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HOMOLOGY OF THE GAUGE GROUP OF EXCEPTIONAL LIE GROUP G_2

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Abstract. We study homology of the gauge group associated with the principal G_2 bundle over the four-sphere using the Eilenberg–Moore spectral sequence and the Serre spectral sequence with the aid of homology and cohomology operations.

1. Introduction

Let G be a compact, connected simple Lie group. The fact that $\pi_3(G) = \pi_4(BG) = Z$ leads to the classification of principal G bundles P_k over S^4 by the integer k in Z. The gauge group $\mathcal{G}_k(G)$ acts freely on the space $\mathrm{Map}(P_k, EG)$ of all G-equivariant maps from P_k to EG and its orbit space is given by the k–component of the space $\mathrm{Map}_k(S^4, BG)$ of maps from S^4 to BG. Since $\mathrm{Map}(P_k, EG)$ is contractible, the classifying space of $\mathcal{G}_k(G)$ is homotopy equivalent to $\mathrm{Map}_k(S^4, BG)$. Then the number of homotopy types of $\mathcal{G}_k(G)$ is finite [7]. Similarly, if $\mathcal{G}^b_k(G)$ is the based gauge group which consists of base point preserving automorphisms on P_k, $B\mathcal{G}^b_k(G)$ is homotopy equivalent to $\Omega^3_k G$ [1].

In this paper we study the mod p homology of the gauge group associated with principal bundle of the exceptional Lie group G_2 by computing the Serre spectral sequence for the following fibration:

$$\mathcal{G}^b_k(G_2) \longrightarrow \mathcal{G}_k(G_2) \longrightarrow G_2.$$

The main result is that the Serre spectral sequence converging to

$$H_*(\mathcal{G}_k(G_2); \mathbb{F}_p)$$

collapses at the E_2-term except for $p = 3, 7$.

2. Preliminaries

Let $E(x)$ be the exterior algebra on x and $\Gamma(x)$ be the divided power Hopf algebra on x which is free over $\gamma_i(x)$ with product $\gamma_i(x)\gamma_j(x) = \binom{i+j}{j}\gamma_{i+j}(x)$

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and with coproduct $\Delta(\gamma_n(x)) = \sum_{i=0}^{n} \gamma_{n-i}(x) \otimes \gamma_i(x)$. Throughout this paper, the subscript of an element always means the degree of an element; for example the degree of a_i is i.

There are only four division algebras over R, that is, the real numbers R, complex numbers C, the quaternions H and the Cayley numbers K. K is R^8 as a vector space and it is the non-associative algebra. The exceptional Lie group G_2 is the group of automorphisms of K. By the Cartan-Killing classification we call G_2 the exceptional Lie group of type (3,11). The following theorem is well-known [14].

Theorem 2.1. *The cohomology of G_2 are given by*

$$H^*(G_2; \mathbb{F}_2) = \mathbb{F}_2[x_3]/(x_3^4) \otimes E(Sq^2 x_3),$$
$$H^*(G_2; \mathbb{F}_3) = E(x_3, x_{11}) \text{ for } p > 2$$

with all x_i primitive, where $\mathcal{P}^1 x_3 = x_{11}$ for $p = 5$.

We have homology operations $Q_{i(p-1)}$ on the n-fold loop space $\Omega^n X$

$$Q_{i(p-1)} : H_q(\Omega^n X; \mathbb{F}_p) \to H_{pq+i(p-1)}(\Omega^n X; \mathbb{F}_p)$$

for $0 \le i \le n-1$ when $p = 2$, and for $0 \le i \le n-1$ and $i + q$ even when $p > 2$. They are natural with respect to n-fold loop maps. In particular, we have $Q_0 x = x^p$. The iterated power Q_i^a denotes the composition of Q_i's a times. If G is a Lie group, G is homotopy equivalent to ΩBG. Hence $Q_{3(p-1)}$ is defined in $H_*(\Omega^3 G; \mathbb{F}_p)$ and $Q_{4(p-1)}$ is defined in $H_*(\Omega^4 G; \mathbb{F}_p)$. These operations satisfy the following properties [6].

Theorem 2.2. *In the path-loop fibration $\Omega^{n+1} X \to P\Omega^n X \to \Omega^n X$, we have the following.*

(a) *If $x \in H_*(\Omega^n X; \mathbb{F}_p)$ is transgressive in the Serre spectral sequence, so is $Q_i x$ and $\tau \circ Q_{i(p-1)} x = Q_{(i+1)(p-1)} \circ \tau x$ for each i, $0 \le i \le n-1$, where τ is the transgression.*

(b) *For $p > 2$ and $n > 1$, $d^{2q(p-1)}(x^{p-1} \otimes \tau(x)) = -\beta Q_{(p-1)} \tau(x)$ if $x \in H_{2q}(\Omega^n X; \mathbb{F}_p)$.*

(c) *For $p = 2$, $Sq_*^1 Q_i x = Q_{i-1} x$ if $x \in H_q(\Omega^n X; \mathbb{F}_2)$ and $q + i$ is even.*

From now on, we will simplify the notation as follows: We denote $\mathbb{F}_p[Q_i^a u : a \ge 0]$ by $\mathbb{F}_p[Q_i^a u]$ and $\mathbb{F}_p[Q_i^{a+1} u : a \ge 0]$ by $\mathbb{F}_p[Q_i^{a+1} u]$ and so on. Now we recall the mod p homology of the four fold loop space of a sphere.

$$H_*(\Omega^4 S^{n+4}; \mathbb{F}_2) = \mathbb{F}_2[Q_1^a Q_2^b Q_3^c \iota_n], n \ge 1,$$
$$H_*(\Omega^4 S^{n+4}; \mathbb{F}_p) = E(Q_{p-1}^a Q_{3(p-1)}^b \iota_n) \otimes \mathbb{F}_p[\beta Q_{p-1}^{a+1} Q_{3(p-1)}^b \iota_n]$$
$$\otimes \mathbb{F}_p[Q_{2(p-1)}^a \beta Q_{3(p-1)}^{b+1} \iota_n] \otimes E(Q_{(p-1)}^a \beta Q_{2(p-1)}^{b+1} \beta Q_{3(p-1)}^{c+1} \iota_n)$$
$$\otimes \mathbb{F}_p[\beta Q_{(p-1)}^{a+1} \beta Q_{2(p-1)}^{b+1} \beta Q_{3(p-1)}^{c+1} \iota_n], n \ge 1.$$

Theorem 2.3 ([3, Theorem 5.14]). *Let X be a path connected H-space. Then the following are true.*

(a) *The Eilenberg Moore spectral sequence collapses at E_2 if and only if* $\ker\sigma = 0$.

(b) *The suspension $\sigma : QH^k(X;\mathbb{F}_p) \to PH^{k-1}(\Omega X;\mathbb{F}_p)$ is injective if $k \not\equiv 2 \mod 2p$.*

(c) *The suspension $\sigma : QH^k(X;\mathbb{F}_p) \to PH^{k-1}(\Omega X;\mathbb{F}_p)$ is surjective if $k-1 \not\equiv -2 \mod 2p$.*

From now on we denote $H_*(\Omega^i S^n;\mathbb{F}_p)$ by $\Omega_i(n)$, $\otimes_{k=1}^r H_*(\Omega^i S^{n_k};\mathbb{F}_p)$ by $\Omega_i(n_1,\ldots,n_r)$, and $\otimes_{k=1}^r H^*(\Omega^i S^{n_k};\mathbb{F}_p)$ by $\Omega^i(n_1,\ldots,n_r)$ for each $i, n \geq 1$.

Theorem 2.4. *The cohomology of the loop space of G_2 are*

$$H^*(\Omega G_2;\mathbb{F}_2) = \mathbb{F}_2[y_2]/(y_2^4) \otimes \Gamma(y_8, y_{10}),$$
$$H^*(\Omega G_2;\mathbb{F}_5) = \bigotimes_{i\geq 0} \mathbb{F}_5[\gamma_{5^i}(y_2)]/((\gamma_{5^i}(y_2))^{25}),$$
$$H^*(\Omega G_2;\mathbb{F}_p) = \Omega^1(3,11) \text{ for odd primes } p \neq 5.$$

Proof. By the Eilenberg Moore spectral sequence for the path loop fibration converging to $H^*(\Omega G_2;\mathbb{F}_2)$, as a Hopf algebra we have

$$\begin{aligned} E_2 &= \mathrm{Tor}_{H^*(G_2;\mathbb{F}_2)}(\mathbb{F}_2,\mathbb{F}_2) \\ &= \mathrm{Tor}_{\mathbb{F}_2[x_3]/(x_3^4)\otimes E(x_5)}(\mathbb{F}_2,\mathbb{F}_2) \\ &= \mathrm{Tor}_{\mathbb{F}_2[x_3]/(x_3^4)}(\mathbb{F}_2,\mathbb{F}_2) \otimes \mathrm{Tor}_{E(x_5)}(\mathbb{F}_2,\mathbb{F}_2) \\ &= E(y_2) \otimes \Gamma(y_{10}) \otimes \Gamma(y_4). \end{aligned}$$

Since the E_2-term concentrates on even dimensions, the spectral sequence collapses at the E_2-term. Hence $E_2 = E_\infty$. Since the Eilenberg-Moore spectral sequence preserves the Steenrod actions, from $Sq^2 x_3 = x_5$, we get $Sq^2 y_2 = y_4$, that is, $y_2^2 = y_4$. From this, we can solve the algebra extension problem and we have

$$H^*(\Omega G_2;\mathbb{F}_2) = \mathbb{F}_2[y_2]/(y_2^4) \otimes \Gamma(y_8) \otimes \Gamma(y_{10}).$$

Now we turn to the odd prime cases. Like the mod 2 case, we use the Eilenberg–Moore spectral sequence converging to $H^*(\Omega G_2;\mathbb{F}_p)$ with

$$\begin{aligned} E_2 &= \mathrm{Tor}_{H^*(G_2;\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p) \\ &= \mathrm{Tor}_{E(x_3)\otimes E(x_{10})}(\mathbb{F}_p,\mathbb{F}_p) \\ &= \Gamma(y_2) \otimes \Gamma(y_{10}). \end{aligned}$$

Since all elements in E_2 are even dimensional, the spectral sequence collapses at the E_2-term, so $E_2 = E_\infty$. For $p \neq 5$, there is no extension problem and we get

$$H^*(\Omega G_2; F_p) = \Gamma(y_2) \otimes \Gamma(y_{10}) = \Omega^1(3,11)$$

as a Hopf algebra. Now we consider the case $p = 5$. In the bar construction, y_2 is represented by $[x_3]$. Since $\mathcal{P}^1 x_3 = x_{11}$ in $H^*(G_2; \mathbb{F}_5)$, we have the following algebra extension:

$$y_2^5 = \mathcal{P}^1 y_2 = \mathcal{P}^1[x_3] = [\mathcal{P}^1 x_3] = [x_{11}] = y_{10}\,.$$

The element $\gamma_{5^i}(y_2)$ is represented by $[x_3|\cdots|x_3]$ (5^i factors) in the bar construction. By the Cartan formula, we also have the following algebra extension:

$$\begin{aligned}(\gamma_{5^i}(y_2))^5 &= \mathcal{P}^{5^i}\gamma_{5^i}(y_2) = \mathcal{P}^{5^i}[x_3|\cdots|x_3]\\ &= [\mathcal{P}^1 x_3|\cdots|\mathcal{P}^1 x_3] = [x_{11}|\cdots|x_{11}]\\ &= \gamma_{5^i}(y_{10}).\end{aligned}$$

Since $\Gamma(y_2) = \bigotimes_{i\geq 0}\gamma_{5^i}(y_2)$ and $\Gamma(y_{10}) = \bigotimes_{i\geq 0}\gamma_{5^i}(y_{10})$ as an algebra, it follows that $\Gamma(y_2)\otimes\Gamma(y_{10})$ in E_∞ produces $\bigotimes_{i\geq 0}\mathbb{F}_5[\gamma_{5^i}(y_2)]/((\gamma_{5^i}(y_2))^{25})$ in $H^*(G_2; \mathbb{F}_5)$ and we get the conclusion. □

To get $H_*(\Omega_0^3 G; \mathbb{F}_p)$, where $\Omega_0^3 G$ is the zero component of $\Omega^3 G$, we need the following result [10].

Theorem 2.5. *The Eilenberg–Moore spectral sequences for the path loop fibrations converging to the mod p (co)homology of the double and the triple loop spaces of any simply connected finite H–space collapse at the E_2-term.*

By Theorem 2.5 and the formal computations of Tor, Ext and Cotor, we get the following two theorems.

Theorem 2.6. *The homology of the double loop space of G_2 are*

$$\begin{aligned}&H_*(\Omega^2 G_2; \mathbb{F}_2) = E(z_1)\otimes\mathbb{F}_2[\beta z_7]\otimes\mathbb{F}_2[Q_1^a z_7]\otimes\Omega_2(11),\\ &H_*(\Omega^2 G_2; \mathbb{F}_5) = E(Q_4^a z_1)\otimes\mathbb{F}_5[Q_8^a z_{48}],\\ &H_*(\Omega^2 G_2; \mathbb{F}_p) = \Omega_2(3,11) \text{ for odd primes } p\neq 5.\end{aligned}$$

Theorem 2.7. *The homology of the triple loop space of G_2 are*

$$\begin{aligned}&H_*(\Omega_0^3 G_2; \mathbb{F}_2) = \mathbb{F}_2[Q_1^a\beta w_6]\otimes\mathbb{F}_2[Q_1^a Q_2^b w_6]\otimes\Omega_3(11),\\ &H_*(\Omega_0^3 G_2; \mathbb{F}_5) = \mathbb{F}_5[Q_8^a(Q_8[1]*[-5])]\otimes E(Q_4^a Q_{12}^b w_{47})\otimes\mathbb{F}_5[\beta Q_4^a Q_{12}^b w_{47}],\\ &H_*(\Omega^3 G_2; \mathbb{F}_p) = \Omega_3(3,11) \text{ for odd primes } p\neq 5.\end{aligned}$$

3. Based gauge group

For the exceptional Lie group G_2 we have

$$\begin{aligned}\pi_3(G_2) &= Z,\\ \pi_4(G_2) &= 0\,.\end{aligned}$$

Exploiting the fibration leading to the 3-connected cover $G_2\langle 3\rangle$, for which the base is G_2 and the fiber is $K(Z,2)$[13], we obtain

$$H^*(G_2\langle 3\rangle;\mathbb{F}_2) = \mathbb{F}_2[y_8]\otimes E(\beta y_8, Sq^2Sq^1y_8),$$
$$H^*(G_2\langle 3\rangle;\mathbb{F}_5) = \mathbb{F}_5[y_{50}]\otimes E(y_{11},\beta y_{50}),$$
$$H^*(G_2\langle 3\rangle;\mathbb{F}_p) = \mathbb{F}_p[y_{2p}]\otimes E(y_{11},\beta y_{2p}) \quad \text{for primes } p\neq 2,\ 5\,.$$

Hence $H_i(G_2\langle 3\rangle;\mathbb{F}_p)=0$ for $1\le i\le 7$ for all primes p and $H_i(\Omega^4G_2\langle 3\rangle;\mathbb{F}_p) = H_i(\Omega^4G_2;\mathbb{F}_p)=0$ for $1\le i\le 3$. Note that $\Omega^4G_2\simeq\Omega^4G_2\langle 3\rangle$. Since $B\mathcal{G}_k^b(G_2)\simeq \Omega_k^3G_2$, we have $\mathcal{G}_k^b(G_2)\simeq\Omega(\Omega_k^3G_2)$. Note $\Omega^3G_2\simeq\Omega_k^3G_2\times Z$ and $\Omega(\Omega_k^3G_2)\simeq \Omega^4G_2$ for any $k\in Z$.

Theorem 3.1. *The mod 2 homology of the four fold loop space of G_2 is*

$$H_*(\Omega^4G_2;\mathbb{F}_2) = \mathbb{F}_2[Q_1^aQ_2^b\beta u_5]\otimes\mathbb{F}_2[Q_1^aQ_2^bQ_3^cu_i : i=5,7].$$

Proof. We compute the $H_*(\Omega^4G_2;\mathbb{F}_2)$ by the almost same method computing the triple loop space in [4]. The group G_2, as a subgroup of $O(7)$, acts on S^7. The action is transitive and the isotropy group is $SU(3)$. So we have the following fibration

$$SU(3)\longrightarrow G_2\longrightarrow S^6.$$

Consider the Serre spectral sequence for the following fibration:

$$\Omega^4SU(3)\longrightarrow\Omega^4G_2\longrightarrow\Omega^4S^6\,.$$

We have

$$H_*(\Omega^4S^6;\mathbb{F}_2) = \mathbb{F}_2[Q_1^aQ_2^bQ_3^c\iota_2],$$
$$H_*(\Omega^4SU(3);\mathbb{F}_2) = \mathbb{F}_2[Q_1^aQ_3^bu_1]\otimes\mathbb{F}_2[Q_1^aQ_2^bQ_4^cu_4],$$
$$d^2Q_3^{a+1}u_1 = Q_4^au_4.$$

Since $H_i(\Omega^4G_2;\mathbb{F}_2)=H_i(\Omega^4G_2\langle 3\rangle;\mathbb{F}_2)=0$ for $1\le i\le 3$, we have

$$\tau(Q_0^aQ_2^b\iota_2) = Q_1^aQ_3^bu_1,\ a,b\ge 0.$$

Note that $\tau(Q_1^{a+1}Q_2^b\iota_2)=0$ since $Q_2^{a+1}Q_3^bu_1=0$. Then the E_∞ term is

$$\mathbb{F}_2[Q_1^{a+1}Q_2^b\iota_2]\otimes\mathbb{F}_2[Q_1^aQ_2^bQ_3^{c+1}\iota_2]\otimes\mathbb{F}_2[Q_1^aQ_2^bQ_4^cu_4]\,.$$

On the other hand, $Sq_*^1Q_1Q_2^{a+1}\iota_2 = Q_0Q_2^{a+1}\iota_2$, $a\ge 0$ by the Nishida relation. Since $d^2Q_3^{a+1}u_1 = Q_4^au_4$, we have $Sq_*^1Q_1Q_2^{a+1}\iota_2 = Q_4^{a+1}u_4, a\ge 0$ by Theorem 2.2 in [4]. By the Nishida relation again, $Q_2Q_1\iota_2 = Q_4u_4$. So if we put $Q_1\iota_2=u_5$, then $Q_0^aQ_1^{b+1}Q_2^c(\iota_2)$ can be expressed as $Q_0^aQ_1^bQ_3^cu_5$. We also put $u_7=Q_3\iota_2$. Then we get $H_*(\Omega^4G_2;\mathbb{F}_2)=\mathbb{F}_2[Q_1^aQ_2^bu_4]\otimes\mathbb{F}_2[Q_1^aQ_2^bQ_3^cu_i : i=5,7]$. This result implies that the Eilenberg–Moore spectral sequence converging to $H^*(\Omega^4G_2;\mathbb{F}_2)$ for the path loop fibration collapses at E_2. Since $H_i(\Omega^4G_2\langle 3\rangle;\mathbb{F}_2)=H_i(\Omega^4G_2;\mathbb{F}_2)$, we have $\beta u_5=u_4$ from the information of $H^*(G_2\langle 3\rangle;\mathbb{F}_2)$. □

Theorem 3.2. *For odd primes p, the mod p homology of the four fold loop space of G_2 are as follows:*

$$H_*(\Omega^4 G_2;\mathbb{F}_5) = E(Q_4^a Q_{12}^b u_7)\otimes \mathbb{F}_5[\beta Q_4^{a+1} Q_{12}^b u_7]\otimes \mathbb{F}_5[Q_8^a Q_{16}^b u_{46}]$$
$$\otimes E(Q_4^a \beta Q_8^b Q_{16}^c u_{46})\otimes \mathbb{F}_5[\beta Q_4^{a+1}\beta Q_8^{b+1} Q_{16}^c u_{46}],$$
$$H_*(\Omega^4 G_2;\mathbb{F}_p) = \Omega_4(3,11) \quad \textit{for odd primes} \quad p\neq 5\,.$$

Proof. Consider the Eilenberg–Moore spectral sequence converging to

$$H^*(\Omega^4 G_2;\mathbb{F}_p)$$

with

$$E_2 \cong \mathrm{Tor}_{H^*(\Omega_0^3 G_2);\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p)\,.$$

Then by Theorem 2.3, the collapse at E_2 depends on whether

$$\sigma : QH^{2kp+2}(\Omega_0^3 G_2;\mathbb{F}_p)\to PH^{2kp+1}(\Omega^4 G_2;\mathbb{F}_p)$$

is injective or not. By the exact sequence of Milnor–Moore [12] and Theorem 2.3, we have that

$$QH^{2kp+2}(\Omega_0^3 G_2;\mathbb{F}_p)\cong PH^{2kp+2}(\Omega_0^3 G_2;\mathbb{F}_p)$$
$$\cong QH^{2kp+3}(\Omega^2 G_2;\mathbb{F}_p)\,.$$

For odd primes p, every primitive element in $H_*(\Omega^2 G_2;\mathbb{F}_p)$ is one of the following types:

$$Q_{(p-1)}^a z_{2i+1},\quad (\beta Q_{(p-1)}^a z_{2i+1})^{p^k},\quad (\beta^2 Q_{(p-1)}^a z_{2i+1})^{p^k}.$$

Since $|Q_{(p-1)}^a z_{2i-1}| = 2p^a i - 1$ and $|\beta Q_{(p-1)}^a z_{2i-1}| = |\beta^2 Q_{(p-1)}^a z_{2i-1}| = 2p^a i - 2$, there is no primitive element with degree $2kp+3$. By duality, there is no indecomposable element with degree $2kp+3$ in $H^*(\Omega^2 G_2;\mathbb{F}_p)$. Hence the Eilenberg–Moore spectral sequence collapses at E_2 and there is no coalgebra extension problem in such a case [8]. Hence by duality, the Eilenberg–Moore spectral sequence converging to $H_*(\Omega^4 G_2;\mathbb{F}_p)$ with

$$E^2 \cong \mathrm{Cotor}_{H_*(\Omega_0^3 G_2;\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p)$$

also collapses at E^2. Then we get the conclusion for $H_*(\Omega^4 G_2;\mathbb{F}_p)$ by the formal cotor calculation since there is no algebra extension problem by the duality. □

4. Full gauge group

Let P_k be a principal G_2 bundle over S^4 classified by the integer k in Z and $\mathcal{G}_k(G_2)$ be the gauge group of the principal G_2 bundle P_k. From [1, Prop. 2.4] we can get

$$B\mathcal{G}_k(G_2)\simeq \mathrm{Map}_{P_k}(S^4, BG_2),$$

where the subscript P_k denotes the component of a map of M into BG_2 which induces P_k. By the natural evaluation map, we have

$$\Omega_k^3 G_2 \longrightarrow \mathrm{Map}_{P_k}(S^4, BG_2)\longrightarrow BG_2\,.$$

Looping again, we get

$$\Omega^4 G_2 \simeq \Omega(\Omega^3_k G_2) \longrightarrow \mathcal{G}_k(G_2) \longrightarrow G_2 .$$

So we have the following fiber sequence

$$\cdots \Omega G_2 \xrightarrow{\Omega h_k} \Omega(\Omega^3_k G_2) \longrightarrow \mathcal{G}_k(G_2) \longrightarrow G_2 \xrightarrow{h_k} \Omega^3_k G_2 \cdots .$$

Main idea to compute $H(\mathcal{G}_k(G_2);\mathbb{F}_p)$ is to exploit the property of the map $(\Omega h_k)_* : H(\Omega G_2;\mathbb{F}_p) \to H(\Omega(\Omega^3_k G_2);\mathbb{F}_p)$. Recall the following p-primary component of homotopy groups of odd spheres [15, *p.* 176].

Proposition 4.1. *Let p be an odd prime. Then we have the following.*

$$\pi_{2m+1+2i(p-1)-2}(S^{2m+1};p) = \mathbb{Z}/(p) \text{ for } 1 \le m < i, \text{ and } i = 2, \dots, p-1.$$

$$\pi_{2m+1+2i(p-1)-1}(S^{2m+1};p) = \mathbb{Z}/(p) \text{ for } 1 \le m, \text{ and } i = 1, 2, \dots, p-1.$$

$$\pi_{2m+1+k}(S^{2m+1};p) = 0 \text{ otherwise for } k < 2p(p-1)-2.$$

From this, we have

$$\pi_i(S^{4j-1};p) = \begin{cases} \mathbb{Z}/(p), & \text{if } i = 2p \text{ and } j = 1, \\ 0, & \text{otherwise for } j > 1 \text{ and } 0 \le i \le 2p. \end{cases}$$

When localized at $p \ge 7$, G_2 splits as $G_2 \simeq_p S^3 \times S^{11}$ [14]. So when localized at $p > 7$, the map $h : (G_2)_{(p)} \to (\Omega^3 G_2)_{(p)}$ is null homotopic since $h : (S^3 \times S^{11})_{(p)} \to (\Omega^3 S^3 \times \Omega^3 S^{11})_{(p)}$ is null homotopic. Here $X_{(p)}$ denote X localized at the prime p. Note that $\pi_6(S^3;p) = \pi_{14}(S^3;p) = 0$ for $p > 7$, but $\pi_6(S^3;3) = \mathbb{Z}/(3)$ and $\pi_{14}(S^3;7) = \mathbb{Z}/(7)$.

Theorem 4.2. *For any primes $p \neq 3, 7$, we have*

$$H_*(\mathcal{G}_k(G_2);\mathbb{F}_p) = H_*(\Omega^4 G_2;\mathbb{F}_p) \otimes H_*(G_2;\mathbb{F}_p)$$

as an algebra for any $k \in Z$.

Proof. Consider the Serre spectral sequence converging to $H_*(\mathcal{G}_k(G_2);\mathbb{F}_p)$ for $\Omega(\Omega^3_k G) \to \mathcal{G}_k(G_2) \to G_2$. For $p = 2$, by the degree reason the first possible nontrivial differentials are $\tau((Sq^2 x_3)^*) = \beta u_5$ and $\tau((x_3^2)^*) = u_5$ where $(Sq^2 x_3)^*$ and $(x_3^2)^*$ are dual homology elements in $H_*(G_2;\mathbb{F}_2)$. Note that $\beta(x_3^2)^* = (Sq^2 x_3)^*$. Consider the following homotopy commutative morphisms of fibrations:

$$\begin{array}{ccccc} \Omega G_2 & \longrightarrow & * & \longrightarrow & G_2 \\ {\scriptstyle \Omega h_k}\downarrow & & \downarrow & & \downarrow \\ \Omega(\Omega^3_k G_2) & \longrightarrow & \Omega B\mathcal{G}_k(G_2) & \longrightarrow & G_2. \end{array}$$

Assume that $\tau((Sq^2 x_3)^*) = \beta u_5$. Then $\tau(Q_0(Sq^2 x_3)^*) = Q_1 \beta u_5$. But

$$Q_0(Sq^2 x_3)^* = 0$$

in $H_*(G_2;\mathbb{F}_2)$, while $Q_1\beta u_5$ is not trivial in $H_*(\Omega^4 G_2;\mathbb{F}_2)$. This is a contradiction. Hence the Serre spectral sequence collapses at E^2 and there is no algebra extension problem by the degree reason.

Similarly for $p=5$, the degree of target primitive of the first possible nontrivial differential should be 2 or 10. But there is no primitive of such degrees in $H_*(\Omega^4 G_2;\mathbb{F}_5)$. So the Serre spectral sequence collapses at E_2 and there is no algebra extension problem by the degree reason. Localized at $p>7$, the map $h_k : (G_2)_{(p)} \to (\Omega^3 G_2)_{(p)}$ is null homotopic by the above argument following Proposition 4.1. So $\Omega h_k : (\Omega G_2)_{(p)} \to (\Omega(\Omega^3_k G_2))_{(p)} \simeq (\Omega^4 G_2)_{(p)}$ is null homotopic for $p>7$. Hence the Serre spectral sequence collapses at E^2 for $p>7$. □

Theorem 4.3. *For $k \not\equiv 0 \mod 3$, the mod 3 homology of $\mathcal{G}_k(G_2)$ is as follow:*

$$\begin{aligned} H_*(\mathcal{G}_k(G_2);\mathbb{F}_3) = {} & E(Q_2^a Q_6^b u_3) \otimes \mathbb{F}_3[\beta Q_2^a Q_6^b u_3] \otimes \mathbb{F}_3[Q_4^a \beta Q_6^{b+1} u_3] \\ & \otimes \mathbb{F}_3[Q_4^a(Q_4\beta u_3)] \otimes E(Q_2^a\beta Q_4^b \beta Q_6^c u_3) \\ & \otimes \mathbb{F}_3[\beta Q_2^a \beta Q_4^b \beta Q_6^c u_3] \otimes \Omega_4(11) \otimes E(e_{11}). \end{aligned}$$

Proof. We have the following homotopy commutative morphism of fibrations:

$$\begin{array}{ccc} \Omega^3_k S^3 & \xrightarrow{f} & \Omega^3_k G_2 \\ \downarrow & & \downarrow \\ \mathrm{Map}_k(S^4, BS^3) & \longrightarrow & \mathrm{Map}_k(S^4, BG_2) \\ \downarrow & & \downarrow \\ BS^3 & \longrightarrow & BG_2. \end{array}$$

Then $H_*(f;\mathbb{F}_3) = f_*$ is onto by Theorem 2.7. Consider the following homotopy commutative morphism of fibrations:

$$\begin{array}{ccccc} \Omega G_2 & \longrightarrow & * & \longrightarrow & G_2 \\ {\scriptstyle \Omega h_k}\downarrow & & \downarrow & & \downarrow{\scriptstyle h_k} \\ \Omega(\Omega^3_k G_2) & \longrightarrow & * & \longrightarrow & \Omega^3_k G_2. \end{array}$$

From the case for S^3 [9, 11], $(h_k)_*(x_3) = k\beta w_4$ up to a choice of a generator. So we have

$$(\Omega h_k)_*(z_2) = (\Omega h_k)_*(\tau(x_3)) = \tau((h_k)_*(x_3)) = \tau(\beta w_4) = \beta u_3 .$$

There is the following homotopy commutative morphism of fibrations:

$$\begin{array}{ccccc} \Omega G_2 & \longrightarrow & * & \longrightarrow & G_2 \\ {\scriptstyle \Omega h_k}\downarrow & & \downarrow & & \downarrow \\ \Omega(\Omega^3_k G_2) & \longrightarrow & \mathcal{G}_k(G_2) & \longrightarrow & G_2. \end{array}$$

Then we have $\tau(e_3) = \beta u_3$ in the Serre spectral sequence for the bottom row fibration converging to $H_*(\mathcal{G}_k(G_2);\mathbb{F}_3)$ where $H_*(G_2;\mathbb{F}_3) = E(e_3, e_{11})$. Differentials from e_{11} are trivial because of degree reason. Hence we get the conclusion. □

For the case of $p = 7$, we follow the procedure used in [5]. Localized at $p = 7$, there are the following p-equivalences [14]:

$$Sp(3) \simeq_7 S^3 \times S^7 \times S^{11} \simeq_7 G_2 \times S^7.$$

Now we consider the exact sequence of homotopy groups:

$$\cdots \to \pi_{11}(Sp(3)) \xrightarrow{(\partial_k)_\#} \pi_{11}(\Omega_k^3 Sp(3)) \longrightarrow \pi_{11}(\mathrm{Map}_k(S^4, BSp(3)) \to \cdots$$

Then the boundary map $(\partial_k)_\#$ can be expressed in terms of the Samelson product $\langle , \rangle$ as follows [2, 15]. For α in $\pi_{11}(Sp(n))$, we have

$$(\partial_k)_\# \alpha = \pm k\langle \alpha, \beta \rangle, \tag{1}$$

where β generates $\pi_3(Sp(3))$ [15, Proposition 2.1]. We recall the following fact.

Proposition 4.4 ([2, Theorem 2]). *The kernel of the homomorphism*

$$\pi_{4n-1}(Sp(n)) \otimes \pi_{4m-1}(Sp(m)) \to \pi_{4n+4m-2}(Sp(n+m-1))$$
$$\alpha \otimes \beta \to \langle \alpha, \beta \rangle$$

induced by the Samelson product $\langle , \rangle$ is precisely divisible by $k_{n+m}/k_n k_m$, where

$$k_r = \begin{cases} (2r-1)!2 & \text{for even } r, \\ (2r-1)! & \text{for odd } r. \end{cases}$$

Theorem 4.5. *For $k \not\equiv 0 \mod 7$, the mod 7 homology of $\mathcal{G}_k$ is as follow:*

$$\begin{aligned} H_*(\mathcal{G}_k(G_2);\mathbb{F}_7) =& E(Q_6^a Q_{18}^b u_{11}) \otimes \mathbb{F}_7[\beta Q_6^a Q_{18}^b u_{11}] \otimes \mathbb{F}_7[Q_{12}^a \beta Q_{18}^{b+1} u_{11}] \\ & \otimes \mathbb{F}_7[Q_{12}^a(Q_{12}\beta u_{11})] \otimes E(Q_6^a \beta Q_{12}^b \beta Q_{18}^c u_{11}) \\ & \otimes \mathbb{F}_7[\beta Q_6^a \beta Q_{12}^b \beta Q_{18}^c u_{11}] \otimes \Omega_4(11) \otimes E(e_3). \end{aligned}$$

Proof. By Proposition 4.4, for $\alpha \in \pi_{11}(Sp(3))$ and $\beta \in \pi_3(Sp(3))$ we get

$$\text{the order of } \langle \alpha, \beta \rangle = 84.$$

So if $p = 7$, then $\partial_k : (Sp(3))_{(7)} \to (\Omega_k^3 Sp(3))_{(7)}$ is not null homotopic for $k \not\equiv 0$ mod 7. Localized at $p = 7$, $Sp(3) \simeq_7 S^3 \times S^7 \times S^{11}$. So $\partial_k : (S^3 \times S^7 \times S^{11})_{(7)} \to (\Omega_k^3 S^3 \times \Omega^3 S^7 \times \Omega^3 S^{11})_{(7)}$ is not null homotopic for $k \not\equiv 0$ mod 7. Note that

$$\pi_i(S^3;7) = 0 \quad \text{for } i = 6, 10, \quad \pi_{14}(S^3;7) = Z/(7),$$
$$\pi_i(S^7;7) = \pi_i(S^{11};7) = 0 \quad \text{for } i = 6, 10, 14.$$

This implies that $\partial_k : (S^3 \times S^{11})_{(7)} \to (\Omega_k^3 S^3 \times \Omega^3 S^{11})_{(7)}$ is not null homotopic for $k \not\equiv 0$ mod 7. Since $G_2 \simeq_7 S^3 \times S^{11}$, $\partial_k : (G_2)_{(7)} \to (\Omega_k^3 G_2)_{(7)}$ is not null homotopic for $k \not\equiv 0 \bmod 7$. Consider the following:

$$\begin{array}{ccccc} S^{11} & \xrightarrow{\iota} & S^3 \times S^{11} & \xrightarrow{\partial_k} & \Omega^3_k S^3 \times \Omega^3 S^{11} = \Omega^3_k G_2 \\ & & & & \downarrow p \\ & & & & \Omega^3_k S^3. \end{array}$$

Let $\partial_k' = p \circ \partial_k$ and $\partial_k'' = p \circ \partial_k \circ \iota$. Since $(\partial_k)_\#$ is nonzero, $(\partial_k'')_\#$ is nonzero. We have $h_\# : \pi_i(S^{11}; 7) \to \pi_i(\Omega^3_k S^3; 7)$ is an isomorphism for $i \leq 11$. Note that $\pi_i(\Omega^3_k S^3; 7) = 0$ for $1 \leq i \leq 10$. By J. H. C. Whitehead Theorem, $(\partial_k'')_* : H_{11}(S^{11}; \mathbb{F}_7) \to H_{11}(\Omega^3_k S^3; \mathbb{F}_7)$ is also an isomorphism. Hence we have

$$(\partial_k)_*(e_{11}) = \begin{cases} 0, & k \equiv 0 \bmod 7 \\ \neq 0, & k \not\equiv 0 \bmod 7, \end{cases}$$

where $H_*(G_2; \mathbb{F}_7) = E(e_3, e_{11})$. There is the following homotopy commutative morphism of fibrations:

$$\begin{array}{ccccc} \Omega G_2 & \longrightarrow & * & \longrightarrow & G_2 \\ \downarrow{\scriptstyle \Omega\partial_k} & & \downarrow & & \downarrow{\scriptstyle \partial_k} \\ \Omega(\Omega^3_k G_2) & \longrightarrow & * & \longrightarrow & \Omega^3_k G_2. \end{array}$$

We have $(\partial_k)_*(e_{11}) = k\beta w_{12}$ for $\beta w_{12} \in H_*(\Omega^3_k G_2; \mathbb{F}_7)$ up to a choice of a generator. Let $\tau(e_{11}) = v_{10}$ for $v_{10} \in H_*(\Omega G_2; \mathbb{F}_7)$. Then we have that

$$(\Omega\partial_k)_*(v_{10}) = (\Omega\partial_k)_*(\tau(e_{11})) = \tau((\partial_k)_*(e_{11})) = \tau(k\beta w_{12}) = k\beta u_{11}\,.$$

Consider the following homotopy commutative morphism of fibrations:

$$\begin{array}{ccccc} \Omega G_2 & \longrightarrow & * & \longrightarrow & G_2 \\ \downarrow{\scriptstyle \Omega\partial_k} & & \downarrow & & \downarrow \\ \Omega(\Omega^3_k G_2) & \longrightarrow & \mathcal{G}_k(G_2) & \longrightarrow & G_2. \end{array}$$

Then we have $\tau(e_{11}) = k\beta u_{11}$ in the Serre spectral sequence converging to $H_*(\mathcal{G}_k(G_2); \mathbb{F}_7)$ for the bottom row fibration. From this we get the conclusion. □

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