[f_1f_1f_2]f_1^2$$
$$+ \frac{b(b-1)(b-2)}{2\cdot 3}[f_1f_2]^{b-3}f_1^3\Big)f_1^d$$
$$+\cdots$$
$$+ \frac{f_2^{n-b+d+r}}{(n-b+d+r)\cdots(n-b+d+2)}\Big(A_{r,1}[f_1f_2]^{b-2r+2}[f_1f_1f_2]^{r-1}+\cdots$$
$$+ A_{r,i}[f_1f_2]^{b-2r+i+1}[f_1f_1f_2]^{r-i}f_1^{i-1}+\cdots+A_{r,r}[f_1f_2]^{b-r+1}f_1^{r-1}\Big)f_1^d$$
$$+\cdots$$
$$+ \frac{f_2^{n+d-1}}{(n+d-1)!/(n-b+d+1)!}\Big(\frac{b(b-1)(b-2)(b-3)}{2\cdot 4}[f_1f_2]^2[f_1f_1f_2]^{b-4}$$
$$+ \frac{b(b-1)(b-2)}{2}[f_1f_2][f_1f_1f_2]f_1^{b-3} + \frac{b(b-1)}{2}[f_1f_2]^2f_1^{b-2}\Big)f_1^d$$
$$+ \frac{f_2^{n+d}}{(n+d)!/(n-b+d+1)!}\left(\frac{b(b-1)}{2}[f_1f_1f_2]f_1^{b-2} + b[f_1f_2]f_1^{b-1}\right)f_1^d$$
$$+ \frac{f_2^{n+d+1}f_1^{b+d}}{(n+d+1)!/(n-b+d+1)!} = 0$$

hold in $V(\lambda)$ for $d \geq 0$ and $0 \leq b \leq n+d+1$, where

$A_{r,1} = \frac{b(b-1)\cdots(b-2r+3)}{2^r r!}$, $A_{r,2} = \frac{b(b-1)\cdots(b-2r+4)}{2^{r-1}(r-1)!}, \ldots,$ $A_{r,r} = \frac{b(b-1)\cdots(b-r+2)}{(r-1)!}$,

and $A_{r,i}$ is given by

$$A_{r+1,i} = \frac{b-2r+i}{2r-i+1}\left(A_{r,i-1} + (b-2r+i+1)A_{r,i}\right)$$

for $2 \leq r \leq [\frac{b}{2}]+1$ and $2 \leq i \leq r$.

For short, we denote these relations by

$$f_2^{n-b+d+1}[f_1f_2]^bf_1^d + R_1(f_1,f_2,n;b,d) = 0$$

for $d \geq 0$ and $0 \leq b \leq n+d+1$. Multiplying by $[f_1f_1f_2]$ from the left and using the relation

$$[f_1f_1f_2]f_2 - f_2[f_1f_1f_2] = 0 \in U_-,$$

we get the following:

Lemma 3.6. *The relations*

$$f_2^{n-b+d+1}[f_1f_2]^b[f_1f_1f_2]^cf_1^d + R_2(f_1,f_2,n;b,c,d) = 0$$

hold in $V(\lambda)$ *for* $c, d \geq 0$ *and* $0 \leq b \leq n+d+1$, *where* $R_2(f_1, f_2, n; b, c, d)$ *is the polynomial with all monomials of the form* $f_2^\alpha [f_1 f_2]^\beta [f_1 f_1 f_2]^\gamma f_1^\delta$ $(\alpha, \beta, \gamma, \delta \geq 0)$, *induced from*

$$[f_1 f_1 f_2]^c R_1(f_1, f_2, n; b, d)$$

by the relations $[f_1 f_1 f_2] f_2 = f_2 [f_1 f_1 f_2]$ *and* $[f_1 f_1 f_2][f_1 f_2] = [f_1 f_2][f_1 f_1 f_2]$. *In particular, if* $c = 0, b = n + d + 1$, *then we have*

$$[f_1 f_2]^{n+d+1} f_1^d + R_2(f_1, f_2, n; n+d+1, 0, d) = 0 \quad (d \geq 0).$$

Lemma 3.7. *The relations*

$$\sum_{r=0}^{c} \frac{(n-c+d+1)!}{(n-c+d+1+r)!} \binom{c}{r} [f_1 f_2]^{n-c+d+1+r} [f_1 f_1 f_2]^{c-r} f_1^{d+r}$$
$$+ R_3(f_1, f_2, n; c, d) = 0$$

hold in $V(\lambda)$ *for* $d \geq 0$ *and* $0 \leq c \leq n + d + 1$, *where* $R_3(f_1, f_2, n; c, d)$ *is the polynomial with all monomials of the form* $f_2^\alpha [f_1 f_2]^\beta [f_1 f_1 f_2]^\gamma f_1^\delta$ $(\alpha, \beta, \gamma, \delta \geq 0)$, *induced from*

$$f_1^c R_2(f_1, f_2, n; n+d+1, 0, d).$$

Proof. Multiplying the relation

$$[f_1 f_2]^{n+d+1} f_1^d + R_2(f_1, f_2, n; n+d+1, 0, d) = 0$$

in Lemma 3.6 by f_1 from the left and using Lemma 3.3, we get the relation inductively. □

Lemma 3.8. *The relations*

$$[f_1 f_1 f_2]^{n+1} f_1^d + R_4(f_1, f_2, n; d) = 0$$

hold in $V(\lambda)$ *for* $d \geq 0$, *where* $R_4(f_1, f_2, n; d)$ *is the polynomial with all monomials of the form* $f_2^\alpha [f_1 f_2]^\beta [f_1 f_1 f_2]^\gamma f_1^\delta$ $(\alpha, \beta, \gamma, \delta \geq 0)$, *induced from*

$$f_1^d \left(\sum_{r=1}^{n+1} \binom{n+1}{r} \frac{1}{r!} [f_1 f_2]^r [f_1 f_1 f_2]^{n+1-r} f_1^r + R_3(f_1, f_2, n; n+1, 0) \right).$$

Proof. If $d = 0$, it is the same as the case of $c = n+1, d = 0$ in Lemma 3.7. Assume that the relation holds for some fixed d. Multiplying by f_1 from the left and using $f_1 [f_1 f_1 f_2] = [f_1 f_1 f_2] f_1$ and Lemma 3.3, we obtain the relation inductively. □

Using the previous lemmas, we can prove the main theorem of this paper.

Theorem 3.9. *The pair* $(S, \mathcal{T}_\lambda)$ *is a Gröbner–Shirshov pair for the finite-dimensional irreducible module* $V(\lambda)$ *over the simple Lie algebra* $\mathfrak{sp}_4$, *where* $\mathcal{T}_\lambda$ *consists of the following elements*:

(a) f_1^{m+1},

(b) $[f_1 f_1 f_2]^{n+1} f_1^d + R_4(f_1, f_2, n; d)$ $(0 \leq d \leq m)$,

(c) $\sum_{r=0}^{c} \frac{(n-c+d+1)!}{(n-c+d+1+r)!}\binom{c}{r}[f_1f_2]^{n-c+d+1+r}[f_1f_1f_2]^{c-r}f_1^{d+r}$
$+R_3(f_1,f_2,n;c,d) \quad (0 \le c \le n, 0 \le d \le m),$

(d) $f_2^{n-b+d+1}[f_1f_2]^b[f_1f_1f_2]^c f_1^d + R_2(f_1,f_2,n;b,c,d)$
$(0 \le d \le m, 0 \le c \le n, 0 \le b \le n-c+d),$

where the polynomials R_2, R_3, R_4 are as given in Lemma 3.6-Lemma 3.8.

Hence the set of monomials of the form

$$f_2^a(f_1f_2)^b(f_1^2f_2)^c f_1^d \qquad (\,0 \le d \le m,\ \ 0 \le c \le n,\ \ 0 \le b \le n-c+d,\ \ 0 \le a \le n-b+d\,) \tag{3.1}$$

forms a linear basis of $V(\lambda)$.

Proof. By Lemma 3.3-Lemma 3.8, we see that the above relations hold in $V(\lambda)$. Note that the set of $(\mathcal{S},\mathcal{T}_\lambda)$-standard monomials is given by:

$$f_2^a(f_1f_2)^b(f_1^2f_2)^c f_1^d$$

$$(\,0 \le d \le m,\ \ 0 \le c \le n, 0 \le b \le n-c+d,\ \ 0 \le a \le n-b+d\,).$$

Hence the number of $(\mathcal{S},\mathcal{T}_\lambda)$-standard monomials is

$$\sum_{d=0}^{m}\sum_{c=0}^{n}\sum_{b=0}^{n-c+d}(n-b+d+1) = \frac{1}{6}(m+1)(n+1)(m+n+2)(m+2n+3).$$

This is exactly the dimension of $V(\lambda)$. Hence by Proposition 2.7, the pair $(\mathcal{S},\mathcal{T}_\lambda)$ is a Gröbner–Shirshov pair for $V(\lambda)$. □

4. Gröbner–Shirshov graphs of irreducible $\mathfrak{sp}_4$-modules

Let $\lambda = m\Lambda_1 + n\Lambda_2$ be a dominant integral highest weight for $\mathfrak{sp}_4$. To give an explicit realization for the monomial bases of $V(\lambda)$, we let $Y^\lambda = \{(i) \mid 1 \le i \le m+n\}$ be the horizontal *frame* of $m+n$ boxes. We define a *semistandard tableau with $m+n$ boxes* to be a function τ of Y^λ into the set $\{1,2,\overline{2},\overline{1},3,4,0,\overline{4},\overline{3}\}$ $(1<2<\overline{2}<\overline{1},\ 3<4<0<\overline{4}<\overline{3})$ such that

$$\tau(i) \in \{1,2,\overline{2},\overline{1}\} \text{ for } 1 \le i \le m,\ \tau(j) \in \{3,4,0,\overline{4},\overline{3}\} \text{ for } m+1 \le j \le m+n,$$
$$\tau(i) \le \tau(i+1) \ \text{ for } i = 1,\dots,m-1,\ m+1,\dots,m+n-1.$$

As usual, we present a semistandard tableau by an array of weakly increasing colored boxes.

To begin with, if $\lambda = m\Lambda_1$, i.e., $n=0$, then the formula (3.1) says that the monomials including the word $(f_1^2f_2)$ do not occur in the set of $(\mathcal{S},\mathcal{T}_\lambda)$-standard monomials. In this case, the monomial basis can be in 1-1 correspondence with the set of *semistandard tableaux with m boxes.* The empty word is expressed as the semistandard tableau τ^λ defined by $\tau^\lambda(i) = 1$ for $1 \le i \le m$.

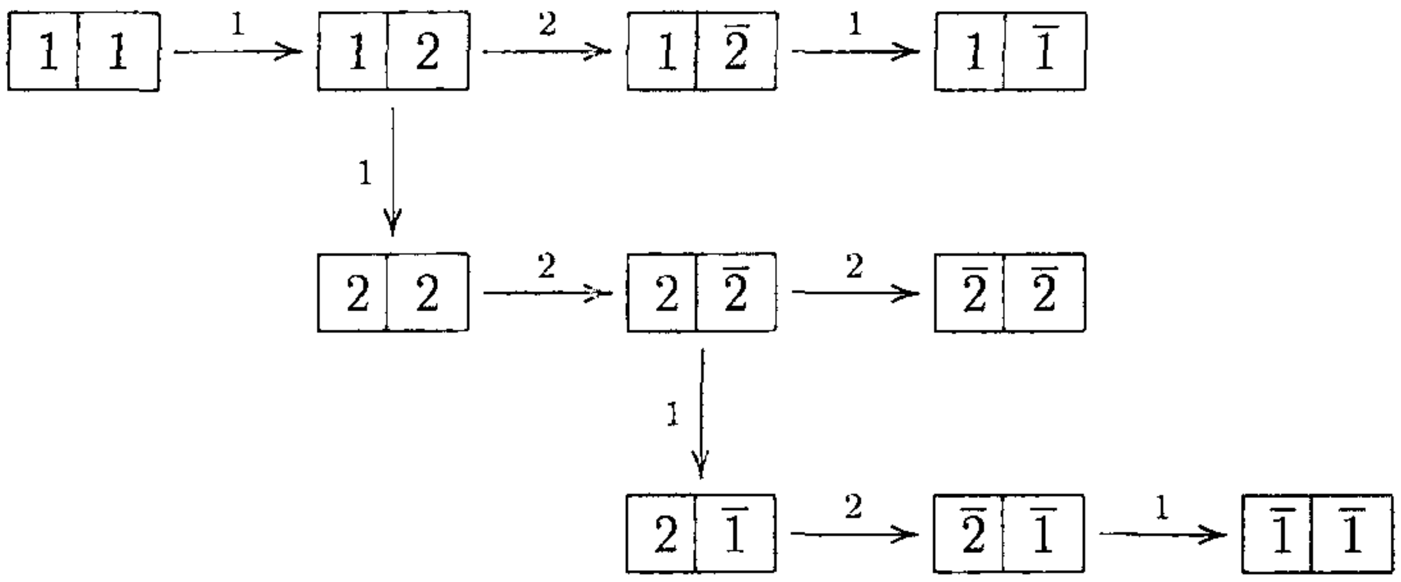

FIGURE 1. The Gröbner–Shirshov graph of $V(2\Lambda_1)$

Example 4.1. First, consider the simplest case $\lambda = \Lambda_1$, which is the vector representation of $\mathfrak{sp}_4$. The monomial basis of $V(\Lambda_1)$ is given by

$$\{1, f_1, f_2 f_1, (f_1 f_2) f_1\}.$$

Bearing in mind the *crystal graph* developed by Kashiwara (cf. [6, Chapter 8]), we realize this basis as follows:

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}\ , \tag{4.1}$$

where $\xrightarrow{i}$ means that the monomial is multiplied by f_i $(i = 1, 2)$ from the left.

We notice that the word f_2 changes the box $\boxed{2}$ to the box $\boxed{\bar{2}}$, and the word f_1 changes $\boxed{\bar{2}}$ to $\boxed{\bar{1}}$, or $\boxed{1}$ to $\boxed{2}$. In the horizontal multiple boxes $(m > 1)$, we require that f_1 acts on the rightmost. We set $1 < 2 < \bar{2} < \bar{1}$.

Next, we consider the case $\lambda = 2\Lambda_1$, which is the adjoint representation of $\mathfrak{sp}_4$. The monomial basis of $V(2\Lambda_1)$ is given by

$$\{1, f_1, f_1^2, f_2 f_1, f_2 f_1^2, (f_1 f_2) f_1, (f_1 f_2) f_1^2, f_2^2 f_1^2, f_2 (f_1 f_2) f_1^2, (f_1 f_2)^2 f_1^2\},$$

and the *Gröbner–Shirshov graph* of $V(2\Lambda_1)$ is given in Figure 1.

Remark. We note that the Gröbner–Shirshov graph has a *tree* structure since each of its tableaux receives only one arrow.

In general, to each $(S, \mathcal{T}_\lambda)$-standard monomial $f_2^a (f_1 f_2)^b f_1^d$ with $0 \le d \le m, 0 \le b \le c + d, 0 \le a \le b + d$, we associate the semistandard tableau τ as follows. Start with the tableau τ^λ and change its entries by the following rules:

(i) Let the word f_1 change the box $\boxed{1}$ on the rightmost to the box $\boxed{2}$, let f_2 change $\boxed{2}$ on the rightmost to $\boxed{\bar{2}}$ and let the word $(f_1 f_2)$ change $\boxed{2} \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$ with preserving boxes semistandard.

(ii) Let the words f_1, $(f_1 f_2)$ and f_2 in $f_2^a (f_1 f_2)^b f_1^d$ act successively on τ^λ changing the boxes in τ^λ.

Then, the monomial basis of $V(m\Lambda_1)$ consisting of $(\mathcal{S}, \mathcal{T}_\lambda)$-standard monomials can be realized as the Gröbner–Shirshov graph with the above rules (i) and (ii) on the set of semistandard tableaux with m boxes. Let $SST(m)$ be the set of all semistandard tableaux with m boxes whose entries are from the set $\{1, 2, \bar{2}, \bar{1}\}$. We know that the cardinality of $SST(m)$ is $\frac{1}{6}(m+1)(m+2)(m+3)$, which is the number of $(\mathcal{S}, \mathcal{T}_\lambda)$-standard monomials. So it is easy to verify that the monomial basis of $V(m\Lambda_1)$ is in 1-1 correspondence with $SST(m)$ by the rules (i) and (ii).

Now, we consider $V(n\Lambda_2)$. Since $m = 0$, the $(\mathcal{S}, \mathcal{T}_\lambda)$-standard monomials cannot have f_1 on the rightmost. This means that the Gröbner–Shirshov graph begins necessarily with f_2-arrow.

We know that the classical Lie algebra $\mathfrak{so}_{2n+1}$ is *dual* to $\mathfrak{sp}_{2n}$. So, bearing in mind the *crystal graph of type* B_2, we can realize the monomial basis of $V(n\Lambda_2)$ over $\mathfrak{sp}_4$ by the Gröbner–Shirshov graph of $V(n\Lambda_1)$ over $\mathfrak{so}_5$.

Example 4.2. In the simplest case $\lambda = \Lambda_2$ of $\mathfrak{sp}_4$, the monomial basis of $V(\Lambda_2)$ is given by

$$\{1, f_2, (f_1 f_2), (f_1^2 f_2), f_2(f_1^2 f_2)\},$$

and its Gröbner–Shirshov graph is simply as follows:

$$\boxed{3} \xrightarrow{\ 2\ } \boxed{4} \xrightarrow{\ 1\ } \boxed{0} \xrightarrow{\ 1\ } \boxed{\bar{4}} \xrightarrow{\ 2\ } \boxed{\bar{3}}\ , \tag{4.2}$$

where $\xrightarrow{i}$ means that the monomial is multiplied by f_i $(i = 1, 2)$ from the left.

We notice that the word f_2 changes the box $\boxed{3}$ to the box $\boxed{4}$, or $\boxed{\bar{4}}$ to $\boxed{\bar{3}}$, and the word f_1 changes $\boxed{4}$ to $\boxed{0}$, or $\boxed{0}$ to $\boxed{\bar{4}}$. We set $3 < 4 < 0 < \bar{4} < \bar{3}$. In the horizontal multiple boxes $(n > 1)$, we require that f_1 and f_2 act on the rightmost among the boxes with less entry.

For the second example, consider the case $\lambda = 2\Lambda_2$ of $\mathfrak{sp}_4$. The monomial basis of $V(2\Lambda_2)$ is given by

$$\begin{gathered}\{\ 1, f_2, (f_1 f_2), f_2^2, (f_1^2 f_2), f_2(f_1 f_2), f_2(f_1^2 f_2), (f_1 f_2)^2, (f_1 f_2)(f_1^2 f_2), \\ f_2^2(f_1^2 f_2), (f_1^2 f_2)^2, f_2(f_1 f_2)(f_1^2 f_2), f_2^2(f_1^2 f_2)^2, f_2^2(f_1^2 f_2)^2\ \},\end{gathered}$$

and the Gröbner–Shirshov graph of $V(2\Lambda_2)$ is given in Figure 2.

In general, to each $(\mathcal{S}, \mathcal{T}_\lambda)$-standard monomial $f_2^a(f_1 f_2)^b(f_1^2 f_2)^c$ with $0 \le c \le n, 0 \le b \le n - c + d, 0 \le a \le n - b + d$, we associate the semistandard tableau τ as follows. Start with the tableau τ^λ where each n box is filled with 3 and change its entries by the following rules:

(i) Let the word $(f_1^2 f_2)$ change $\boxed{3} \xrightarrow{\ 2\ } \boxed{4} \xrightarrow{\ 1\ } \boxed{0} \xrightarrow{\ 1\ } \boxed{\bar{4}}$, let $(f_1 f_2)$ change $\boxed{3} \xrightarrow{\ 2\ } \boxed{4} \xrightarrow{\ 1\ } \boxed{0}$, and let f_2 change $\boxed{3}$ to $\boxed{4}$, or $\boxed{\bar{4}}$ to $\boxed{\bar{3}}$ (the former is prior), with preserving boxes semistandard.

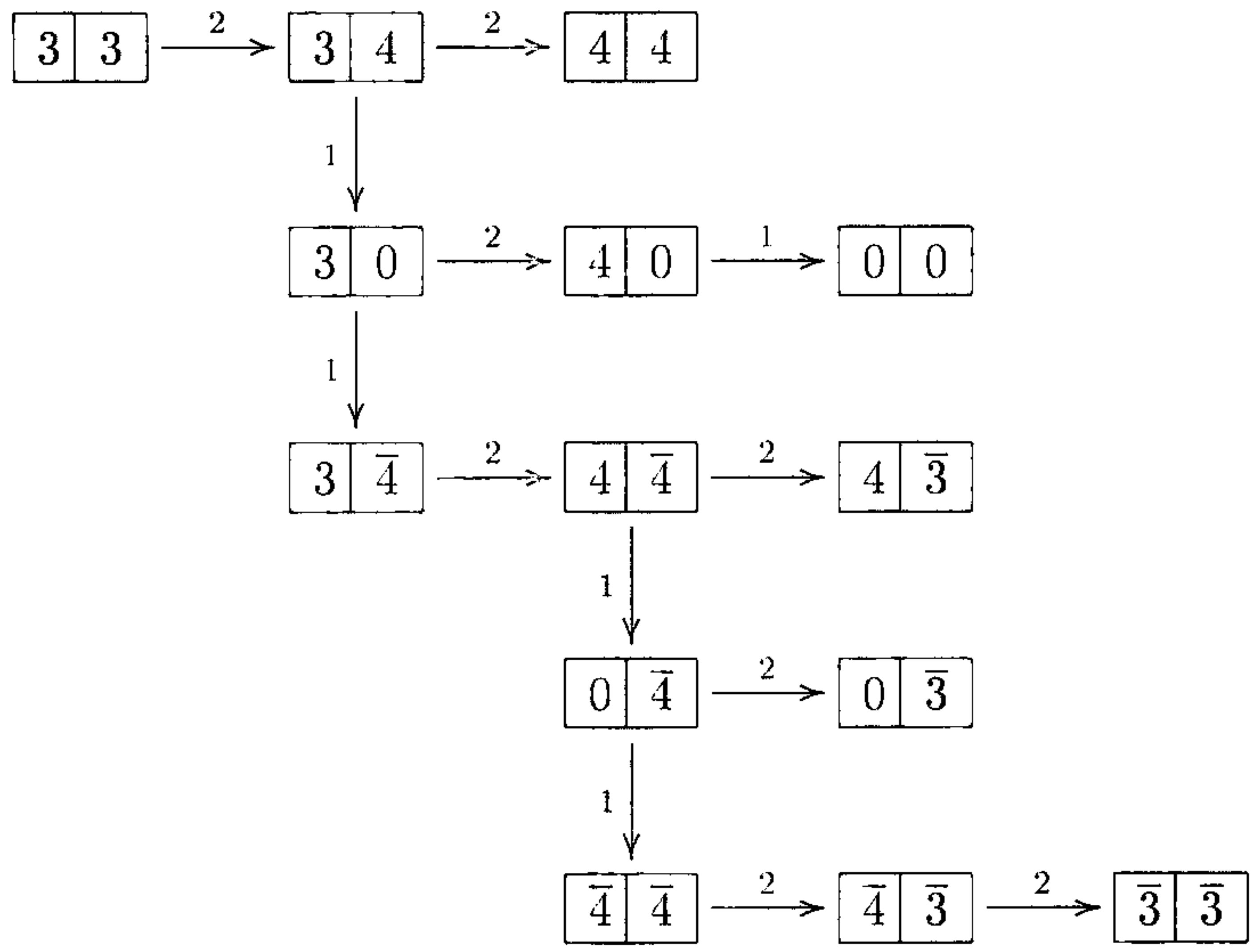

FIGURE 2. The Gröbner–Shirshov graph of $V(2\Lambda_2)$

(ii) Let the words $(f_1^2 f_2)$, $(f_1 f_2)$ and f_2 in $f_2^a(f_1 f_2)^b(f_1^2 f_2)^c$ act successively on τ^λ changing the boxes in τ^λ.

Then it can be verified that the monomial basis of $V(n\Lambda_2)$ consisting of $(\mathcal{S}, \mathcal{T}_\lambda)$-standard monomials can be realized as the Gröbner–Shirshov graph with the above rules (i) and (ii) on a set of semistandard tableaux with n boxes. Note that the whole semistandard tableaux do not appear in the graph if $n > 1$.

We combine the above two simple cases to realize the monomial bases of the general $\mathfrak{sp}_4$-modules $V(\lambda)$ with highest weight $\lambda = m\Lambda_1 + n\Lambda_2$. We would like to identify the monomial basis consisting of $(\mathcal{S}, \mathcal{T}_\lambda)$-standard monomials with a set of semistandard tableaux with $m+n$ boxes. Consider the empty word as the semistandard tableau τ^λ with $m+n$ boxes defined by

$$\tau^\lambda(i) = 1 \quad \text{for } 1 \le i \le m, \quad \text{and } \tau^\lambda(j) = 3 \quad \text{for } m+1 \le j \le m+n.$$

To each $(\mathcal{S}, \mathcal{T}_\lambda)$-standard monomial $f_2^a(f_1 f_2)^b(f_1^2 f_2)^c f_1^d$ with $0 \le d \le m, 0 \le c \le n, 0 \le b \le n - c + d, 0 \le a \le n - b + d$, we associate the semistandard tableau τ as follows. Start with the tableau τ^λ and change its entries by the following rules:

(I-i) Let the word f_1 change the box $\boxed{1}$ to the box $\boxed{2}$,

(I-ii) let the word $(f_1^2 f_2)$ change $\boxed{3} \xrightarrow{2} \boxed{4} \xrightarrow{1} \boxed{0} \xrightarrow{1} \boxed{\bar{4}}$,

(I-iii) let $(f_1 f_2)$ change $\boxed{3} \xrightarrow{2} \boxed{4} \xrightarrow{1} \boxed{0}$, or $\boxed{2} \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$

(If the former case is possible, then the action is applied), and

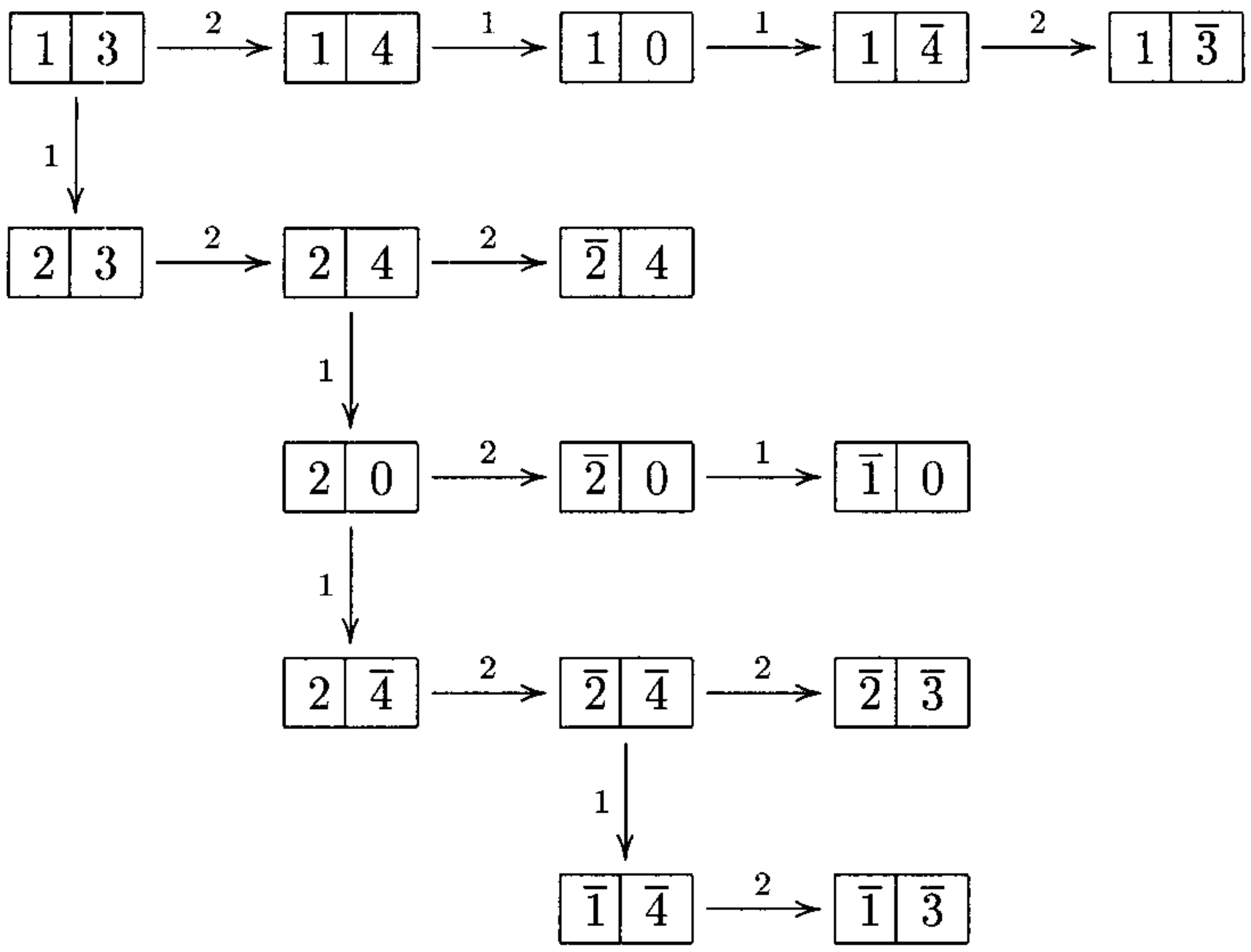

FIGURE 3. The Gröbner–Shirshov graph of $V(\Lambda_1+\Lambda_2)$

(I-iv) let f_2 change $\boxed{3}$ to $\boxed{4}$, or $\boxed{2}$ to $\boxed{\overline{2}}$, or $\boxed{\overline{4}}$ to $\boxed{\overline{3}}$ (If the former case is possible, then the action is applied).

(II) Let the words f_1, $(f_1^2f_2)$, (f_1f_2) and f_2 in $f_2^a(f_1f_2)^b(f_1^2f_2)^cf_1^d$ act successively on τ^λ changing the boxes in τ^λ. Then,

Proposition 4.3. *The monomial basis of $V(\lambda)$ consisting of $(\mathcal{S},\mathcal{T}_\lambda)$-standard monomials can be realized as the Gröbner–Shirshov graph with the above rules* (I-i,ii,iii,iv) *and* (II) *on a set of semistandard tableaux with $m+n$ boxes.*

Proof. We know the Gröbner–Shirshov graphs (4.1) and (4.2) of $V(\Lambda_1)$ and $V(\Lambda_2)$, respectively. The above rules (I-i,ii,iii,iv) and (II) on a set of semistandard tableaux with $m+n$ boxes are just the translations of the rules that $(\mathcal{S},\mathcal{T}_\lambda)$-standard monomials are given by $f_2^a(f_1f_2)^b(f_1^2f_2)^cf_1^d$ with $0\le d\le m, 0\le c\le n, 0\le b\le n-c+d, 0\le a\le n-b+d$, in order for 1-arrows and 2-arrows to be compatible with each other. □

Example 4.4. For the simplest case $\lambda=\Lambda_1+\Lambda_2$, the monomial basis of $V(\lambda)$ is given by

$$\{\, 1, f_1, f_2, f_2f_1, (f_1f_2), (f_1f_2)f_1, f_2^2f_1, (f_1^2f_2), (f_1^2f_2)f_1, f_2(f_1f_2)f_1, f_2(f_1^2f_2),$$
$$f_2(f_1^2f_2)f_1, (f_1f_2)^2f_1, (f_1f_2)(f_1^2f_2)f_1, f_2^2(f_1^2f_2)f_1, (f_1f_2)(f_1^2f_2)f_1 \,\},$$

and the Gröbner–Shirshov graph of $V(\Lambda_1+\Lambda_2)$ is given in Figure 3.

For the second example $\lambda=2\Lambda_1+\Lambda_2$, the monomial basis of $V(\lambda)$ is given by

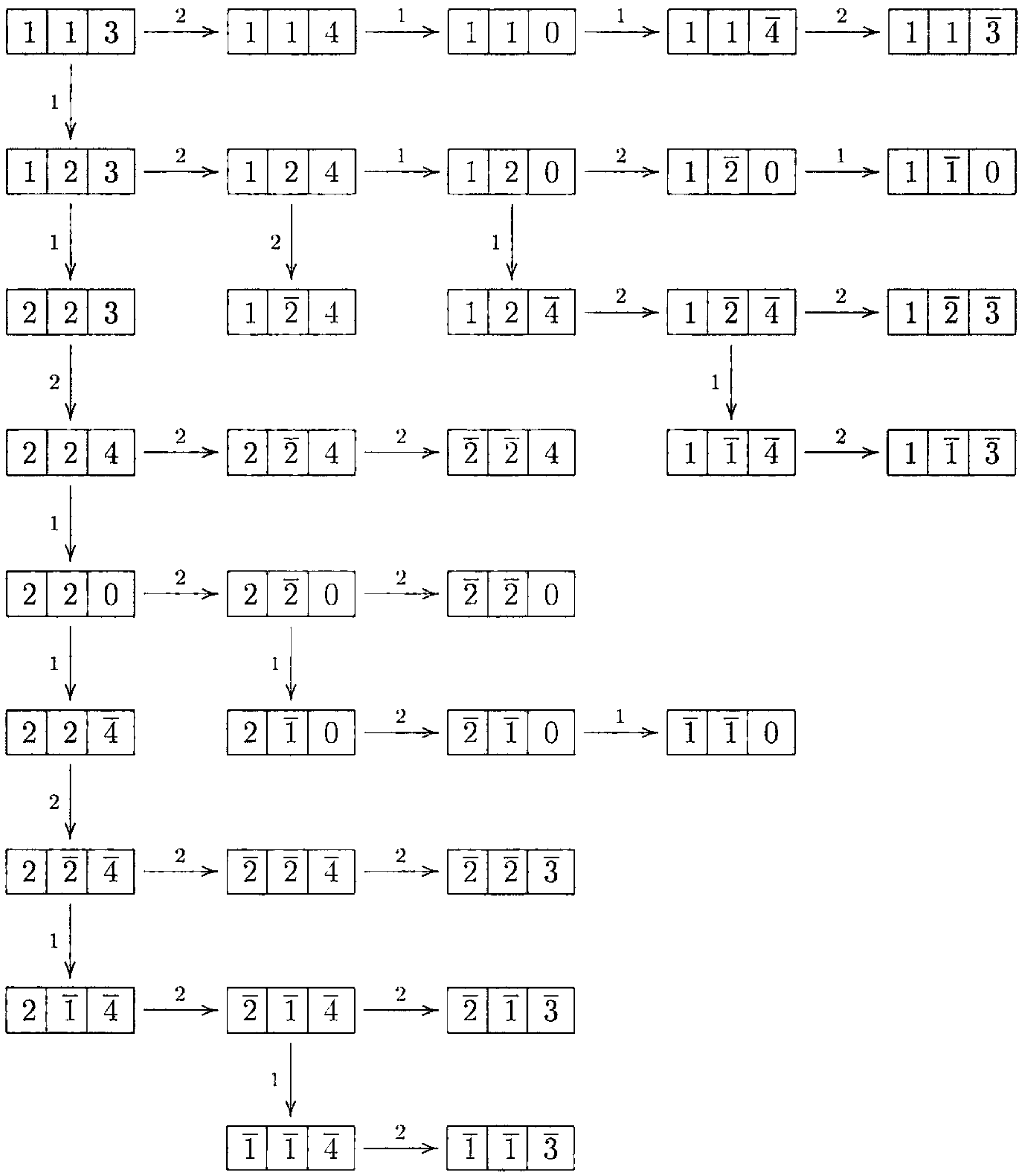

FIGURE 4. The Gröbner Shirshov graph of $V(2\Lambda_1 + \Lambda_2)$

$\{ 1, f_1, f_2, f_1^2, f_2 f_1, (f_1 f_2), f_2 f_1^2, (f_1 f_2) f_1, f_2^2 f_1, (f_1^2 f_2), (f_1 f_2) f_1^2, f_2^2 f_1^2,$
$(f_1^2 f_2) f_1, f_2 (f_1 f_2) f_1, f_2 (f_1^2 f_2), (f_1^2 f_2) f_1^2, f_2 (f_1 f_2) f_1^2, f_2^3 f_1^2, f_2 (f_1^2 f_2) f_1,$
$(f_1 f_2)^2 f_1, f_2 (f_1^2 f_2) f_1^2, (f_1 f_2)^2 f_1^2, f_2^2 (f_1 f_2) f_1^2, (f_1 f_2)(f_1^2 f_2) f_1, f_2^2 (f_1^2 f_2) f_1,$
$(f_1 f_2)(f_1^2 f_2) f_1^2, f_2^2 (f_1^2 f_2) f_1^2, f_2 (f_1 f_2)^2 f_1^2, f_2 (f_1 f_2)(f_1^2 f_2) f_1, f_2 (f_1 f_2)(f_1^2 f_2) f_1^2,$
$f_2^3 (f_1^2 f_2) f_1^2, (f_1 f_2)^3 f_1^2, (f_1 f_2)^2 (f_1^2 f_2) f_1^2, f_2^2 (f_1 f_2)(f_1^2 f_2) f_1^2, f_2 (f_1 f_2)^2 (f_1^2 f_2) f_1^2 \}$,
and the Gröbner–Shirshov graph of $V(2\Lambda_1 + \Lambda_2)$ is given in Figure 4.

5. Irreducible modules over $\mathfrak{so}_5$

Since the orthogonal Lie algebra $\mathfrak{so}_5$ is dual to $\mathfrak{sp}_4$, we have exactly the dual results by interchanging two generators f_1 and f_2. Explicitly,

Theorem 5.1. *The pair $(\mathcal{S}, \mathcal{T}_\lambda)$ is a Gröbner–Shirshov pair for the finite-dimensional irreducible $\mathfrak{so}_5$-module $V(\lambda)$ with highest weight $\lambda = m\Lambda_1 + n\Lambda_2$, where*

$$\mathcal{S} = \{\ [[f_2 f_1] f_1],\ [f_2 f_2 f_2 f_1],\ [[f_2 f_2 f_1][f_2 f_1]]\ \},$$

and $\mathcal{T}_\lambda$ consists of the following elements:

(a) f_2^{n+1},

(b) $[f_2 f_2 f_1]^{m+1} f_2^d + R_4(f_2, f_1, m; d) \quad (0 \le d \le n)$,

(c) $\displaystyle\sum_{r=0}^{c} \frac{(m-c+d+1)!}{(m-c+d+1+r)!}\binom{c}{r}[f_2 f_1]^{m-c+d+1+r}[f_2 f_2 f_1]^{c-r} f_2^{d+r}$
$+R_3(f_2, f_1, m; c, d) \quad (0 \le c \le m, 0 \le d \le n)$,

(d) $f_1^{m-b+d+1}[f_2 f_1]^b [f_2 f_2 f_1]^c f_2^d + R_2(f_2, f_1, m; b, c, d)$
$(0 \le d \le n, 0 \le c \le m, 0 \le b \le m-c+d)$,

where the polynomials R_2, R_3, R_4 are as given in Lemma 3.6-Lemma 3.8.

Hence the set of monomials of the form

$$f_1^a (f_2 f_1)^b (f_2^2 f_1)^c f_2^d \quad (\, 0 \le d \le n,\ 0 \le c \le m,\ 0 \le b \le m-c+d,\ 0 \le a \le m-b+d\,)$$

forms a linear basis of $V(\lambda)$.

Remark. The Gröbner–Shirshov graph of $V(m\Lambda_1 + n\Lambda_2)$ over $\mathfrak{so}_5$ is obtained from that of $V(n\Lambda_1 + m\Lambda_2)$ over $\mathfrak{sp}_4$ by merely interchanging 1-arrows with 2-arrows.

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