J. Korean Math. Soc. **45** (2008), No. 3, pp. 683–693

RESIDUAL EMPIRICAL PROCESS FOR DIFFUSION PROCESSES

Sangyeol Lee and In-Suk Wee

Abstract. In this paper, we study the asymptotic behavior of the residual empirical process from diffusion processes. For this task, adopting the discrete sampling scheme as in Florens-Zmirou [9], we calculate the residuals and construct the residual empirical process. It is shown that the residual empirical process converges weakly to a Brownian bridge.

1. Introduction

The diffusion process has long been popular in analyzing random phenomena occurring in a variety of fields such as finance, engineering, physical and medical sciences. During the past decades, the theory for diffusion processes has been enriched in a remarkable way, and the diffusion process has become a representative keyword for stochastic analysis, especially in stochastic finance. See, for instance, Karatzas and Shreve [11] and Shiryayev [26]. Despite of the importance, there has been a tendency that statistical inferences for diffusion models have not drawn much attention from researchers as classical statistical problems. However, nowadays it is becoming a core area of statistics, and relevant fundamental results are available from the literature: see the books by Prakasa Rao [23], Lipster and Shiryayev [1] and Kutoyants [15].

From experiences in empirical analysis of financial time series data, practitioners notice that time series data is not well fitted by diffusion models. For this reason, they introduced other class of stochastic models like jump diffusion processes and Lévy processes. See Sato [24], Barndorff-Nielsen, Mikosch and Resnick [1], Schoutens [25], and Cont and Tankov [6]. Also, see Chan [5], Eberlin, and Raible [8], and Hong and Wee [10]. These models are now widely accepted as a promising alternative to diffusion models for modeling financial time series. Meanwhile, it is well known that time series often suffer from structural changes in underlying models owing to changes of monetary policy and critical social events (see, for instance, Lee, Ha, Na, and Na [16], Lee and Na [17], Lee, Toktsu, and Maekawa [20], and Lee, Nishiyama, and Yoshida [18]).

Received September 16, 2006; Revised August 1, 2007.

2000 *Mathematics Subject Classification.* 60J60, 62F12.

Key words and phrases. diffusion process, discrete scheme, residual empirical process, weak convergence to a Brownian bridge, model check test.

Hence, it is somewhat natural to ask whether or not the diffusion models are suitable for modeling time series data.

In this paper, motivated by the issue mentioned above, we consider the goodness of fit test problem for diffusion models. For this task, we study the empirical process from diffusion processes since it is one of the most popular tools to check the validity of proposed statistical models (see Thode [28] for the Gauusian goodness of fit test), and provides rich statistical theories. For a general review of the empirical process, we refer to Durbin [7] and the textbooks, such as Shorack and Wellner [27], van der Vaart and Wellner [30], and van de Geer [29].

In handling the continuous time stochastic process for statistical inference, one can choose one of the two approaches: the one is to use all continuous sample path as in Kutoyants [15] and the other is to use the sampling scheme as in Flrores-Zmirou [9], Yoshida [31] and Kessler [12]. Here, we adopt the latter since it makes the situation much more tractable in defining the residuals and investigating the asymptotic behavior of the residual empirical process. The idea to employ such a residual empirical process is based on the fact that if the data is generated from a pure diffusion model, the residuals obtained from the sampled observations should behave like normal r.v.'s and thus, through the normality test one can judge the well-fitness of the diffusion model. The residual empirical process has been studied in time series models by many authors. We refer to Boldin ([3], [4]), Kreiss [14], Ling [22], Lee and Wei [21], Lee and Taniguch [19], Koul [13] and the papers cited therein. However, to our knowledge, there are no literatures handling the residual empirical process in diffusion models, which, therefore, deserves special attention considering its wide applicability.

In Section 2, we state the main result. It is shown that the residual empirical process converges weakly to a Brownian bridge under regularity conditions. In Section 3, we provide the proof for the main theorem.

2. Main result

Let us consider the stochastic differential equation

$$dX_t = a(X_t;\theta)dt + \sigma dW_t, \quad X_0 = x_0, \ t \geq 0, \tag{2.1}$$

where θ is a p-dimensional unknown parameter, σ is a constant, a is a real valued function, and $\{W_t; t \geq 0\}$ is a standard Wiener process. This model has been studied by Florens-Zmirou [9] and the result is summarized in Prakasa Rao [23], pages 143–144 and 153–158. For instance, the model in (2.1) includes the Ornstein-Uhlenbeck process

$$dX_t = (\alpha - \mu X_t)dt + \sigma dW_t.$$

We assume that the following conditions hold.

(A1) There exist constants $C, m > 0$ such that for any θ, x, y,

$$\begin{aligned}
|a(x;\theta) - a(y;\theta)| &\leq C|x-y|, \\
\sup_{\theta' \in N_\theta} ||\dot{a}(x;\theta')|| &\leq C(1+|x|^m), \\
\sup_{\theta' \in N_\theta} ||\ddot{a}(x;\theta')|| &\leq C(1+|x|^m),
\end{aligned}$$

where $\dot{a} = \partial a/\partial\theta$, $\ddot{a} = \partial a^2/\partial\theta^2$, and N_θ is an open neighborhood of θ.

(A2) $\sup_t E|X_t|^\gamma < \infty$ for all $\gamma > 0$.

The issue here is to test the adequacy of the model in (1) for a give time series data. Suppose that $\{X_t\}$ is observed at discrete times $t_i = ih_n, i = 1, \ldots, n$, where $\{h_n\}$ is a sequence of positive real numbers such that $h_n \to 0$ and $nh_n \to \infty$. Let $\hat{\theta}_n$ be an estimator of θ such that $\sqrt{nh_n}(\hat{\theta}_n - \theta)$ is asymptotically normal. A sufficient condition for the normality condition can be found in Florens-Zmirou [9] and Kessler [12]. We implicitly assume those without specification. Besides, we impose the condition on the sequence $\{h_n\}$:

(A3) $nh_n^2 \to 0$ and $(nh_n)^{1/2}/\log n \to \infty$ as $n \to \infty$.

By noticing

(2.2)
$$\begin{aligned}
X_{t_i} - X_{t_{i-1}} &= h_n a(X_{t_{i-1}};\theta) + \int_{t_{i-1}}^{t_i} (a(X_s;\theta) - a(X_{t_{i-1}};\theta))ds + \sigma(W_{t_i} - W_{t_{i-1}}) \\
&\simeq h_n a(X_{t_{i-1}};\theta) + \sigma\sqrt{h_n} r_i,
\end{aligned}$$

where r_i are iid standard normal r.v.'s. we define the residuals

$$\hat{r}_i = \{X_{t_i} - X_{t_{i-1}} - h_n a(X_{t_{i-1}};\hat{\theta}_n)\}/\hat{\sigma}_n\sqrt{h_n}, \tag{2.3}$$

where

$$\hat{\sigma}_n^2 = \frac{1}{nh_n}\sum_{i=1}^{n}(X_{t_i} - X_{t_{i-1}} - h_n a(X_{t_{i-1}};\hat{\theta}_n))^2, \tag{2.4}$$

which is a consistent estimator of σ^2 (cf. Flores-Zmirou [9]). The residual empirical process is then defined by

$$Y_n(x) = \frac{1}{\sqrt{n_h}}\sum_{i=1}^{n_h}\{I(\hat{r}_i \leq x) - \Phi(x)\}, \ x \in \mathbf{R}, \tag{2.5}$$

where n_h is the largest integer that does not exceed nh_n. The main goal of this section is to figure out the limiting distribution of Y_n.

Put

$$\begin{aligned}
\eta_{ni} &= \frac{W_{t_i} - W_{t_{i-1}}}{\sqrt{h_n}}, \\
\Delta_{ni} &= \int_{t_{i-1}}^{t_i} \{a(X_s;\theta) - a(X_{t_{i-1}};\theta)\}ds, \\
d_{ni} &= a(X_{t_{i-1}};\hat{\theta}_n) - a(X_{t_{i-1}};\theta).
\end{aligned}$$

Note that Δ_{ni} can be viewed as a model bias in the regression model in (2.1). By (A1),

$$\begin{aligned}
E|\Delta_{ni}| &\le C' \int_{t_{i-1}}^{t_i} E\left|X_s - X_{t_{i-1}}\right| ds \\
&\le C' \int_{t_{i-1}}^{t_i} E^{1/2}\left|X_s - X_{t_{i-1}}\right|^2 ds \;=\; O(h_n^{\frac{3}{2}}), \quad C' > 0,
\end{aligned} \tag{2.6}$$

where we have used Lemma 3.4.2 of Prakara Rao [23], p.156. Similarly, we can show that for every integer $k \ge 1$,

$$E|\Delta_{ni}|^{2k} = O(h_n^{2k+1}). \tag{2.7}$$

Write that $Y_n(x) = I_n(x) + II_n(x) + III_n(x)$, where

$$\begin{aligned}
I_n(x) &= \frac{1}{\sqrt{n_h}} \sum_{i=1}^{n_h} \{I(\eta_{ni} \le x) - \Phi(x)\}, \\
II_n(x) &= \frac{1}{\sqrt{n_h}} \sum_{i=1}^{n_h} \left\{ \Phi\left(\frac{\hat{\sigma}_n}{\sigma}x - \frac{\Delta_{ni}}{\sigma\sqrt{h_n}} - \frac{\sqrt{h_n}}{\sigma}d_{ni}\right) - \Phi(x) \right\}, \\
III_n(x) &= \frac{1}{\sqrt{n_h}} \sum_{i=1}^{n_h} \left\{ I\left(\eta_{ni} \le \frac{\hat{\sigma}_n}{\sigma}x - \frac{\Delta_{ni}}{\sigma\sqrt{h_n}} - \frac{\sqrt{h_n}}{\sigma}d_{ni}\right) \right. \\
&\quad \left. -\Phi\left(\frac{\hat{\sigma}_n}{\sigma}x - \frac{\Delta_{ni}}{\sigma\sqrt{h_n}} - \frac{\sqrt{h_n}}{\sigma}d_{ni}\right) + \Phi(x) - I(\eta_{ni} \le x) \right\}.
\end{aligned}$$

Since by (A1)-(A3) and the mean value theorem,

$$\sqrt{n}(\hat{\sigma}_n^2 - \sigma^2) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\eta_{ni}^2 - 1)\sigma^2 + o_P(1),$$

we can write that

$$\begin{aligned}
II_n(x) &= \frac{1}{\sqrt{n_h}} \sum_{i=1}^{n_h} \left(\frac{\hat{\sigma}_n}{\sigma} - 1\right) x\phi(x) + \sum_{i=1}^{3} II_{ni}(x), \\
&= \sum_{i=1}^{3} II_{ni}(x) + o_P(1),
\end{aligned}$$

where

$$\begin{aligned} II_{n1}(x) &= \frac{-1}{\sqrt{n_h}} \sum_{i=1}^{n_h} \frac{\Delta_{ni}}{\sigma\sqrt{h_n}} \phi(x), \\ II_{n2}(x) &= \frac{-1}{\sqrt{n_h}} \sum_{i=1}^{n_h} \frac{\sqrt{h_n}}{\sigma} d_{ni} \phi(x), \\ II_{n3}(x) &= \frac{1}{2\sqrt{n_h}} \sum_{i=1}^{n_h} \left(\frac{\Delta_{ni}}{\sigma\sqrt{h_n}} + \frac{\sqrt{h_n}}{\sigma} d_{ni} \right)^2 \phi^{'}(\xi_{ni}(x)), \end{aligned}$$

where $\xi_{ni}(x)$ is a number lying between x and $\frac{\hat{\sigma}_n}{\sigma} x - \frac{\Delta_{ni}}{\sigma\sqrt{h_n}} + \frac{h_n}{\sigma} d_{ni}$. Note that by (2.6) and (A.3),

$$\sup_x |II_{n1}(x)| = o_P(1). \tag{2.8}$$

Using Taylor's theorem, we can write $d_{ni} = d_{ni}^{(1)} + d_{ni}^{(2)}$ with

$$d_{ni}^{(1)} = \dot{a}(X_{t_{i-1}}; \theta)(\hat{\theta_n} - \theta) \text{ and } d_{ni}^{(2)} = \frac{1}{2}(\hat{\theta_n} - \theta)^{'} \ddot{a}(X_{t_{i-1}}; \theta_{ni}^*)(\hat{\theta_n} - \theta),$$

where θ_{ni}^* is an intermediate point of $\hat{\theta_n}$ and θ. Since by (A1),

$$\sqrt{n_h} ||\hat{\theta}_n - \theta||^2 \max_{1 \le i \le n_h} \sup_{\theta' \in N_\theta} ||\ddot{a}(X_{t_{i-1}}; \theta^{'})|| = o_P(1),$$

we have

$$\sup_x \left| \frac{1}{\sqrt{n_h}} \sum_{i=1}^{n_h} \frac{\sqrt{h_n}}{\sigma} d_{ni}^{(2)} \right| = o_P(1). \tag{2.9}$$

Also,

$$\frac{1}{\sqrt{n_h}} \sum_{i=1}^{n_h} \frac{\sqrt{h_n}}{\sigma} d_{ni}^{(1)} \phi(x) = o_P(1).$$

Hence,

$$\sup_x |II_{n2}(x)| = o_P(1). \tag{2.10}$$

In a similar fashion, we can see that

$$\sup_x |II_{n3}(x)| = o_P(1). \tag{2.11}$$

Combining (2.8), (2.10), (2.11), and the fact that

$$\sup_x |III_n(x)| = o_P(1), \tag{2.12}$$

of which proof is in Section 3, we have the following.

Theorem 2.1. *Assume that (A1) – (A3) hold. Then, as $n \to \infty$,*

$$Y_n(x) = \frac{1}{\sqrt{n_h}} \sum_{i=1}^{n_h} \{I(\eta_{ni} \leq x) - \Phi(x)\} + \xi_n(x),$$

where $\sup_x |\xi_n(x)| = o_P(1)$. *Thus,* $Y_n(\Phi^{-1}(u))$ *converges weakly to a Brownian bridge* $W^o(u)$.

Remark. One can ask whether n_h in Y_n can be replaced by the sample size n since it is always desired to use the whole observations. According to our analysis, the answer is negative since the model bias terms Δ_{ni} cannot be regarded negligible uniformly. Recall that h_n should not be so small since nh_n must go to infinity, and so there is a limit to reducing the model bias in view of (2.6). Using n_h may not cause a serious trouble in actual practice as far as a large number of observations are utilized. If the sample size is small, the whole statistical inference may not be reliable since the $\hat{\theta}_n$ itself is unreliable.

Usually, the estimator of scale parameter affects the limiting distribution as seen in Durbin [7]. In our case, however, it disappears as we saw in Theorem 2.1. This happens since the $\hat{\sigma}_n^2$ is $\sqrt{n}$-consistent unlike the estimator of θ, and behaves as if it were a super efficient estimator since only n_h number of observations are involved in the residual empirical process.

One can utilize the above result to perform a goodness of fit test. For example, one can use the Kolmogorov-Smirnov test and Cramer-von Mises test since by the continuous mapping theorem,

$$KS_n := \sup_{1 \leq u \leq 1} |Y_n(\Phi^{-1}(u)| \to \sup_{0 \leq u \leq 1} |W^o(u)|$$

and

$$CV_n := \int_0^1 |Y_n(\Phi^{-1}(u)|^2 du \to \int_0^1 |W^0(u)|^2 du \text{ in distribution.}$$

We reject the null hypothesis $\mathcal{H}_0$: $\{X_t\}$ follows the diffusion model in (2.1) if KS_n and CV_n are large.

Here, we only considered the case that the dispersion part is a constant. In fact, one may consider the diffusion model with more general dispersion coefficients, say,

$$dX_t = a(X_t;\theta)dt + b(X_t : \sigma)dW_t. \tag{2.13}$$

This model is used widely in financial time series analysis, and becomes Feller's square model when $a(X_t;\theta) = \alpha(\beta - X_t)$ and $b(X_t;\sigma) = \sigma\sqrt{X_t}$. Though one can easily guess that our result will extend to this model, a careful analysis is needed according to the result in Lee and Taniguch [19] which shows that the varying dispersion components in ARCH models affect the limiting distribution of the residual empirical process. We leave the task of extension as a future study.

3. Proof

In this section, we prove $\sup_x |III_n(x)| = o_P(1)$. Observe that by the monotonicity of the indicator function,

$$I\left(\eta_{ni} \le \frac{\hat{\sigma}_n}{\sigma}x + \frac{\sqrt{h_n}}{\sigma}d_{ni}^{(1)} - \max_{1\le i\le n_h}\left\{\frac{\sqrt{h_n}}{\sigma}|d_{ni}^{(2)}| + \frac{|\Delta_{ni}|}{\sigma\sqrt{h_n}}\right\}\right)$$
$$\le I\left(\eta_{ni} \le \frac{\hat{\sigma}_n}{\sigma}x - \frac{\Delta_{ni}}{\sigma\sqrt{h_n}} + \frac{\sqrt{h_n}}{\sigma}d_{ni}\right)$$
$$\le I\left(\eta_{ni} \le \frac{\hat{\sigma}_n}{\sigma}x + \frac{\sqrt{h_n}}{\sigma}d_{ni}^{(1)} + \max_{1\le i\le n_h}\left\{\frac{\sqrt{h_n}}{\sigma}|d_{ni}^{(2)}| + \frac{|\Delta_{ni}|}{\sigma\sqrt{h_n}}\right\}\right).$$

Also, note that from (2.7),

$$E\{\sqrt{n}\max_{1\le i\le n_h}|\Delta_{ni}|\}^4 = O((nh_n^2)^3) = o(1). \tag{3.1}$$

From (2.9), (3.1) and Taylor's theorem, we have that

$$\frac{1}{\sqrt{n_h}}\sum_{i=1}^{n_h}\Phi\left(\frac{\hat{\sigma}_n}{\sigma}x - \frac{\Delta_{ni}}{\sigma\sqrt{h_n}} + \frac{\sqrt{h_n}}{\sigma}d_{ni}\right)$$
$$= \frac{1}{\sqrt{n_h}}\sum_{i=1}^{n_h}\Phi\left(\frac{\hat{\sigma}_n}{\sigma}x + \frac{\sqrt{h_n}}{\sigma}d_{ni}^{(1)}\right) + \rho_n(x)$$
$$= \frac{1}{\sqrt{n_h}}\sum_{i=1}^{n_h}\Phi\left(\frac{\hat{\sigma}_n}{\sigma}x + \frac{\sqrt{h_n}}{\sigma}d_{ni}^{(1)} \pm \max_{1\le i\le n_h}\left\{\frac{\sqrt{h_n}}{\sigma}|d_{ni}^{(2)}| + \frac{|\Delta_{ni}|}{\sigma\sqrt{h_n}}\right\}\right) + \tilde{\rho}_n(x),$$

where $\sup_x|\rho_n(x)| = o_P(1)$ and $\sup_x|\tilde{\rho}_n(x)| = o_P(1)$. In view of this and the following two facts:

$$\sup_x\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{I(\eta_{ni} \le x + \xi_{ni}) - \Phi(x+\xi_{ni}) + \Phi(x) - I(\eta_{ni}\le x)\right\}\right| = o_P(1)$$

for any double array of r.v.'s $\{\xi_{ni}\}$ with $\sqrt{n}\max_{1\le i\le n}|\xi_{ni}| = o_P(1)$ (cf. Lemma 2.2 of Lee and Wei [21]), and

$$\sup_x\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{I(\eta_{ni} \le \frac{\hat{\sigma}_n}{\sigma}x) - \Phi(\frac{\hat{\sigma}_n}{\sigma}x) + \Phi(x) - I(\eta_{ni}\le x)\right\}\right| = o_P(1)$$

(cf. Billingsley [2], p. 106), we can see that (2.12) follows if

$$\sup_x|\widetilde{III}_n(x)| = o_P(1), \tag{3.2}$$

where

$$\widetilde{III}_n(x) = \frac{1}{\sqrt{n_h}}\sum_{i=1}^{n_h}\left\{I\left(\eta_{ni} \le x + \frac{\sqrt{h_n}}{\sigma}d_{ni}^{(1)}\right) - \Phi\left(x + \frac{\sqrt{h_n}}{\sigma}d_{ni}^{(1)}\right) + \Phi(x) - I(\eta_{ni}\le x)\right\}.$$

Let x_j be such that $-\infty = x_0 < \cdots < x_{N_n} = \infty$ and $\Phi(x_j) = \frac{j}{N_n}, j = 0, \ldots, N_n$, where $N_n = n^2$. Since

$$\max_{1\le j\le N_n} \sup_{x_j\le x\le x_{j+1}} \left| \frac{1}{\sqrt{n_h}} \sum_{i=1}^{n_h} \{I(\eta_{ni} \le x_j) - \Phi(x_j) + \Phi(x) - I(\eta_{ni} \le x)\} \right| = o_P(1),$$

(3.2) follows if

$$\max_{1\le j\le N_n} |III'_{nj}| = o_P(1), \tag{3.3}$$

where

$$III'_{nj} = \frac{1}{\sqrt{n}_h} \sum_{i=1}^{n_h} \left\{ I\left(\eta_{ni} \le x_j + \frac{\sqrt{h}_n}{\sigma} d_{ni}^{(1)}\right) - \Phi\left(x_j + \frac{\sqrt{h}_n}{\sigma} d_{ni}^{(1)}\right) + \Phi(x_j) - I(\eta_{ni} \le x_j) \right\}.$$

Since $\sqrt{n}_h(\hat{\theta}_n - \theta) = O_P(1)$, (4.3) holds if for any $K > 0$,

$$\begin{aligned} \sup_{\|\mathbf{s}\|\le K} \max_{1\le j\le N_n} \Bigg| \frac{1}{\sqrt{n}_h} \sum_{i=1}^{n_h} \{ & I(\eta_{ni} \le x_j + n^{-1/2} \mathbf{s}' \dot{a}(X_{t_{j-1}}; \theta)) \\ & -\Phi(x_j + n^{-1/2} \mathbf{s}' \dot{a}(X_{t_{j-1}}; \theta)) + \Phi(x_j) - I(\eta_{ni} \le x_j)\} \Bigg| = o_P(1). \end{aligned} \tag{3.4}$$

Partitioning $\mathcal{S}_k = \{\mathbf{y} \in \mathcal{R}^p; \|\mathbf{y}\| \le K\}$ by vertices $(y_{1\nu_1}, \ldots, y_{p\nu_p})$ with $y_{q,\nu_q} = -K + \frac{2K\nu_q}{n^2}, \nu_q = 1, \ldots, n^2$, one can obtain the disjoint rectangles $Q_1, \ldots, Q_{\tau_n}$ with $Q_l \cap S_k \neq \phi$ and $S_k \subset \cup_{l=1}^{\tau_n} Q_l$, where $\tau_n = O(n^{2p})$. Then, one can readily show that (4.4) holds if

$$\begin{aligned} W_n : &= \max_{1\le l\le \tau_n} \max_{1\le j\le N_n} \Bigg| \frac{1}{\sqrt{n_h}} \sum_{i=1}^{n_h} \{ I(\eta_{ni} \le x_j + n^{-1/2} u_{il}^{\pm}) \\ & \qquad -\Phi(x_j + n^{-1/2} u_{il}^{\pm}) + \Phi(x_j) - I(\eta_{ni} \le x_j)\} \Bigg| = o_P(1), \end{aligned}$$

where

$$u_{il}^{+} = \sup_{\mathbf{s}\in Q_l} \mathbf{s}' \dot{a}(X_{t_{i-1}}; \theta) \text{ and } u_{il}^{-} = \inf_{\mathbf{s}\in Q_l} \mathbf{s}' \dot{a}(X_{t_{i-1}}; \theta).$$

Here, we only prove the u_{il}^{+} case. Put

$$e_{ni} = I(\eta_{ni} \le x_j + n^{-1/2} u_{il}^{+}) - \Phi(x_j + n^{-1/2} u_{il}^{+}) + \Phi(x_j) - I(\eta_{ni} \le x_j).$$

In view of the fact that $\sum_{i=1}^{n} |u_{il}^{+}| = O_P(n)$ for any $\delta > 0$, there exists $B > 0$ such that $P(V_n) > 1 - \delta$, where $V_n = (\sum_{i=1}^{n_h} |u_{il}^{+}| \le Bn_h)$. Then if we put

$$e'_{ni} = e_{ni} I(\sum_{k=1}^{i} |u_{kl}^{+}| \le Bn_h),$$

$\{e'_{ni}, \mathcal{F}_{n_i}\}$ forms a sequence of martingale differences where $\mathcal{F}_{n_i} = \sigma(\eta_{n_1}, \ldots, \eta_{n_i})$. Note that

$$\begin{aligned}\sum_{i=1}^{n_h} E((e'_{ni})^2|\mathcal{F}_{n_{i-1}}) &\le n^{-1/2}K\sum_{i=1}^{n_h}|u_{il}^+|I(\sum_{k=1}^{i}|u_{kl}^+| \le Bn_h)\\ &\le B'\sqrt{n}h_n,\ B' > 0.\end{aligned}$$

Applying Freedman's inequality for martingales, we obtain that for any $\lambda > 0$,

$$\begin{aligned}P(|\sum_{i=1}^{n_h} e'_{ni}| > \lambda\sqrt{n_h}) &\le 2\exp\left\{-(1/2)n_h\lambda^2/(B'\sqrt{n}h_n + (2/3)\lambda\sqrt{n_h})\right\}\\ &= O(e^{-\kappa\sqrt{n_h}}),\end{aligned}$$

where κ is a positive constant. Since $P(e_{ni} \neq e'_{ni}$ for some $i = 1, \ldots, n_h, V_n) = 0$, we have

$$\begin{aligned}P(W_n > \lambda) &\le P(W_n > \lambda, V_n) + P(V_n^c)\\ &\le \sum_{l=1}^{r_n}\sum_{j=1}^{N_n} P\left(\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n} e_{n_i}\right| > \lambda, V_n\right) + \delta\\ &\le \sum_{l=1}^{r_n}\sum_{j=1}^{N_n} P\left(\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n} e'_{n_i}\right| > \lambda\right) + \delta\\ &= o(1) + \delta\end{aligned}$$

by the second condition of (A3). Since δ is arbitrarily chosen, we have

$$\lim_{n\to\infty} P(W_n > \lambda) = 0.$$

This completes the proof. □

Acknowledgements. We wish to acknowledge that this work was supported by Korea Research Foundation Grant 2003-070-C00008.

References

[1] O. E. Barndorff-Nielsen, T. Mikosch, and S. I. Resnick, *Lévy Processes*, Theory and Applications, Birkhäuser Boston, Inc., Boston, MA, 2001.

[2] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, Inc., New York-London-Sydney, 1968.

[3] M. V. Boldin, *An estimate of the distribution of the noise in an autoregressive scheme*, Teor. Veroyatnost. i Primenen. **27** (1982), no. 4, 805–810.

[4] ———, *On empirical processes in heteroscedastic time series and their use for hypothesis testing and estimation*, Math. Methods Statist. **9** (2000), no. 1, 65–89.

[5] T. Chan, *Pricing contingent claims on stocks driven by Lévy processes*, Ann. Appl. Probab. **9** (1999), no. 2, 504–528.

[6] P. Cont and P. Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall/CRC Financial Mathematics Series, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[7] J. Durbin, *Weak convergence of the sample distribution function when parameters are estimated*, Ann. Statist. **1** (1973), 279–290.
[8] E. Eberlin and S. Raible, *Term structure models driven by general Lévy processes*, Math. Finance **9** (1999), no. 1, 31–53.
[9] D. Florens-Zmirou, *Approximate discrete-time schemes for statistics of diffusion processes*, Statistics **20** (1989), no. 4, 547–557.
[10] D. Hong and I.-S. Wee, *Convergence of jump-diffusion models to the Black-Scholes model*, Stochastic Anal. Appl. **21** (2003), no. 1, 141–160.
[11] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Graduate Texts in Mathematics, 113. Springer-Verlag, New York, 1988.
[12] M. Kessler, *Estimation of an ergodic diffusion from discrete observations*, Scand. J. Statist. **24** (1997), no. 2, 211–229.
[13] H. L. Koul, *Weighted Empirical Processes in Dynamic Nonlinear Models*, Lecture Notes in Statistics, 166. Springer-Verlag, New York, 2002.
[14] J. P. Kreiss, *Estimation of the distribution function of noise in stationary processes*, Metrika **38** (1991), no. 5, 285–297.
[15] Y. Kutoyants, *Statistical Inference for Ergodic Diffusion Processes*, Springer Series in Statistics. Springer-Verlag London, Ltd., London, 2004.
[16] S. Lee, J. Ha, O. Na, and S. Na, *The cusum test for parameter change in time series models*, Scand. J. Statist. **30** (2003), no. 4, 781–796.
[17] S. Lee and O. Na, *Test for parameter change in stochastic processes based on conditional least-squares estimator*, J. Multivariate Anal. **93** (2005), no. 2, 375–393.
[18] S. Lee, Y. Nishiyama, and N. Yoshida, *Test for parameter change in diffusion processes by cusum statistics based on one-step estimators*, Ann. Inst. Statist. Math. **58** (2006), no. 2, 211–222.
[19] S. Lee and M. Taniguchi, *Asymptotic theory for ARCH-SM models: LAN and residual empirical processes*, Statist. Sinica 15 (2005), no. 1, 215–234.
[20] S. Lee, Y. Tokutsu, and K. Maekawa, *The cusum test for parameter change in regression models with ARCH errors*, J. Japan Statist. Soc. **34** (2004), no. 2, 173–188.
[21] S. Lee and C. Z. Wei, *On residual empirical processes of stochastic regression models with applications to time series*, Ann. Statist. **27** (1999), no. 1, 237–261.
[22] S. Ling, *Weak convergence of the sequential empirical processes of residuals in nonstationary autoregressive models*, Ann. Statist. **26** (1998), no. 2, 741–754.
[23] B. L. S. Prakasa Rao, *Statistical Inference for Diffusion Type Processes*, Kendall's Library of Statistics, 8. Edward Arnold, London; Oxford University Press, New York, 1999.
[24] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Translated from the 1990 Japanese original. Revised by the author. Cambridge Studies in Advanced Mathematics, 68. Cambridge University Press, Cambridge, 1999.
[25] W. Schoutens, *Lévy Processes in Finance*, John Wiley & Sons, Inc., New York, 2003.
[26] A. N. Shiryayev, *Essentials of Stochastic Finance*, Facts, models, theory. Translated from the Russian manuscript by N. Kruzhilin. Advanced Series on Statistical Science & Applied Probability, 3. World Scientific Publishing Co., Inc., River Edge, NJ, 1999.
[27] G. R. Shorack and J. Wellner, *Empirical Processes with Applications to Statistics*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986.
[28] H. C. Jr. Thode, *Testing for Normality*, Statistics: Textbooks and Monographs, 164. Marcel Dekker, Inc., New York, 2002.
[29] S. van de Geer, *Empirical Processes in M-Estimators*, Cambridge University Press, Cambridge, 1999.

[30] A. W. van der Vaat and J. A. Wellner, *Weak Convergence and Empirical Processes*, With applications to statistics. Springer Series in Statistics. Springer-Verlag, New York, 1996.

[31] N. Yoshida, *Estimation for diffusion processes from discrete observation*, J. Multivariate Anal. **41** (1992), no. 2, 220–242.

SANGYEOL LEE
DEPARTMENT OF STATISTICS
SEOUL NATIONAL UNIVERSITY
SEOUL 151-742, KOREA
E-mail address: sylee@stats.snu.ac.kr

IN-SUK WEE
DEPARTMENT OF MATHEMATICS
KOREA UNIVERSITY
SEOUL 136-701, KOREA
E-mail address: iswee@korea.ac.kr