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# Modified Equivalent Radius Approach in Evaluating Stress-Strain Relationship in Torsional Test

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## Abstract

Determination of stress-strain relationship in torsional tests is complicated due to nonuniform stress-strain variation occurring linearly with the radius in a soil specimen in torsion. The equivalent radius approach is adequate when calculating strain at low to intermediate strains, however, the approach is less accurate when performing the test at higher strain levels. The modified equivalent radius approach was developed to account for the problem more precisely. This approach was extended to generate the plots of equivalent radius ratio versus strain using modified hyperbolic and Ramberg-Osgood models. Results showed the effects of soil nonlinearity on the equivalent radius ratio curves were observed. Curve fitting was also performed to find the stress-strain relationship by fitting the theoretical torque-rotation relationship to measured torque-rotation relationship.

***Key words*** : Shear Stress, Shear Strain, Torsional Shear Test, Equivalent Radius Approach, Curve Fitting, Nonlinear Model

## 1. Introduction

The successful performance of dynamic response analyses is dependent on the incorporation of suitably representative soil properties, and these properties are nonlinearly strain-dependent. Considerable work has been conducted over the past decades in soil dynamics to obtain accurate dynamic properties of soil deposit, assess the dynamic behavior of soils and develop representative constitute relations.

The torsional shear (TS) test is one of the most effective methods for measuring the cyclic (dynamic) properties of soil over a wide range of strains ($10^{-5}$%-1%). It measures the rotation of the specimen and torque applied to the specimen so that the stress-strain hysteresis loop is determined by means of converting these data (rotation and torque) to stress and stain with the calibration factors. These calibration factors are somewhat difficult to determine because of the nonuniform stress-strain distribution in the radial direction of the horizontal plane of a soil specimen during torsional loading as illustrated in Fig. 1. Chen and Stokoe (1979) developed equivalent radius approach to investigate this nonuniform distribution of strain in soil specimen.

Sasanakul (2005) showed that the equivalent radius approach is not accurate at higher strains and it does not effect soil nonlinearity. The modified equivalent radius approach was developed to account for the problems (2005).

This paper presented the modified equivalent radius approach using hyperbolic, modified hyperbolic, and Ramberg-Osgood models to generate equivalent radius ratio, $R_{eq}$ values for shear modulus. Curve fitting techniques are also used to match the theoretical torque-rotation relationship with the measured torque-rotation relationship to obtain the best fit model parameters. These model parameters are then used to develop the stress-strain relationship for the soil. The application of curve fitting was illustrated.

## 2. Nonlinear Soil Behavior in Torsional Shear Test

In torsional shear test, slow cyclic torsional loading with a given frequency, generally below 10 Hz, is applied at the top of the specimen. Instead of determining the resonant frequency, the stress-strain hysteresis loop is determined from measuring the torque-twist response of the specimen. Proximitors are used to measure the angle of twist while the voltage applied to coil is calibrated to yield torque. Shear modulus is calculated from the slope of a line through the end points of the hysteresis loop. Details of description of resonant column and torsional shear (RC/TS) device are presented by Ni (1987), Kim (1991), and Darendelli (1997). The schematic diagram of torsional shear apparatus is illustrated in Fig. 2.

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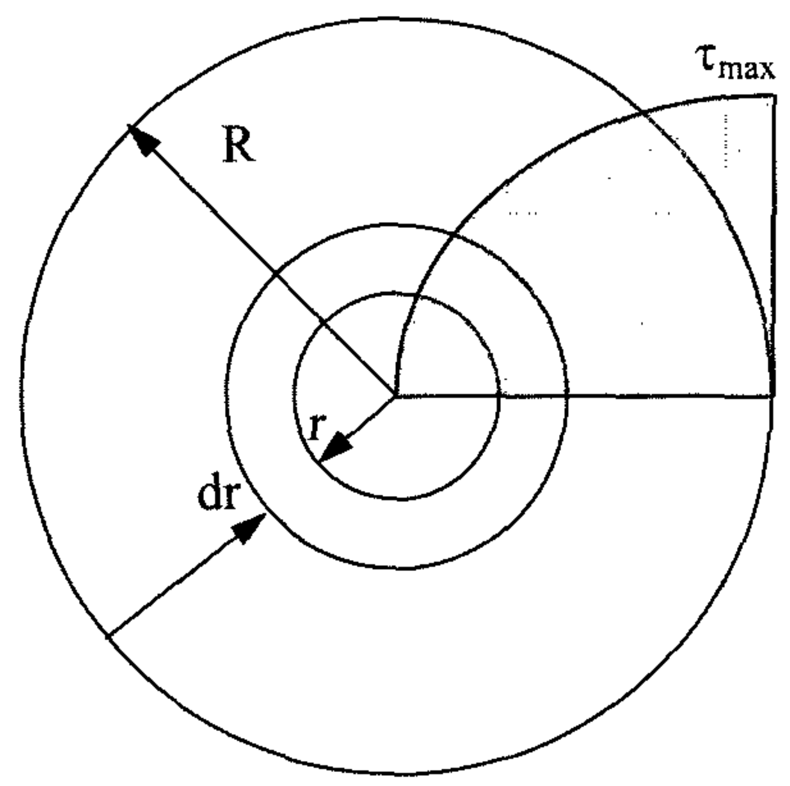



Fig. 1. Transverse section of shearing stress distribution in soil column for TS test

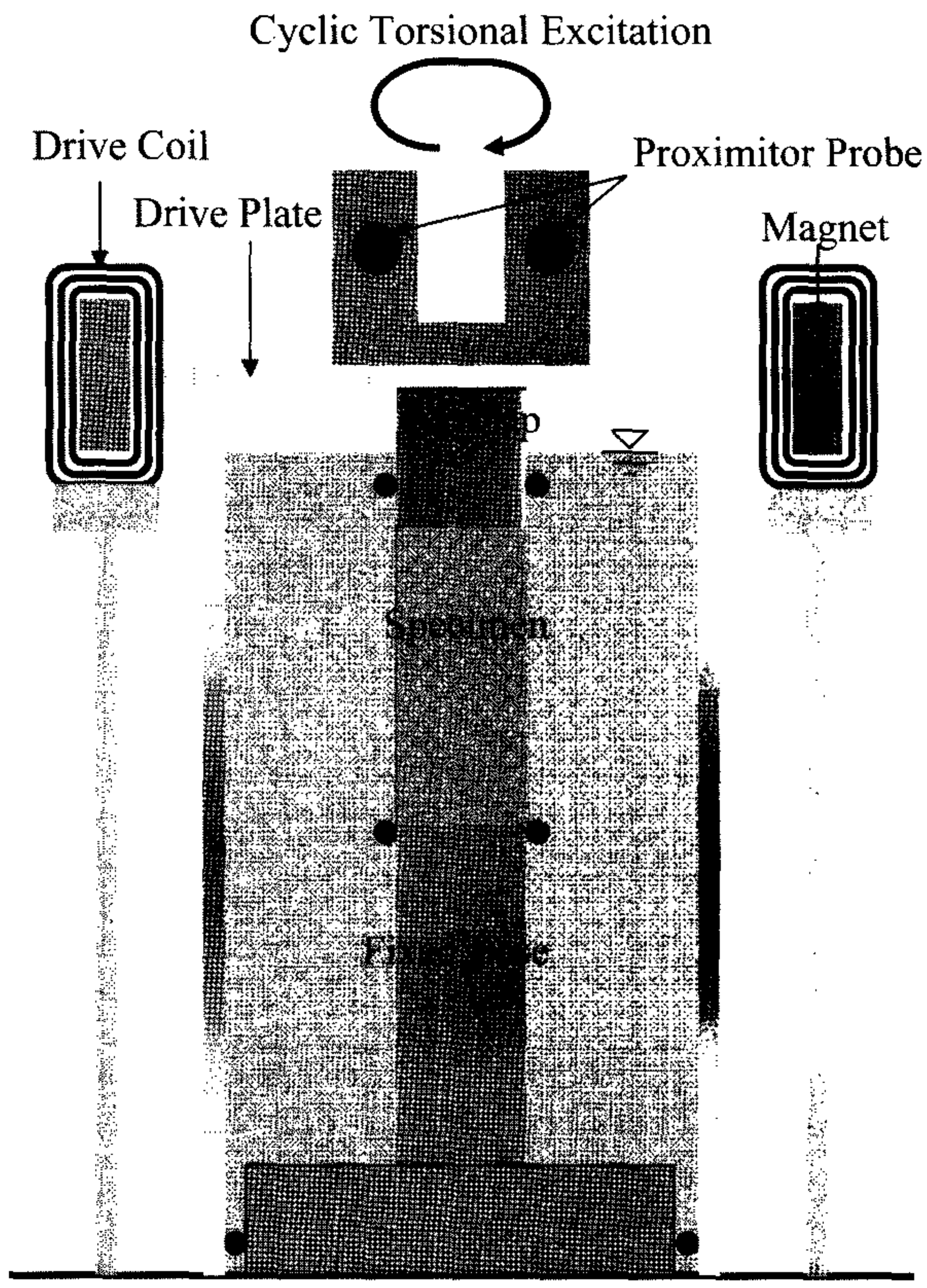



Fig. 2. Schematic diagram of torsional shear apparatus

One of the weaknesses of the RC/TS method is the fact that stress-strain distribution is not uniform in the radial direction of the horizontal plane of a soil specimen during torsion. The theory of elasticity was utilized originally to measure the small strain soil behavior. However, soils behave nonlinearly when subjected to strain levels as small as $10^{-5}$%. The uniform stress-strain effect becomes significant when testing on soil at medium to high strain levels. Using hollow rather solid specimens can minimize this effect however the hollow specimens are not commonly used because of trimming and handling difficulties (Saada, 1988).

The shearing strain varies in radial direction, and may be expressed as a function of the distance from the longitudinal axis as shown in Fig. 3.

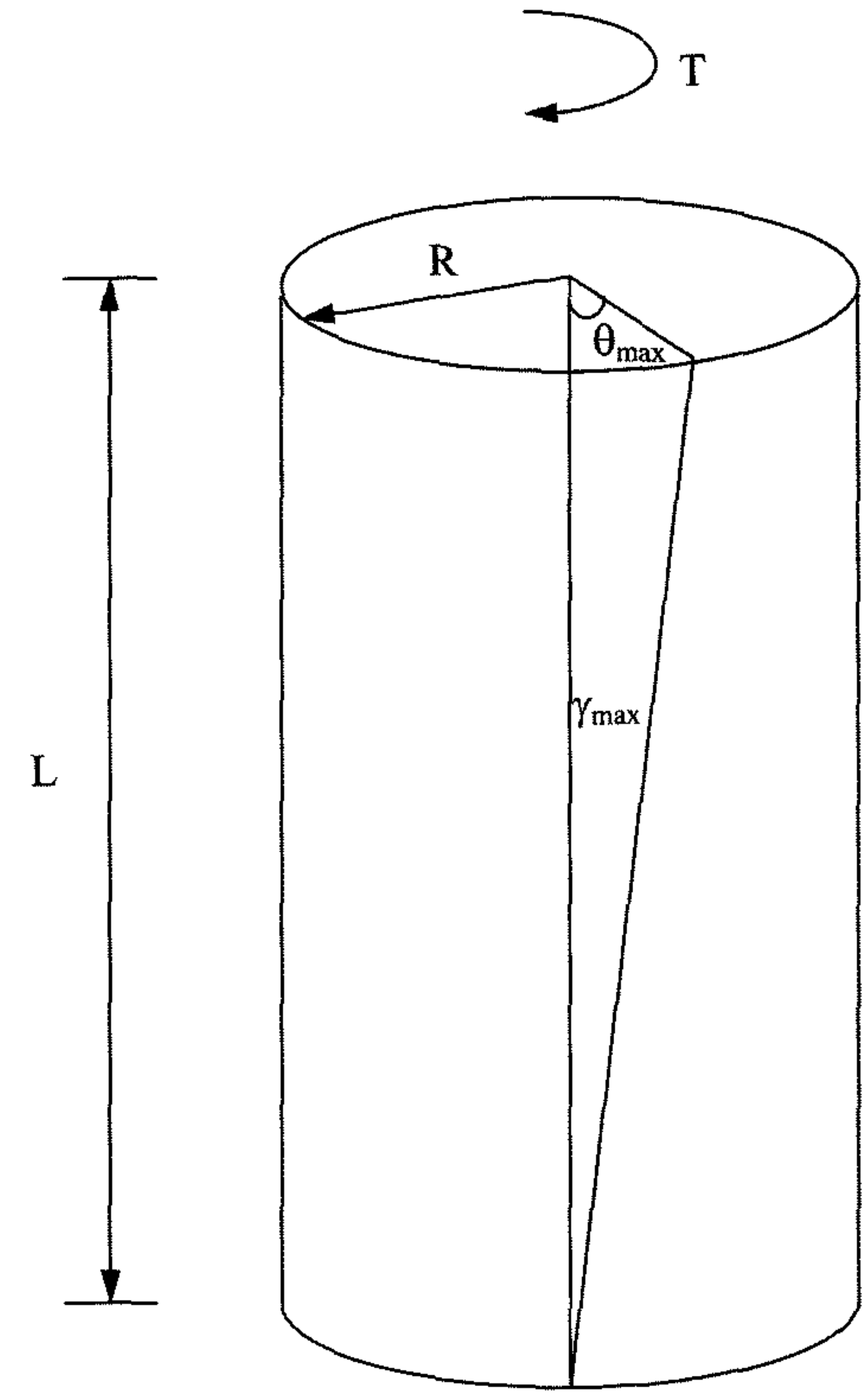



Fig. 3. Longitudinal section of shearing stress in soil column for TS test

The equivalent shearing strain, $g_{eq}$ is represented by:

$$\gamma_{ed} = r_{ed} \times \theta_{max} / L \tag{1}$$

where $r_{eq}$ is equivalent radius, $\theta_{max}$ is angle of twist at the top of the specimen, and $L$ is length of the specimen. By linearizing the problem, the effective shear modulus, $G_{eff}$ is expressed by:

$$G_{eff} = \frac{T \times L}{I_P \times \theta_{max}} \tag{2}$$

where $T$ is torque applied to soil specimen and $I_p$ is area polar moment of inertia.

Therefore, considerations are needed to identify a specific strain ($\gamma_{eq}$) associated with the effective shear modulus ($G_{eff}$) at given angle of twist ($\theta_{max}$).

## 3. Equivalent Radius Approach

To evaluate stress-strain relationship in TS tests, Hardin and Drenevich (1972) recommended using the average stress and strain in the soil specimen. The average shear stress and strain are obtained from the following equations:

$$\gamma \approx 2/3R(\theta/L) \tag{3}$$

$$\tau \approx 2/3R(T/I_P) \tag{4}$$

This approach is based on the summation of linear varia-

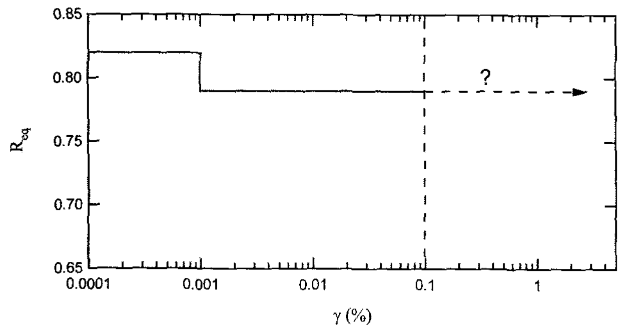



Fig. 4. Equivalent radius ratios, $R_{eq}$ values according to Chen and Stokoe (1979)

tion of strain over radius of soil specimen but does not account for the nonlinear stress-strain soil behavior.

The equivalent radius approach proposed by Chen and Stokoe (1979) is based on the assumption that the representative stress and strain in soil specimen in torsion occurs at a radius called the equivalent radius, $r_{eq}$. They correct the radius ($R$) in term of the equivalent radius ratio ($R_{eq}$), which is defined as the ratio of the equivalent radius, $r_{eq}$ and the radius, R This approach is considered reasonable in practice since the nonuniformity of strain can be minimized by a simple correction for the radius of soil specimen. Their suggestion $R_{eq}$ value varies from 0.82 for strains below $10^{-3}$% to 0.79 for strains at $10^{-1}$% for a solid specimen shown in Fig. 4.

Shear strain is calculated from:

$$\gamma = R_{eq} \times \gamma_{\max} \tag{5}$$

Shear stress, $\tau$, is calculated using:

$$\tau = R_{eq} \times \tau_{\max} \tag{6}$$

There are several weaknesses of using the equivalent radius approach (Sasanakul, 2005). Their suggested $R_{eq}$ values are lacking in express in terms of reference strain, $g_r$, which is defined by $\tau_{\max}/G_{\max}$. The reference strain is useful for expressing mathematical form of the stress-strain relationship and as a measure of the relative values of $G_{\max}$ and $\tau_{\max}$ (Pyke, 2004). Hence, soils with larger values of reference strain have greater shear strengths relative to their small strain modulus and show more elastic, less nonlinear, stress-strain behavior than soils with smaller values of reference strain. The other weakness of Chen and Stokoe's approach is that soil nonlinearity is not taken into account. The strain level is limited up to $10^{-1}$% and a single value of $R_{eq}$ does not accurately represent the actual $R_{eq}$ for a wide range of strains (Sasanakul, 2005).

## 4. Modified Equivalent Radius Appraoch

### 4.1 $R_{eq}$ Based on Hyperbolic Model

A more general approach to account for nonuniform stress-strain than the conventional equivalent radius approach, the modified stress integration approach was developed by Sasanakul (2005). The nonuniform stress-strain was accounted for more precisely by integrating the stress over the radius of soil specimen to obtain a twist-torque relationship. The twist-torque relationship was calculated using closed form integration developed by Sasanakul (2005). The closed form solution was obtained assuming the stress-strain relation is hyperbolic. The stress-strain relationship and the closed form solution of torque-rotation relationship are presented in Eqs. 7 and 8, respectively.

$$\tau = \frac{G_{\max}\gamma}{1+\left(\frac{\gamma}{\gamma_r}\right)} \tag{7}$$

where, $G_{\max}$ = shear modulus at low strain and

$\gamma_r$ = reference strain.

$$T = \frac{1}{3}\pi G_{\max}\gamma_r R\left[2R^2 - 3R\left(\frac{\gamma_r L}{\theta}\right) + 6\left(\frac{\gamma_r L}{\theta}\right)^2\right] + 2\pi G_{\max}\gamma_r^4\left(\frac{L}{\theta}\right)^3\left[\ln(\gamma_L) - \ln\left(\gamma_r + \frac{\theta R}{L}\right)\right] \tag{8}$$

where, $T$ = torque,

$r$ = radius of soil specimen,

$L$ = height of soil specimen, and

$\theta$ = rotation.

Then the effective shear modulus, $G_{eff}$ can be calculated using Equation (2).

The steps to determine the value of $R_{eq}$ are as follows. For a given, $\gamma_1$, the corresponding value, $G_1$, can be obtained using Eq. (7) and $G$ versus $\gamma$ curve is shown in Fig. 5(a). The $G_{eff}$ versus $\theta$ curve is presented in Fig. 5(b). A value $\theta_1$ that is associated with the $G_{eff}$ value equal to $G_1$ is identified in Fig. 5(b). Then using the following equation, a value of $R_{eq}$ is found for $\theta_1$ and $\gamma_1$ as shown in Fig. 5(c).

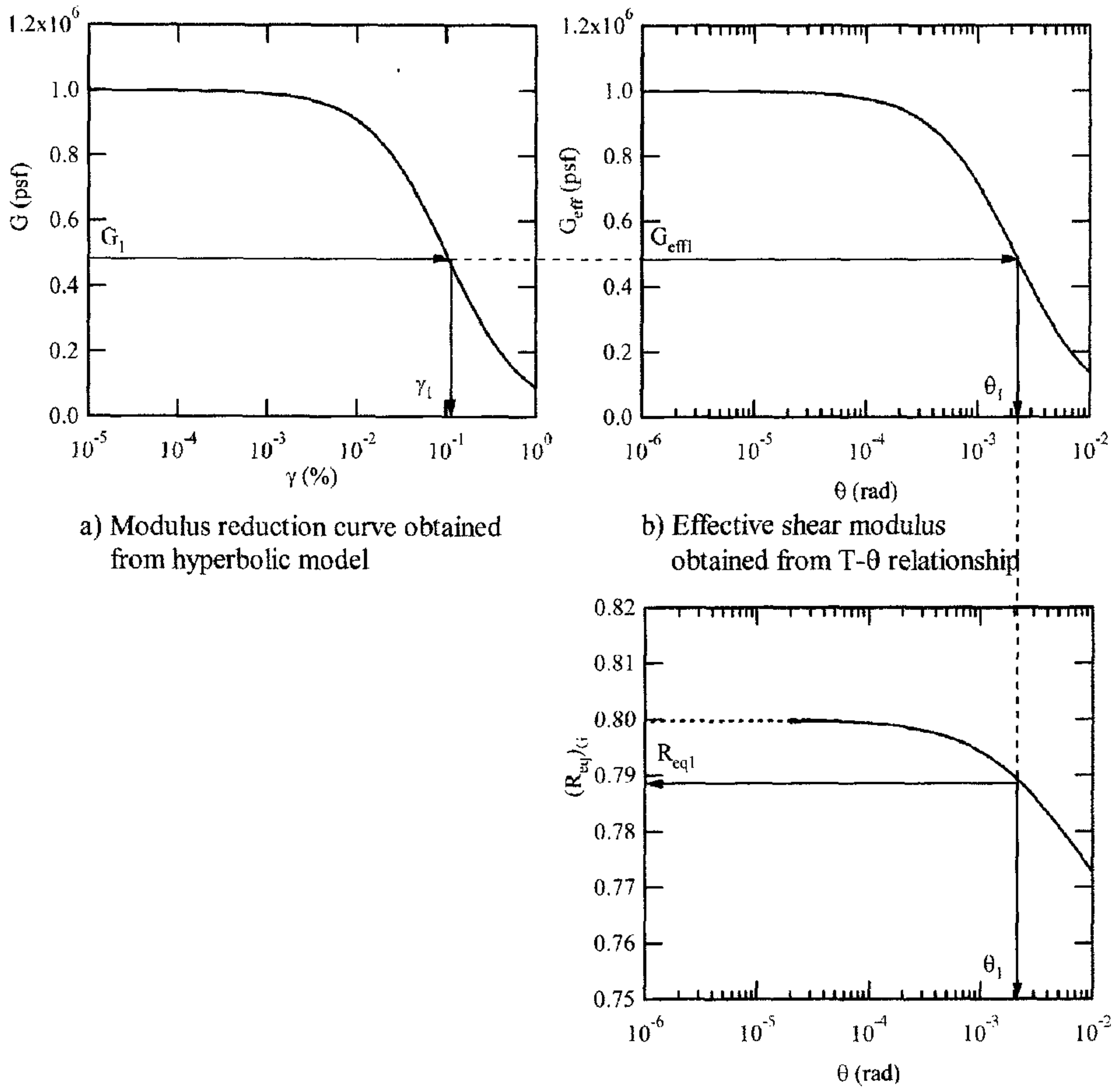
$$R_{eq} = \frac{1}{R}\frac{\gamma L}{\theta} \tag{9}$$

Therefore, for a given twist-torque relationship the equivalent shearing strain, $\gamma_{eq}$ can be obtained using the $R_{eq}$ curve. This approach effectively accounts for the variation of $R_{eq}$ over the range of strains and soil nonlinearity. The $R_{eq}$ curves also consider the value of reference strain, $\gamma_r$. The $R_{eq}$ curve based on hyperbolic model is plotted in terms of normalized rotation, $\theta/\theta_r$ in Fig. 6 (Sasanakul, 2005). The reference rotation, $\theta_r$, is defined as:

$$\theta_r = \frac{\gamma_r L}{R} \tag{10}$$

### 4.2 $R_{eq}$ Based on Modified Hyperbolic Model

The modified equivalent radius approach was extended to




a) Modulus reduction curve obtained from hyperbolic model

b) Effective shear modulus obtained from T-θ relationship

c) $(R_{eq})_G$ curve based on shear modulus

Fig. 5. Determination of $R_{eq}$ based on shear modulus (From Sasanakul, 2005)

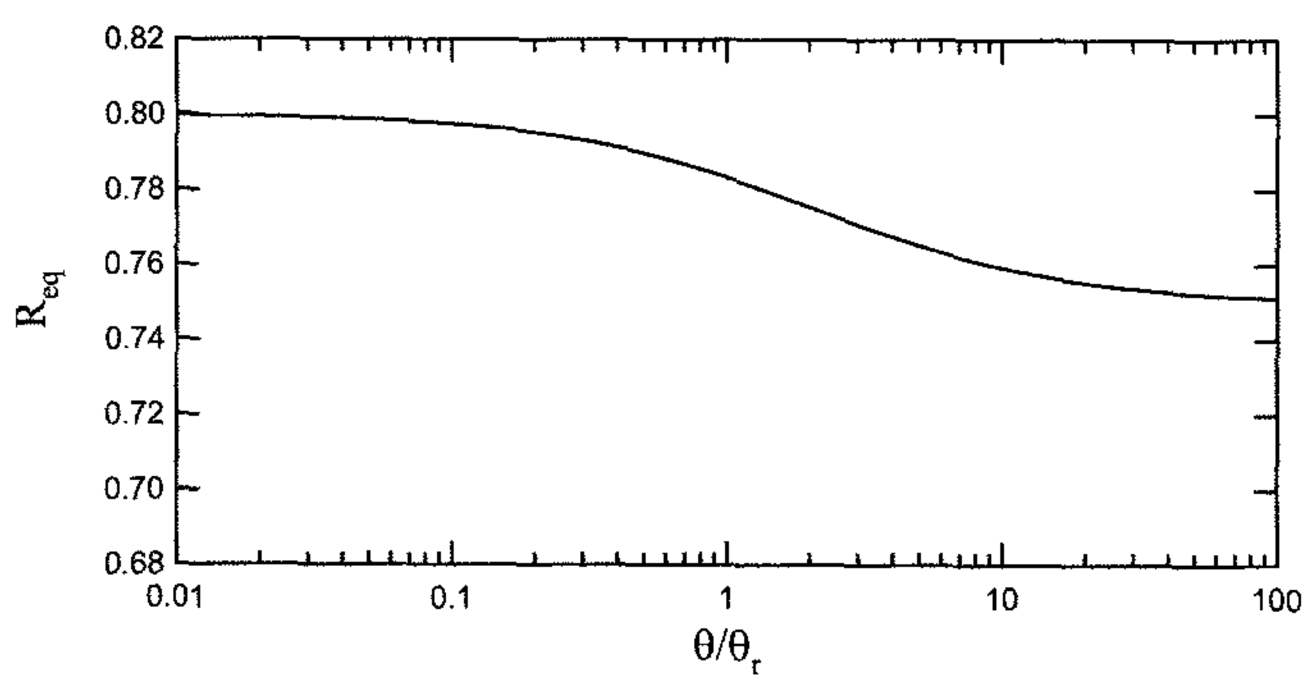



Fig. 6. $R_{eq}$ curves based on hyperbolic model (From Sasanakul, 2005)

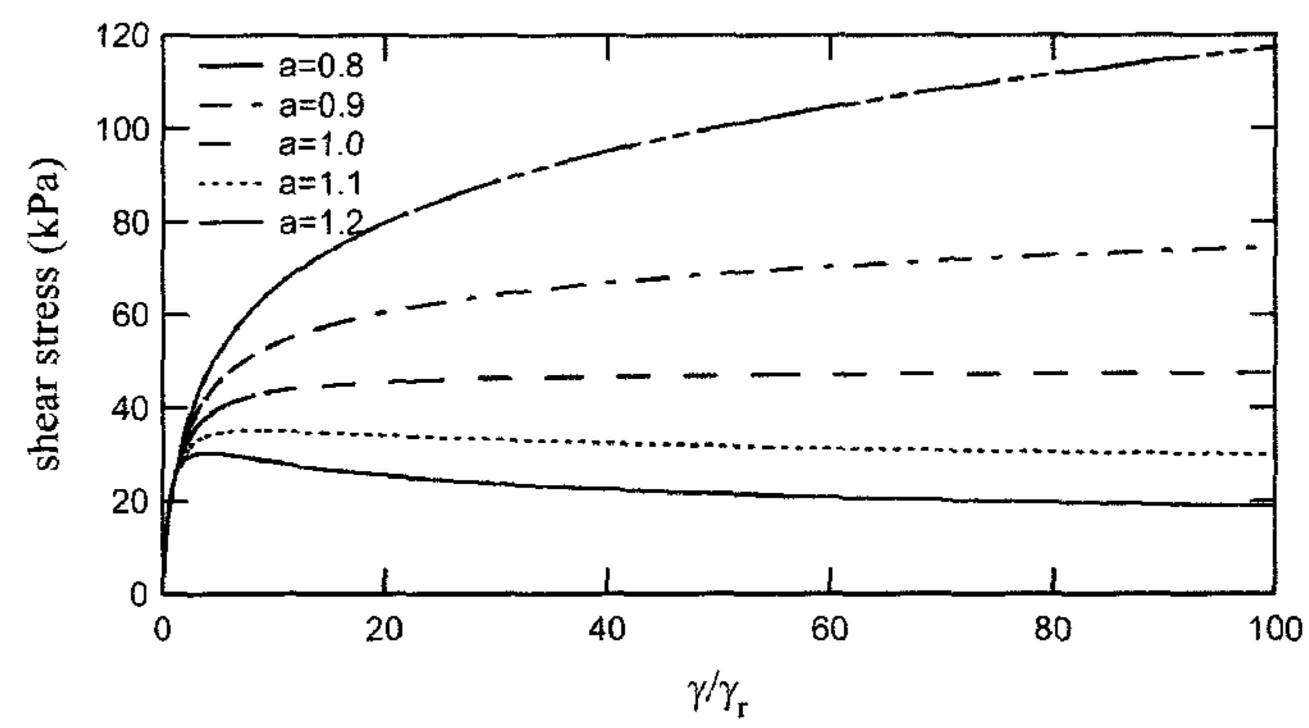



Fig. 7. Stress strain curves for modified hyperbolic model with different curvature coefficients

generate $R_{eq}$ curves for shear modulus using modified hyperbolic model. To obtain $R_{eq}$ using modified hyperbolic model, the approach starts by relating the shear stress, $\tau$ acting on a circular cross section to strain, $\gamma$ using:

$$\frac{\tau}{\gamma}=\frac{G_{\max}}{1+\left(\frac{\gamma}{\gamma_r}\right)} \quad (11)$$

where, $G_{\max}$ = shear modulus at low strain (47,880 kPa)

$a$ = curvature coefficient, and

$\gamma_r$ = reference strain

The stress strain curves for different curvature coefficients are presented in Fig. 7.

The $T$-$\theta$ relationships were then determined numerically using Eq. (12):

$$T=\int_A \mathrm{dM}=\int \tau r\mathrm{dM}=\int_0^R 2\pi r^2\tau\mathrm{dr} \quad (12)$$

where, $M$ = resultant moment over the entire cross section area

$A$ = cross section area, and

$t$ = shear stress obtained using Eq. (10)

After obtaining $G_{eff}$ using Eq. (2), the same procedures presented in Fig. 5 are applied to determine $R_{eq}$ curves. Fig. 8

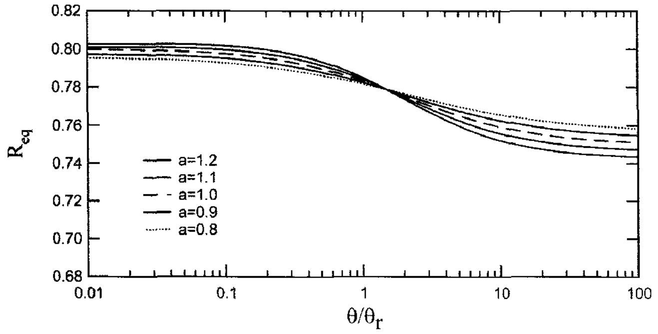



Fig. 8. $R_{eq}$ curves based on modified hyperbolic model

presents the $R_{eq}$ curves based on shear modulus with modified hyperbolic model using different curvature coefficients. Rotations are normalized with respect to reference rotation, $\theta_r$, as defined in Eq. (10). As shown in Fig. 8, the ranges of $R_{eq}$ value are a little wider in high strain levels.

### 4.3 $R_{eq}$ Based on Ramberg-Osgood Model

The modified equivalent radius approach was also extended to generate $R_{eq}$ curves for shear modulus using Ramberg-Osgood model. The approach starts by numerically relating the shear stress, $\tau$ acting on a circular cross section to strain, $\gamma$ using:

$$\gamma=\frac{\tau}{G_{max}}\left(1+\alpha\left|\frac{\tau}{G_{max}\gamma_r}\right|^{b-1}\right) \tag{12}$$

The model parameters for Ramberg-Osgood model are: $G_{max}$, $\gamma_r$, $\alpha$, and $\beta$. The reference strain is the same as the reference strain used in the hyperbolic model. The stress strain curves for different model parameters ($\alpha$ and $\beta$) are presented in Fig. 9.

Torque for Ramberg-Osgood model was also obtained numerically (Sasanakul, 2005). The Ramberg-Osgood model presents g as a function of shear stress, $\tau$. For a given twist, $\theta$, the relationship between radius, $r$ and shear strain, $\gamma$ can be obtained using the following equation.

$$r=\frac{L}{\theta}\gamma \tag{13}$$

Then, radius, $r$ can be expressed as:

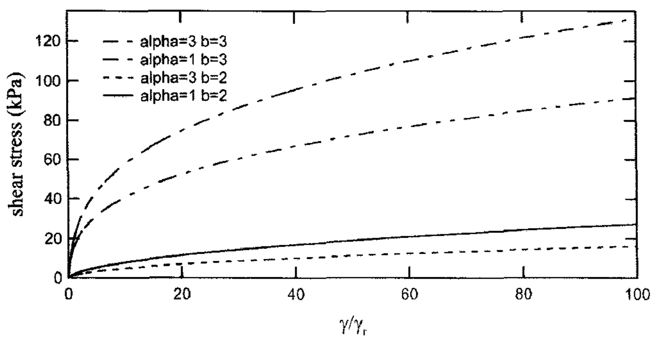



Fig. 9. Stress strain curves for Ramberg-Osgood model with different model parameters ($\alpha$, $\beta$)

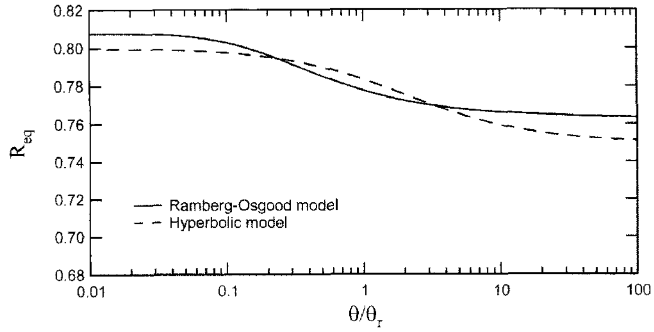



Fig. 10. $R_{eq}$ curves based on Ramberg-Osgood model

$$r=\frac{L}{\theta}\frac{\tau}{G_{max}}\left[1+\alpha\left|\frac{\tau}{G_{max}\gamma_r}\right|^{b-1}\right] \tag{14}$$

Torque is calculated numerically integrating the stress-strain relationship using:

$$T=2\pi\int_0^R \tau(r)r^2dr \tag{18}$$

where, $\tau(g)$ is stress-strain relationship based on Eq. (12).

After obtaining effective shear modulus, $G_{eff}$ using Eq. (2), the same procedures presented in Fig. 5 are also applied to determine $R_{eq}$ curve based on Ramberg-Osgood model. Fig. 10 presents the $R_{eq}$ curve relative to normalized rotation based on damping ratio with Ramberg-Osgood model parameters (alpha = 3, b = 2.5). At high strains, the $R_{eq}$ values based on Ramberg-Osgood model are a little higher than the $R_{eq}$ values based on hyperbolic model. At low strains, the opposite result was obtained. But the differences are not significant.

## 5. Curve Fitting

### 5.1 Curve Fitting for $T$-$\theta$ Relationships and Model Selection

The theoretical $T$-$\theta$ relationship calculated from the assumed soil stress-strain model is fit to the measured $T$-$\theta$ relationship by adjusting values of the model parameters. For the hyperbolic model, the closed form solution was used to obtain the $T$-$\theta$ relationship using Eq. (8). Numerical integrations were used to calculate $T$-$\theta$ relationship for the modified hyperbolic model and the Ramberg-Osgood model using Eq. (12) and Eq. (18). The Igor curve fitting program was employed to find the best nonlinear soil models. The chi-square value is an indicator of the quality of the fit. The best values of the model parameters are the ones that minimize the value of chi-square. The chi-square is defined using Eq. (19):

$$\sum\left(\frac{y-y_i}{\sigma_i}\right)^2 \tag{19}$$

where $y$ is fitted value for a given point, $y_i$ is original data value for the point, and $\sigma_i$ is standard deviation of each data value.

Igor employs the Lavenberg-Marquardt algorithm to find the minimum value of chi-square using nonlinear, least square curve fitting. It starts from an initial guesses of the model parameters. The initial values must be provided manually and the curve fitting algorithm finds the best fit starting from these initial values. As the fitting procedure proceeds, the chi-square value decreases. The fit is ended when the difference between experimental data and fitted data are minimum. The initial values must be adjusted until the lowest chi-square value is obtained. The details of curve fitting techniques are described by Sasanakul (2005).

### 5.2 Application of Curve Fitting

An application of curve fitting was illustrated using sand soils. The grain size distribution of the tested sand soils is shown in Fig. 11 and the basic soil properties are summarized in Table 1. The measured $T$-$\theta$ relationship from TS test

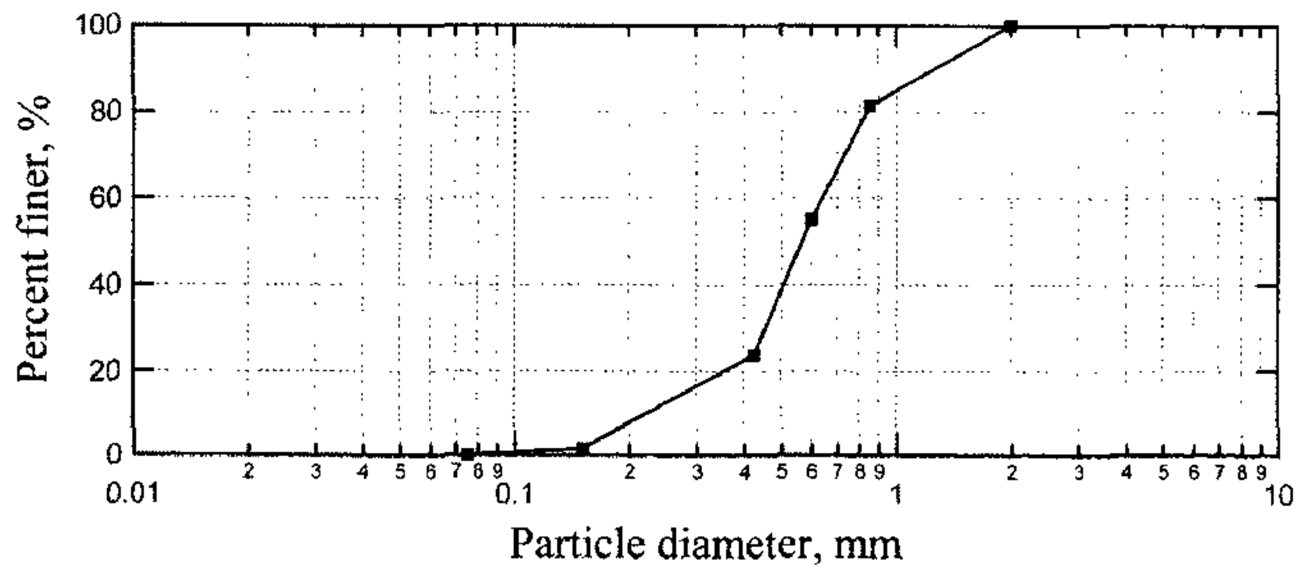



Fig. 11. Grain size distribution of the tested sand soil

Table 1. Soil properties index of tested sand specimen

| USGS | Curvature coefficient, $C_c$ | Uniformity coefficient, $C_u$ | PI | Median particle size, $D_{50}$, mm |
|---|---|---|---|---|
| SP | 1.423 | 3.1 | NP | 0.57 |

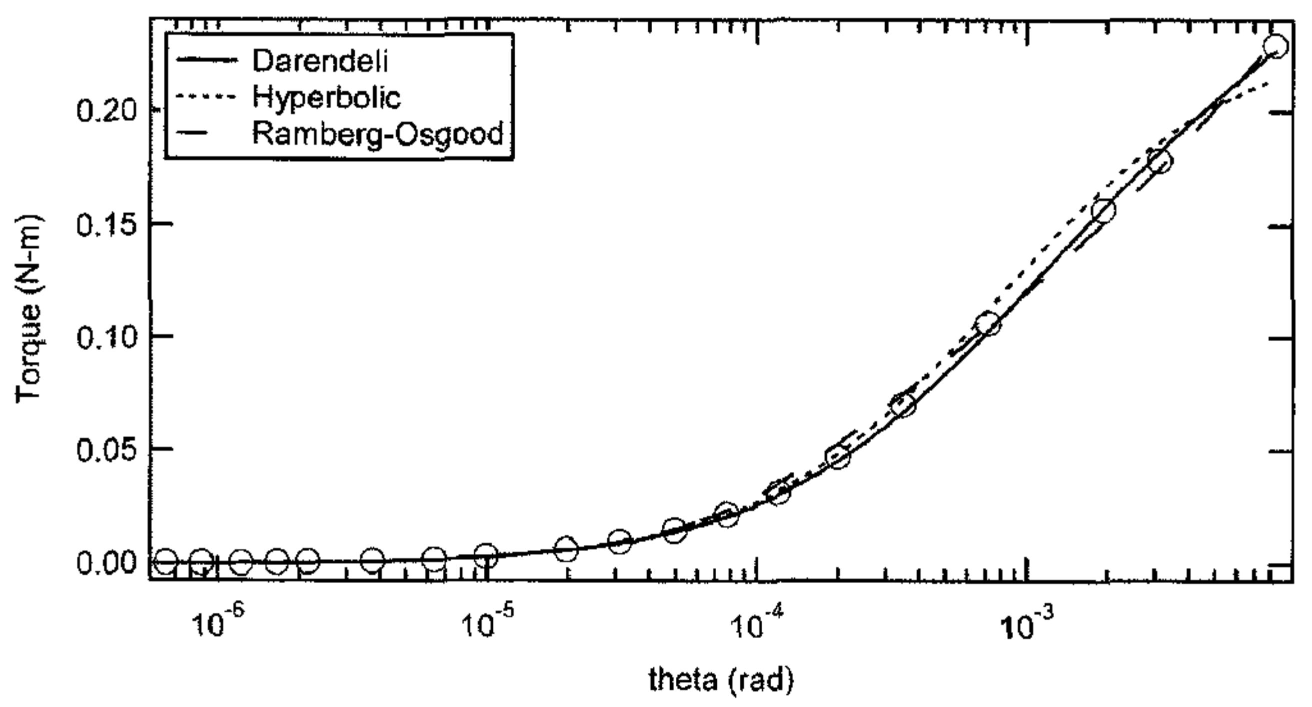



Fig. 13. Measured rotation vs torque with fit curves using the three nonlinear models

on the soil samples at 25.3 kPa confining pressures is presented in Fig. 12. Curve fitting was performed on each measured $T$-$\theta$ relationship using three nonlinear models.

The low strain shear modulus, $G_{max}$ was evaluated from low-strain resonant column (RC) test. The fitted $T$-$\theta$ relationships based on each nonlinear model are also plotted in Fig. 12. The model parameters and chi-square values determined from the curve fitting are presented in Table 2. As shown in Table 2, the modified hyperbolic model provides the best fit result. Therefore, the modified hyperbolic model (a=0.903 and $\gamma_r$ = 0.043%) was selected as the best model to predict the $\tau$-$\gamma$ relationships for sand soils in this study. As shown in Fig. 12, all of the assumed soil models perform very well in the linear range indicating that the value $G_{max}$ used for curve fitting is close to measured value of $G_{max}$. Fig. 12 showed that the hyperbolic model fits the $T$-$\theta$ relationship well at low to medium strain levels but typically fit the data poorly at high strain levels. On the other hand, Ramberg-Osgood model generally fit the $T$-$\theta$ relationship at high strains better than the other two models shown in Fig. 12.

## 6. Conclusions

The modified equivalent radius approach resolves the weaknesses of conventional equivalent radius approach. It considers the reference strain, $\gamma_r$ and soil nonlinearity to represent the $R_{eq}$ curves for a wide range of strains.

The modified equivalent radius approach was applied using the hyperbolic, the modified hyperbolic, and Ramberg-Osgood model to determine the values of $R_{eq}$ for shear modulus. The results based on modified hyperbolic model varying curvature coefficient, a, show that the range of $R_{eq}$ is a little wider at higher strain levels than hyperbolic models. In case of the Ramberg-Osgood model, the $R_{eq}$ values at high strains are a little higher than the $R_{eq}$ values based on hyperbolic model and the opposite result was obtained at low strains.

Curve fitting is performed to evaluate the stress-strain relationship from the twist-torque data. The model parameters providing the best fit of the theoretical $T$-$\theta$ relationship to the measured $T$-$\theta$ relationship are used to develop the stress-strain curve of the soil tested. In this study, sand soil was tested and modified hyperbolic model provides the best fit result.

Table 2. Curve fitting result from TS tests on sand soil

| $G_{max}$(kPa) | Hyperbolic Model | | Modified Hyper Model | | | Ramberg-Osgood Model | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\gamma_r$ (%) | chi-square | $\gamma_r$ (%) | a | chi-square | $\gamma_r$ (%) | $\alpha$ | b | chi-square |
| 39,622 | 0.052 | 2.37E-04 | 0.043 | 0.903 | 2.52E-05 | 0.029 | 1.903 | 3.867 | 7.45E-05 |

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