Vector and Scalar Modes in Coherent Mode Representation of Electromagnetic Beams

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It is shown that the two mode representations, one with vector modes and the other with scalar modes, for the cross spectral density matrices of electromagnetic beams are equivalent to each other. In particular, we suggest a method to find the vector modes from the scalar modes and formulate the cross spectral density matrix as a correlation matrix.

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The unified theory of coherence and polarization serves as the natural ground for the statistical theory of random electromagnetic beams. [1] Accordingly, the coherent mode representation for the scalar field is also generalized to the vector field, however, the procedure to determine vector modes becomes rather involved compared with scalar modes, and requires solving a vector integral equation. [2-3] On the other hand, a different mode representation has been introduced, which consists of two scalar coherent mode representations for the diagonal elements and one bi-modal expansion for the off-diagonal elements of the cross spectral density matrix. [4] In this paper, we consider the connection between these two mode representations. In particular, we suggest a method to find the vector modes from the scalar modes and formulate the cross spectral density matrix as a correlation matrix.

Let us consider a random electromagnetic beam propagating close to the z-direction. Its coherence properties and polarization properties may be expressed in terms of its 2×2 cross-spectral density matrix [1]

$$\vec{W}(\vec{r}_1,\vec{r}_2,\omega) \equiv \left[W_{ij}(\vec{r}_1,\vec{r}_2,\omega)\right] = \left[\left\langle E_i^*(\vec{r}_1,\omega)E_j(\vec{r}_2,\omega)\right\rangle\right]. \qquad (1)$$

Here, E_i and E_j are components of a typical realization of a monochromatic electric field of frequency ω along two mutually orthogonal directions perpendicular to the z-direction, asterisk denotes the complex conjugate and the sharp brackets denote ensemble average, in the sense of coherence theory in the space-frequency domain [5].

The coherent mode representation for a scalar field may be extended to an electromagnetic beam and the cross spectral density matrix can be expanded as an absolutely and uniformly convergent Mercer-type series [2]

$$\vec{W}(\vec{r}_1,\vec{r}_2,\omega) = \sum_{k=1}^{K}\lambda_k(\omega)\vec{E}_k^*(\vec{r}_1,\omega)\vec{E}_k^T(\vec{r}_2,\omega), \qquad (2)$$

where $\lambda_k(\omega)$ and $\vec{E}_k(\vec{r},\omega)=\left[E_{i,k}(\vec{r},\omega)\right]$ are the eigenvalues and eigenvectors of the Fredholm integral equation

$$\int_D \vec{E}_k^T(\vec{r}_1,\omega)\vec{W}(\vec{r}_1,\vec{r}_2,\omega)d^2r_1 = \lambda_k(\omega)\vec{E}_k^T(\vec{r}_2,\omega), \qquad (3)$$

where the integration is carried out over the domain D. This coherent mode expansion is similar to the Karhunen-Loeve expansion for a complex random process [6]. The eigenvalues are nonnegative and the eigenfunctions satisfy the orthogonality condition

$$\int_D \vec{E}_k^T(\vec{r},\omega)\vec{E}_l^*(\vec{r},\omega)d^2r = \delta_{kl}. \qquad (4)$$

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For our convenience, we write the two-component vector $\vec{E}_k(\vec{r},\omega)$ in the form of a column matrix, i.e.,

$$\vec{E}_k(\vec{r},\omega) = \begin{pmatrix} E_{x,k}(\vec{r},\omega) \\ E_{y,k}(\vec{r},\omega) \end{pmatrix}, \tag{5}$$

and consequently

$$\vec{E}_k^T(\vec{r},\omega) = \left(E_{x,k}(\vec{r},\omega), \quad E_{y,k}(\vec{r},\omega) \right). \tag{6}$$

Once the vector coherent mode representation is obtained, the ensemble realization for the given cross spectral density matrix is immediately accomplished and the corresponding ensemble may be expressed by

$$\left\{ \vec{E}(\vec{r},\omega; a_1, a_2, \cdots, a_K) \right\} = \left\{ \sum_{k=1}^{K} a_k \vec{E}_k(\vec{r},\omega) \right\}, \tag{7}$$

where the stochastic variables a_k satisfy

$$\left\langle a_k^* a_l \right\rangle = \lambda_k(\omega) \delta_{kl}. \tag{8}$$

Instead of the vector coherent mode representation, the scalar mode representations have been used to describe the stochastic, planar, electromagnetic sources [4]. The diagonal elements of the cross spectral density matrix have coherent mode representations which may be written explicitly as

$$W_{xx}(\vec{r}_1, \vec{r}_2, \omega) = \sum_{n=1}^{N} \lambda_n^{(x)}(\omega) \phi_n^*(\vec{r}_1, \omega) \phi_n(\vec{r}_2, \omega), \tag{9}$$

and

$$W_{yy}(\vec{r}_1, \vec{r}_2, \omega) = \sum_{m=1}^{M} \lambda_m^{(y)}(\omega) \psi_m^*(\vec{r}_1, \omega) \psi_m(\vec{r}_2, \omega), \tag{10}$$

where the eigenvalues and the eigenfunctions satisfy the following integral eigenvalue equations:

$$\int_D W_{xx}(\vec{r}_1, \vec{r}_2, \omega) \phi_n(\vec{r}_1, \omega) d^2 r_1 = \lambda_n^{(x)}(\omega) \phi_n(\vec{r}_2, \omega), \tag{11}$$

$$\int_D W_{yy}(\vec{r}_1, \vec{r}_2, \omega) \psi_m(\vec{r}_1, \omega) d^2 r_1 = \lambda_m^{(y)}(\omega) \psi_m(\vec{r}_2, \omega). \tag{12}$$

The off-diagonal elements of the cross spectral density matrix do not have coherent mode representations, however, they may possess bi-modal expansions given by

$$W_{xy}(\vec{r}_1, \vec{r}_2, \omega) = \sum_{n=1}^{N} \sum_{m=1}^{M} \Lambda_{nm}(\omega) \phi_n^*(\vec{r}_1, \omega) \psi_m(\vec{r}_2, \omega), \tag{13}$$

$$W_{yx}(\vec{r}_1, \vec{r}_2, \omega) = \sum_{n=1}^{N} \sum_{m=1}^{M} \Lambda_{nm}^*(\omega) \psi_m^*(\vec{r}_1, \omega) \phi_n(\vec{r}_2, \omega), \tag{14}$$

where the coefficients $\Lambda_{nm}(\omega)$ are obtained by

$$\Lambda_{nm}(\omega) = \int_D d^2 r_1 \int_D d^2 r_2 \phi_n(\vec{r}_1, \omega) W_{xy}(\vec{r}_1, \vec{r}_2, \omega) \psi_m^*(\vec{r}_2, \omega). \tag{15}$$

We may realize an ensemble of the electromagnetic beam in terms of two sets of coherent mode functions $\{\phi_n(\vec{r},\omega)\}$ and $\{\psi_m(\vec{r},\omega)\}$. We write the electric field vector as

$$\vec{E}(\vec{r},\omega) = \begin{pmatrix} \sum_{n=1}^{N} a_n \phi_n(\vec{r},\omega) \\ \sum_{m=1}^{M} b_m \psi_m(\vec{r},\omega) \end{pmatrix}, \tag{16}$$

where a_n and b_m are random complex coefficients. We can construct a correlation matrix from the ensemble given by Eq. (16) as

$$W_{xx}(\vec{r}_1, \vec{r}_2, \omega) = \sum_{n=1}^{N} \sum_{n'=1}^{N} \left\langle a_n^* a_{n'} \right\rangle \phi_n^*(\vec{r}_1, \omega) \phi_{n'}(\vec{r}_2, \omega), \tag{17}$$

$$W_{yy}(\vec{r}_1, \vec{r}_2, \omega) = \sum_{m=1}^{M} \sum_{m'=1}^{M} \left\langle b_m^* b_{m'} \right\rangle \psi_m^*(\vec{r}_1, \omega) \psi_{m'}(\vec{r}_2, \omega), \tag{18}$$

$$W_{xy}(\vec{r}_1, \vec{r}_2, \omega) = \sum_{n=1}^{N} \sum_{m=1}^{M} \left\langle a_n^* b_m \right\rangle \phi_n^*(\vec{r}_1, \omega) \psi_m(\vec{r}_2, \omega). \tag{19}$$

On comparing Eqs. (9), (10), and (13) with Eqs. (17), (18), and (19), we require the random coefficients to satisfy

$$\left\langle a_n^* a_{n'} \right\rangle = \lambda_n^{(x)} \delta_{nn'}, \tag{20}$$

$$\left\langle b_m^* b_{m'} \right\rangle = \lambda_m^{(y)} \delta_{mm'}, \tag{21}$$

$$\left\langle a_n^* b_m \right\rangle = \Lambda_{nm}. \tag{22}$$

The equations (20), (21), and (22) imply that $\{a_n\}$ and $\{b_m\}$ are uncorrelated within themselves, however, there exist cross-correlations between $\{a_n\}$ and $\{b_m\}$. We then raise a question whether there exist $(N+M)$ random coefficients satisfying the correlations given by Eqs. (20), (21), and (22). In other words, can we find a new set of random coefficients, say $\{C_n\}$, which are completely uncorrelated? Once we find such random coefficients, an ensemble realization can be obtained by simply taking $c_n = \sqrt{\lambda_n} e^{i\theta_n}$, where λ_n is the strength of

the corresponding mode and θ_n is a randomly distributed real number in $[0, 2\pi)$. The answer to this question is "yes" and it turns out that finding a new set of uncorrelated coefficients is equivalent to constructing the vector modes from two coherent mode representations and a bi-modal expansion. The existence of those random coefficients may be proved by constructing the vector modes explicitly out of the scalar modes as below.

We now show that two coherent mode representations for the diagonal elements and two bi-modal expansions for the off-diagonal elements of the cross spectral density matrix uniquely determine the vector coherent mode representation of the cross spectral density matrix. For this purpose, we write an arbitrary electric field vector of the electromagnetic beam in terms of the mode functions $\{\phi_n(\vec{r}, \omega)\}$ and $\{\psi_m(\vec{r}, \omega)\}$

$$\vec{E}(\vec{r}, \omega) = \begin{pmatrix} \sum_{n=1}^{N} \alpha_n \phi_n(\vec{r}, \omega) \\ \sum_{m=1}^{M} \beta_m \psi_m(\vec{r}, \omega) \end{pmatrix}, \tag{23}$$

where $\{\alpha_n\}$ and $\{\beta_m\}$ are complex numbers, in general. We now calculate

$$I(\vec{E}) \equiv \int_D d^2 r_1 \int_D d^2 r_2 \vec{E}^T(\vec{r}_1, \omega) \vec{W}(\vec{r}_1, \vec{r}_2, \omega) \vec{E}^*(\vec{r}_2, \omega). \tag{24}$$

On using two coherent mode representations and two bi-modal expansions for elements of the cross spectral density matrix, after a straightforward calculation, we obtain the following expression of $I(\vec{E})$:

$$I(\vec{E}) = \sum_{n=1}^{N} \lambda_n^{(x)} \alpha_n^* \alpha_n + \sum_{n=1}^{N} \sum_{m=1}^{M} \Lambda_{mn} \alpha_n^* \beta_m + \sum_{n=1}^{N} \sum_{m=1}^{M} \Lambda_{nm} \alpha_n \beta_m^* + \sum_{m=1}^{M} \lambda_m^{(y)} \beta_m^* \beta_m. \tag{25}$$

This expression can be expressed in the matrix form as

$$I(\vec{E}) = \vec{A}^T \vec{\Omega} \vec{A}^*, \tag{26}$$

where

$$\vec{A} = \begin{pmatrix} \vec{\alpha} \\ \vec{\beta} \end{pmatrix},$$

$$\vec{\Omega} = \begin{pmatrix} \vec{L}^{(x)} & \vec{\Lambda} \\ \vec{\Lambda}^+ & \vec{L}^{(y)} \end{pmatrix},$$

$$\vec{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \alpha_N \end{pmatrix}, \quad \vec{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \beta_M \end{pmatrix},$$

$$\vec{L}^{(x)} = \begin{pmatrix} \lambda_1^{(x)} & 0 & \cdot & \cdot & 0 \\ 0 & \lambda_2^{(x)} & 0 & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \lambda_N^{(x)} \end{pmatrix},$$

$$\vec{L}^{(y)} = \begin{pmatrix} \lambda_1^{(y)} & 0 & \cdot & \cdot & 0 \\ 0 & \lambda_2^{(y)} & 0 & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \lambda_M^{(y)} \end{pmatrix},$$

$$\vec{\Lambda} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \cdot & \cdot & \Lambda_{1M} \\ \Lambda_{21} & \Lambda_{22} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \Lambda_{N1} & \cdot & \cdot & \cdot & \Lambda_{NM} \end{pmatrix}. \tag{27}$$

The rank of the matrix $\vec{\Omega}$ is $(N+M)$ and it is Hermitian. The non-negative definite character of the cross spectral density matrix is now transferred to the matrix $\vec{\Omega}$. It is evident that $\vec{\Omega}$ has $(N+M)$ eigenvectors and the corresponding eigenvalues and the ensemble realization can be obtained exactly same way as in Eqs. (7) and (8). For elucidating the procedure to get vector modes from scalar modes explicitly, we present one simple example below.

We consider each diagonal element has only one mode respectively and the corresponding coherent mode representations are given by

$$W_{xx}(\vec{r}_1, \vec{r}_2, \omega) = \lambda(\omega) \phi^*(\vec{r}_1, \omega) \phi(\vec{r}_2, \omega), \tag{28}$$

and

$$W_{yy}(\vec{r}_1, \vec{r}_2, \omega) = \lambda(\omega) \psi^*(\vec{r}_1, \omega) \psi(\vec{r}_2, \omega), \tag{29}$$

where we set $\lambda^{(x)} = \lambda^{(y)} \equiv \lambda$ for convenience. The off-diagonal elements of the cross spectral density matrix should necessarily be written as

$$W_{xy}(\vec{r}_1, \vec{r}_2, \omega) = \Lambda(\omega) \phi^*(\vec{r}_1, \omega) \psi(\vec{r}_2, \omega), \tag{30}$$

$$W_{yx}(\vec{r}_1, \vec{r}_2, \omega) = \Lambda^*(\omega) \psi^*(\vec{r}_1, \omega) \phi(\vec{r}_2, \omega). \tag{31}$$

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The corresponding matrix $\bar{\Omega}$ may be written as

$$\bar{\Omega} = \begin{pmatrix} \lambda & \Lambda \\ \Lambda^* & \lambda \end{pmatrix}, \tag{32}$$

and the eigenvalues are found to be

$$\lambda_\pm = \lambda \pm |\Lambda|. \tag{33}$$

The corresponding eigenvectors are also easily obtained as

$$\vec{A}_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm |\Lambda|/\Lambda \end{pmatrix}, \tag{34}$$

and finally the two vector modes are given by

$$\vec{E}_\pm(\vec{r},\omega) = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi(\vec{r},\omega) \\ \pm \left(|\Lambda|/\Lambda^*\right)\psi(\vec{r},\omega) \end{pmatrix}. \tag{35}$$

When $\Lambda = 0$, $\bar{\Omega}$ becomes diagonal and two eigenvalues are the same. Any two mutually orthogonal vectors can be taken as the corresponding eigenvectors and the simplest ones may be

$$\vec{A}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{A}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{36}$$

The two vector modes are then given by

$$\vec{E}_1(\vec{r},\omega) = \begin{pmatrix} \phi(\vec{r},\omega) \\ 0 \end{pmatrix}, \quad \vec{E}_2(\vec{r},\omega) = \begin{pmatrix} 0 \\ \psi(\vec{r},\omega) \end{pmatrix}. \tag{37}$$

In this paper, we considered the connection between the vector mode representation for an entire cross spectral density matrix and the two scalar mode representations for two diagonal matrix elements of the cross spectral density matrix of an electromagnetic beam. We explicitly constructed the vector modes out of the two kinds of scalar modes and showed that the number of vector modes is given by the sum of two numbers of scalar modes. These results have been demonstrated through simple examples.

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