Journal of the Korea Society of Mathematical Education Series D:
Research in Mathematical Education
Vol. 12, No. 4, December 2008, 271–281

韓國數學教育學會誌 시리즈 D:
<數學教育研究>
제 12 권 제 4 호 2008 년 12 월, 271–281

A Quarter a Century of Discovering and Inspiring Young Gifted Mathematicians: All the Best from Colorado Mathematical Olympiad[1]

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(Received July 12, 2008. Accepted November 1, 2008)

Quarter a century ago, I founded the Colorado Mathematical Olympiad. The Colorado Mathematical Olympiad is the largest essay-type in-person mathematical competition in the United States, with 600 to 1,000 participants competing annually for prizes. In this article, I explain what it is, how it works, give examples of problems and solutions, and share with the reader careers of some of the Olympiad's winners.

Keywords: Colorado Mathematical Olympiad, mathematical competition, problem solving, transformation geometry
ZDM Classification: D50, G50, U40
MSC2000 Classification: 97D50, 97G50, 97U40

I. THE EVENT

There are many types of mathematical competitions throughout the world. Some expect participants to merely state "good" answers, others are multiple-choice competitions. Some competitions are oral, and completed in a matter of a couple of hours. Others go on for a week or weeks. Over the past 100+ years throughout the world, the word "Olympiad" came to mean the particular type of competition where complete essays-solutions are expected for every problem and an adequate time is offered for solving them.

The First Colorado Springs Mathematical Olympiad took place 25 years ago, on April

[1] This article is an extended version of the paper (Soifer, 2008b) presented at Topic Study Group 6 (Activities and Programs for Gifted Students) of the 11th International Congress on Mathematical Education (ICME-11) held at the Universidad Autonoma de Nuevo Leon (UANL), Monterrey, Mexico, July 6–13, 2008.

27, 1984. The Colorado Mathematical Olympiad is the largest essay-type in-person mathematical competition in the United States, with 600 to 1,000 participants competing annually for prizes, such as: gold, silver and bronze medals, college scholarships, computers, computer software, calculators, books, memorabilia, etc. All prizes are made possible by our various sponsors, such as CASIO, Wolfram Research, Texas Instruments, City of Colorado Springs, University of Colorado, School Districts 11, 20, 3 and others, and various high and middle schools from all over the State of Colorado, etc.

Participants are offered 5 problems and given 4 hours to solve them. Problems vary a great deal in difficulty, and we arrange them from the easiest problem to the most difficult problem. This is an individual competition, and we expect all work to be shown. We specifically warn the Olympians right on the problem sheet that "we will give no credit for answers submitted without supporting work. Conversely, a minor error that leads to an incorrect answer will not substantially reduce your credit."

Our Olympiad is genuinely and principally an essay-type competition. We generously reward originally and creativity. Our goal is to allow every student the opportunity to participate in the Olympiad. We offer only individual competition and put no restrictions on the number of participants from a school.

We try to bring the Olympiad problems closer to "real" mathematics: many problems require construction of examples (rather than just analytical reasoning). Sometimes we mimic a "real" mathematical research by offering a series of problems, increasing in difficulty, and leading to generalizations and deeper results. These problems demonstrate ways in which mathematical research works.

We use the least standard, the most interesting, decisively unknown to participants, problems. We do not have a large enough organization to consider an alternative of offering the competition separately for every grade. Thus, we offer the same problems to everyone who comes from junior high school students to seniors. This approach made finding of acceptable problems much more difficult, but positively improved the quality of the problems. We create problems that require, for their solutions, a minimal amount of knowledge and a great deal of creativity, originality, and analytical thinking.

Often the problems we select stem from mathematical research we do. We "simply" notice a fragment of the research, which utilizes a beautiful idea. We then translate thus found mathematical gem into the language of an engaging story — and a new Olympiad problem is ready!

In this talk, I will show you a selection of our original exciting problems: All the Best from the Quarter a Century of the Colorado Mathematical Olympiad. (Veterans of competitions, of course, noticed my homage to Australian Mathematics Competitions in the subtitle of my talk). Well, maybe not all the best — as my time will be limited — but some of the best! The problems will illustrate how we identify and inspire gifted young mathematicians and why these particular problems aid us in this task.

We are approaching the Quarter a Century mark — the 25th Colorado Mathematical Olympiad will take place on April 18, 2008, with Award Presentation Ceremonies (a 4-hour program) following a week later.

II. THE PROBLEMS

Allow me to give just 3 examples from the past few years. Since we offer problems of increasing difficulty, I will offer you here one easy, one medium, and one hard problem - in this order.

Problem 2006. 2. A horse! A horse! My kingdom for a horse!

The Good, the Bad, and the Ugly divide a pile of gold coins and a horse The pile consists of 2006 gold coins, and they draw in turn 1, 2 or 3 coins from the pile. The Good gets the first turn, the Bad draws second, and the Ugly takes last. The Ugly does not trust the Bad and never draws the same number of coins as the Bad has drawn immediately before him.

The one who takes the last coin is left behind, while the other two cross the prairie together on horseback. Who can guarantee himself the ride out on the horse regardless of how the others draw coins? How can he do this?

Solution: The Good can guarantee himself a horse ride. Observe that $2006 = 6k + 2$ for some integer k. The Good starts by taking 1 coin. Since the Ugly will not draw the same as the Bad, the number of coins drawn by the Bad and the Ugly combined on any round add up to 3, 4 or 5. Thus, by always drawing the difference between 6 and the sum of the draws by the Bad and the Ugly, the Good will eventually bring the number of coins to 1 after his turn. Thus, the Bad will pick up the last coin, and the Good and Ugly will ride off together.

Problem 2007. 4. Looking for the Positive

A number is placed in each angle of a regular 2007-gon so that the sum of any 10 consecutive numbers is positive. Prove that one can choose an angle with the number a in it, such that when we label all 2007 numbers clockwise $a = a_1, a_2, \cdots, a_{2007}$, each sum $a_1, a_1 + a_2, \cdots, a_1 + a_2 + \cdots + a_{2007}$ will be positive.

Proof in two parts:

1) Let us prove that the sum S of all 2007 given numbers is positive. By permuting the

summands cyclically, we get:

$$\begin{aligned} a_1 + a_2 + \cdots + a_{2007} &= S \\ a_2 + a_3 + \cdots + a_1 &= S \\ &\vdots \\ a_{10} + a_{11} + \cdots + a_9 &= S \end{aligned}$$

By adding together these 10 equalities, we get a positive sum in the left-hand side (as the sum of the 2007 columns, each column positive due to being the sum of 10 consecutive numbers). On the right we get $10S$, which is thus positive, and so is S.

2) Consider the following array of $2007 \times 2 = 4014$ sums, where we add one summand at a time moving clockwise:

$$\begin{aligned} s_{1,1} &= a_1 \\ s_{1,2} &= a_1 + a_2 \\ &\vdots \\ s_{1,2007} &= a_1 + a_2 + \cdots + a_{2007} \\ s_{1,2008} &= a_1 + a_2 + \cdots + a_{2007} + a_1 \\ s_{1,2009} &= a_1 + a_2 + \cdots + a_{2007} + a_1 + a_2 \\ &\vdots \\ s_{1,4014} &= a_1 + a_2 + \cdots + a_{2007} + a_1 + a_2 + \cdots + a_{2007} \end{aligned}$$

Choose $1 \le k \le 2007$ such that the sum $s_{1,k}$ is the lowest among the first 2007 sums; if the same lowest sum appears more than once, we choose $s_{1,k}$ with the largest k. This sum is in fact the lowest among all 4014 sums because the latter 2007 sums are equal to the earlier 2007 sums plus S, where S is the sum of all given numbers ($S > 0$ as shown in Part 1 of the proof). We are done because

$$\begin{aligned} 0 < s_{1,k+1} - s_{1,k} &= a_{k+1} \\ 0 < s_{1,k+2} - s_{1,k} &= a_{k+1} + a_{k+2} \\ &\vdots \\ 0 < s_{1,k+2007} - s_{1,k} &= a_{k+1} + a_{k+2} + \cdots + a_{2007} + a_1 + \cdots + a_k \end{aligned}$$

Problem 2004. 5. Chess 7×7

The Problem has two parts:

(a) Each member of two 7-member chess teams is to play once against each member of the opposing team. Prove that as soon as 22 games have been played, we can choose 4 players and seat them at a round table so that each pair of neighbors has already played.

(b) Prove that 22 is best possible *i.e.*, after 21 games the result cannot be guaranteed.

This problem occurred to me while I was reading a wonderful unpublished 1989 manuscript of a monograph "Aspects of Ramsey Theory" by Hans Jürgen Prömel and Bernd Voigt (*cf.* Prömel & Voigt, 1995). I found a mistake in a lemma, and constructed a counterexample for this lemma's statement. Finding this counterexample is exactly Part (b) here. Part (a) is a corrected particular case of that lemma, translated, of course, into a language of a nice "real" story. I found 3 wonderful solutions of Part (a). Here is one of them.

Solution of Part (a): This solution harnesses the power of combinatorics.

In the selection and editing process, Dr. Col. Bob Ewell suggested to use a 7×7 table to record the games played. We number the players in both teams. For each player of the first team we allocate a row of the table and for each player of the second team a column. We place a checker in the table in location (i, j) if the player i of the first team played the player j of the second team (Figure 5.2)

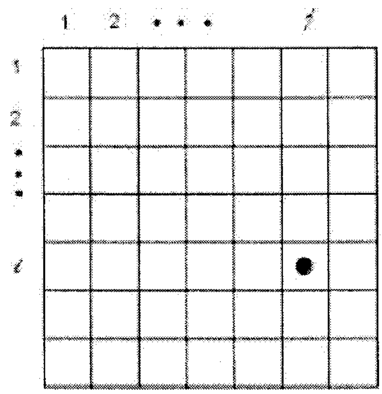

Figure 5.2. Chess 7×7

If the required four players are found, this would manifests itself in the table as a rectangle formed by four checkers (a checkered rectangle)! The problem thus translates into the new language as follows:

A 7×7 table with 22 checkers must contain a checkered rectangle.

Assume that a table has 22 checkers but does not contain a checkered rectangle. Since

22 checkers are contained in 7 rows, by Pigeonhole Principle, there is a row with at least 4 checkers in it. Observe that interchanging rows or columns does not affect the property of the table to have or not to have a checkered rectangle. By interchanging rows we make the row with at least 4 checkers first. By interchanging columns we make all checkers to appear consecutively from the left of the first column. We consider two cases.

1) Top column contains exactly 4 checkers.

Draw a bold vertical line L after the first 4 columns (Figure 5.3.). To the left from L, top row contains 4 checkers, and all other rows contain at most 1 checker each, for otherwise we would have a checkered rectangle (that includes the top row).

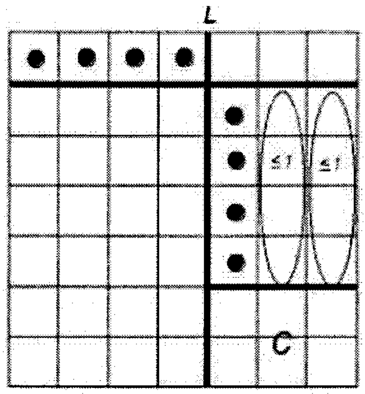

Figure 5.3. Top column contains exactly 4 checkers

Therefore, to the left from L we have at most $4+6=10$ checkers. This leaves at least 12 checkers to the right of L, thus at least one of the three columns to the right of L contains at least 4 checkers; by interchanging columns and rows we put them in the position shown in Figure 5.3. Then each of the two right columns contains at most 1 checker total in the rows 2 through 5, for otherwise we would have a checkered rectangle. We thus have at most $4+1+1=6$ checkers to the right of L in rows 2 through 5 combined. Therefore, in the lower right 2×3 part C of the table we have at least $22-10-6=6$ checkers - thus C is completely filled with checkers and we get a checkered rectangle in C in contradiction with our assumption.

2) Top column contains at least 5 checkers.

Draw a bold vertical line L after the first 5 columns (Figure 5.4). To the left from L, top row contains 5 checkers, and all other rows contain at most 1 checker each, for otherwise we would have a checkered rectangle (that includes the top row). Therefore, to the left from L we have at most $5 + 6 = 11$ checkers. This leaves at least 11 checkers to the right of L, thus at least one of the two columns to the right of L contains at least 6 checkers; by interchanging columns and rows we put 5 of these 6 checkers in the position shown in Figure 5.3.

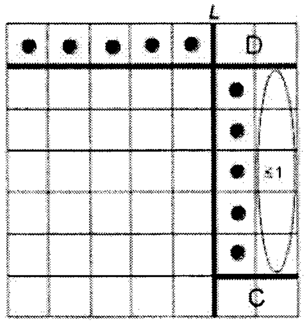

Figure 5.4. Top column contains at least 5 checkers

Then the last column contains at most 1 checker total in the rows 2 through 6, for otherwise we would have a checkered rectangle. We thus have at most 5 + 1 = 6 checkers to the right of L in rows 2 through 6 combined. Therefore, the upper right 1×2 part C of the table plus the lower right 1×2 part D of the table have together have at least $22-1-6=5$ checkers — but they only have 4 cells, and we thus get a contradiction.

Solution of (b): Glue a cylinder (!) out of the 7×7 board, and put 21 checkers on all squares of the first, second, and forth diagonals (Figure 5.8. shows the cylinder with one checkered diagonal; Figure 5.9. shows, in a flat representation, the cylinder with all three cylinder diagonals).

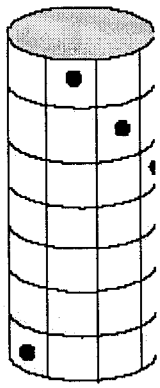

Figure 5.8. Cylinder with one checkered diagonal

Assume that 4 checkers form a rectangle on our 7×7 board. Since these 4 checkers lie on 3 diagonals, by Pigeonhole Principle 2 checkers lie on the same (checkers-covered) diagonal D of the cylinder. But this means that on the cylinder our 4 checkers form a square! Two other (opposite) checkers a and b thus must be symmetric to each other with respect to D, which implies that the diagonals of the cylinder that contain a and b must be symmetric with respect to D - but no 2 checker-covered diagonals in our checker placement are symmetric with respect to D. (To see it, observe Figure 5.10. which shows the top rim of the cylinder with bold dots for checkered diagonals: square distances between the checkered diagonals, clockwise, are 1, 2, and 4) This contradiction implies that there are no checkered rectangles in our placement. Done!

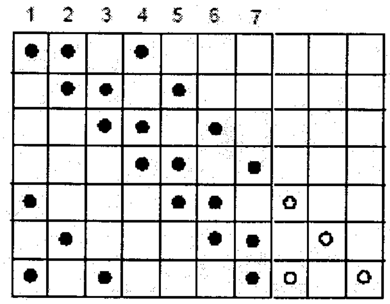

Figure 5.9. Cylinder with all three cylinder diagonals

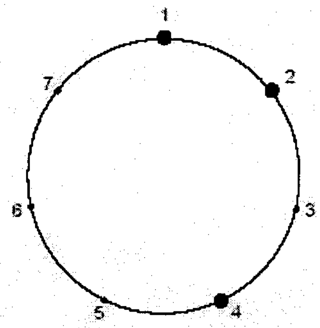

Figure 5.10. Cylinder with bold dots for checkered diagonals

Remark on Problem (b). Obviously, any solution of problem (b) can be presented in a form of 21 checkers on a 7×7 board (left 7×7 part with 21 black checkers in Figure 5.9). It is less obvious, that the solution is unique: by a series of interchanges of rows and columns, any solution of this problem can be brought to precisely the one I presented! Of course, such interchanges mean merely renumbering of players of the same team. The uniqueness of the solution of problem (b) is precisely another way of stating the uniqueness of the projective plane1 of order 2, so called "Fano Plane" 2 denoted by PG(2,2). The Fano plane is an abstract construction, with symmetry between points and lines: it has 7 points and 7 lines (think of rows and columns of our 7×7 board as lines and points respectively!), with 3 points on every line and 3 lines through every point (Figure 5.11.)

1. A finite projective plane of order n is defined as a set of $n^2 + n + 1$ points with the properties that:

   a) Any two points determine a line,
   b) Any two lines determine a point,
   c) Every point has $n + 1$ lines through it, and
   d) Every line contains $n + 1$ points.

2. Named after Gino Fano (1871–1952), the Italian geometer who pioneered the study of finite geometries.

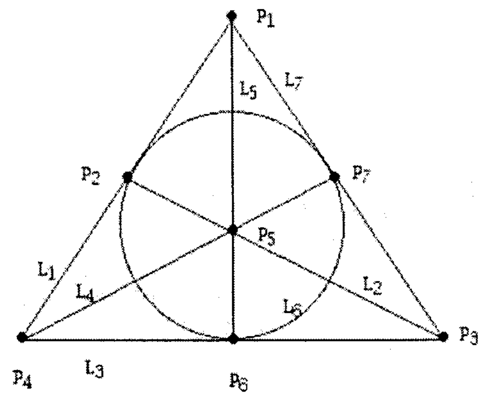

Figure 5.11. Incidence matrix of the Fano Plane

Observe that if in our 7×7 board we replace checkers by 1 and the rest of the squares by zeroes, we would get the incidence matrix of the Fano Plane.

III. INTEREST EARNED ON THE CAPITAL

What has our investment capital been to date? In its first Quarter a Century, Colorado Mathematical Olympiad (CMO) has entertained some 17,000 students. They have written over 85,000 essays, and were awarded over $240,000 in prizes. The Olympiad has been a unique joint effort of school districts, schools, institutions of higher education, business community and local and State governments.

In March 1989, I was hosting Paul Erdös for the first time. We did mathematics. We also talked about life, refugees, anti-Semitism, and, of course, Colorado Mathematical Olympiad (CMO), for which he contributed a problem. Paul said, "The Olympiad would not be important by itself, but it creates a new enthusiasm for mathematics, and this is very important." This enthusiasm "paid interest":

The winner of the First CMO Russell Shaffer, achieved a perfect score at MIT, then on the National Science Foundation Scholarship earned his Ph. D. degree in Theoretical Computer Science from Princeton University, and is now a researcher at MCI.

The winner of the Second CMO, Richard Wolniewitz, earned his Ph. D. degree in Computer Science from the University of Colorado at Boulder. He owns his own software engineering company.

The winner of the Third, Forth, and Fifth CMO's David Hunter, earned his first degree from Princeton University, and a Ph. D. in Statistics. He is a Professor at Pennsylvania State University.

The co-winner of the Fifth CMO, Gideon Jaffe, earned his first degree in Mathematics and Drama from Harvard University, and his Ph. D. degree in Philosophy from University of California at Berkeley. He is a Professor of Philosophy at Berkeley.

Matt Kahle was a "C–" student in geometry in high school. This did not stop him from winning the Eighth and the Ninth CMO's. He earned his first two degrees in Mathematics from Colorado State University (he was not admitted to my University for having a "C–" grade average, in spite of my assurances that one day we would be proud of such an alumnus). Matt earned his Ph. D. in Mathematics from the University of Washington, Seattle, and is now a Post-Doctoral Fellow at Stanford University.

These are just a few examples of the "interest" we gained on the "investment" of the enthusiasm for mathematics among the middle and high school students. Now, after 25 years of its existence, Colorado Mathematical Olympiad has become a part of life in the State of Colorado, an event eagerly awaited by students, their parents and their schools. The road to this success has not always been covered by roses – indeed, sometimes we have encountered thorns as well. Read more about the history of CMO and its problems in Soifer (1994) and more in the forthcoming in Soifer (2009) and start your own Olympiads, big and small, for just your class, or for your school, neighborhood, city, region, and country. Good luck!

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