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쇼케이적분에서 퍼지 프리인벡스에 관한 연구

On fuzzy preinvexity in Choquet integrals

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요 약

우리는 퍼지 인벡스 집합, 퍼지 프리인벡스 함수, 퍼지 유사-프리인벡스 함수 와 퍼지 로그 프리인벡스 함수를 생각한다. 무로푸시 등은 쇼케이적분과 그 응용에 관한 연구를 계속해오고 있다. 이 논문에서는 다음과 같은 쇼케이적분에서의 성질들을 조사한다: 퍼지 프리인벡스성, 퍼지 유사-프리인벡스성 과 퍼지 로그 프리인벡스성, 즉, 쇼케이 적분에 의해 정의되는 범함수의 성질들임. 더욱이 쇼케이적분의 제센 형태 부등식을 증명한다.

Abstract

We consider fuzzy invex sets, fuzzy preinvex functions, fuzzy quasi-preinvex functions, and fuzzy logarithmic preinvex functions. Murofushi et al. have been studied Choquet integrals and their properties. In this paper, we study some characterizations in Choquet integrals as follows: fuzzy preinvexity, fuzzy quasi-preinvexity, and fuzzy logarithemic preinvexity, that mean some characterizations of functionals defined by Choquet integrals. Furthermore, we discuss Jensen's type inequality in Choquet integrals.

Key Words : fuzzy invex set, fuzzy preinvex functions, fuzzy quasi-preinvex functions, fuzzy logarithmic preinvex functions, Choquet integrals, fuzzy measures.

1. Introduction

Noor([5,6]) introduced the concept of fuzzy invex sets, fuzzy preinvex functions, fuzzy quasi-preinvex functions, and fuzzy logarithmic preinvex functions which are applied in optimization theory. We note that these functions are more general than convex fuzzy functions. Murofushi et al. have been studied Choquet integrals which are applied in capacity theory, decision making theory, and utility theory(see[1,2,3,4,7]).

In this paper, we investigate some characterizations of functionals defined by Choquet integrals as in the optimization tool which is used in capacity theory, decision making theory, and utility theory.

In section 2, we list various definitions and notation which are used in the proof of our results. In section 3, by using these definitions and their properties, we study some characterizations in Choquet integrals as follows: fuzzy preinvexity, fuzzy quasi-preinvexity, and fuzzy logarithemic preinvexity, that mean some characterizations of functionals defined by Choquet integrals. Furthermore, we discuss Jensen's type inequality in Choquet integrals.

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2. Prelimaries and definitions

Throughout this paper, we assume that X is a locally compact Hausdorff space,

Ω is a σ-algebra of X M is the class of measurable functions of X M^+ is the class of non-negative measurable functions in M and O is the class of open subsets of X K is the class of continuous functions on X with compact support, and K^+ is the class of non-negative functions in K

A fuzzy measure μ on Ω is a set function $\mu:\Omega\to[0,\infty]$ satisfying

(1) $\mu(\emptyset)=0$,
(2) $\mu(A)\le\mu(B)$,

whenever $A,B\in\Omega$, $A\subset B$. A fuzzy measure μ is said to be lower semi-continuous if for every increasing sequence A_n of measurable sets, we have

$$\mu(\cup_{n=1}^{\infty}A_n)=\lim_{n\to\infty}\mu(A_n).$$

A fuzzy measure μ is said to be upper semi-continuous if for every decreasing sequence $\{A_n\}$ of measurable sets and $\mu(A_1)<\infty$, we have

$$\mu(\cap_{n=1}^{\infty} A_n)=\lim_{n\to\infty}\mu(A_n).$$

If μ is both lower and upper semi-continuous, it is said to be continuous(see [1,2,3,4,8]).

Definition 2.1 ([1,2,3,4,8]) (1) The Choquet integral of a measurable function $f\in M^+$ on A with respect to a fuzzy measure μ is defined by

$$(C)\int_A f\, d\mu=\int_0^{\infty}\mu(\{x|f(x)>r\}\cap A)\,dr$$

where the integral on the right-hand side is an ordinary one.

(2) A measurable function $f\in M^+$ is called integrable if the Choquet of f is defined and its value is finite.

Definition 2.2 ([1,2,3,4,8]) Let $f,g\in M^+$. Then we say that f is comonotonic to g in symbol $f\sim g$ if and only if

$$f(x)<f(x') \Rightarrow g(x)\le g(x')$$

for all $x,x'\in X$.

Theorem 2.3 ([1,2,3,4,8]) Let $f,g,h\in M^+$. Then we have

(1) $f\sim f$,
(2) $f\sim g \Rightarrow g\sim f$,
(3) $f\sim a$ for all $a\in R^+$,
(4) $f\sim g$ and $f\sim h \Rightarrow f\sim(g+h)$.

Theorem 2.4 ([1,2,3,4,8]) Let $f,g\in M^+$.

(1) If $f\le g$, then $(C)\int f\,d\mu\le(C)\int g\,d\mu$.

(2) If $A\subset B$ and $A,B\in\Omega$, then $(C)\int_A f\,d\mu \le (C)\int_B f\,d\mu$.

(3) If $f\sim g$ and $a,b\in R^+$, then $(C)\int (af+bg)\,d\mu = a(C)\int f\,d\mu+b(C)\int g\,d\mu$.

(4) If $(f\vee g)(x)=\max\{f(x),g(x)\}$ and $(f\wedge g)(x)=\min\{f(x),g(x)\}$ for all $x\in X$, then

$$(C)\int (f\vee g)\,d\mu\ge (C)\int f\,d\mu\vee(C)\int g\,d\mu$$

and $(C)\int (f\wedge g)\,d\mu\le (C)\int f\,d\mu\wedge(C)\int g\,d\mu$.

Now, we introduce the concepts of fuzzy invex sets, fuzzy preinvex functions, fuzzy quasi-preinvex functions, and fuzzy logarithmic preinvex functions. Let $\vec{H}$ be a vector space over the field R^+, and $\vec{K}$ a nonempty subset of $\vec{H}$.

Definition 2.5 ([5]) (1) A subset $\vec{K}$ of $\vec{H}$ is said to be fuzzy invex at g with respect to η if for each $f\in\vec{K}$

$$g+t\eta(f,g)\in\vec{K} \quad t\in[0,1]$$

where $\eta:\vec{K}\times\vec{K}\to\vec{H}$ is a mapping.

(2) $\vec{K}$ is said to be fuzzy invex set with respect to η if $\vec{K}$ is fuzzy invex at g for each $f\in\vec{K}$.

Definition 2.6 ([5]) Let $\vec{K}$ be a fuzzy invex subset of $\vec{H}$ with respect to η. A functional $\Phi:\vec{K}\to R^+$ is said to be fuzzy preinvex at $g\in\vec{K}$ with respect to η if

$$\Phi(g+t\eta(f,g))\le(1-t)\Phi(g)+t\Phi(f)$$

for all $t\in[0,1]$ and $f\in\vec{K}$.

Definition 2.7 ([5]) (1) Let $\vec{K}$ be a fuzzy invex set. A functional Φ is said to be fuzzy quasi preinvex at $g\in\vec{K}$ with respect to η if

$$\Phi(g+t\eta(f,g))\le\max\{\Phi(g),\Phi(f)\}$$

for all $f\in\vec{K}$ and $t\in(0,1)$.

(2) Φ is said to be fuzzy quasi preinvex with respect to η on $\vec{K}$ if it is fuzzy quasi preinvex at each $f\in\vec{K}$ with respect to η.

Definition 2.8 ([5]) A functional $\Phi:\vec{K}\to R^+$ is said to be a fuzzy logarithmic preinvex at $g\in\vec{K}$ with respect to η if

$$\Phi(g+t\eta(f,g))\le(\Phi(g))^{1-t}(\Phi(f))^{t}$$

for $t\in(0,1)$ $f,g\in\vec{K}$.

(2) Φ is said to be fuzzy quasi preinvex with respect to η on $\vec{K}$ if it is fuzzy quasi preinvex at each $f\in\vec{K}$ with respect to η.

3. Fuzzy preinvexity in Choquet integrals

We consider the following classes of Choquet integrable functions;

$$\vec{F}=\{f\in K^+|\ (C)\int f\,d\mu<\infty\}$$

and for each $g\in\vec{F}$

$$\vec{F}_g=f\in\vec{F}\,|\ f\sim g$$

At first, we will prove that for each $g\in\vec{F}$ the classes $\vec{F}_g$ are fuzzy invex set at g. By using these classes, we investigate discuss the followings in the special case of $\vec{H}=\vec{F}$ and $\vec{K}=\vec{F}_g$.

Theorem 3.1 Let $\eta:\vec{F}_g\times\vec{F}_g\to\vec{F}$ be a mapping. If $g\in\vec{F}$ and $\eta(f,g)$ is comonotonic to g for all $f\in\vec{F}$ then $\vec{F}_g$ is fuzzy invex set at g with respect to η.

Proof. For each $f\in\vec{F}_g$, by the hypothesis of η, we have

$$g\sim g \text{ and } g\sim\eta(f,g).$$

It is easily to see that $g\sim t\eta(f,g)$ for all $t\in[0,1]$. Then, by Theorem 2.3 (4), we have

$$g\sim g+t\eta(f,g)\in\vec{F}_g.$$

Thus, $\overrightarrow{F_g}$ is fuzzy invex set at g with respect to the mapping η.

We remark that in general, $\overrightarrow{F_g}$ is not a fuzzy invex set. In fact, if we take $g=c$(=constant), then clearly we have that $\overrightarrow{F_g}=\overrightarrow{F}$ because every constant function is co-monotonic to each $f\in\overrightarrow{F_g}$. Now, we consider the following special class of Choquet integrable functions;

$$\overrightarrow{F_g^*}=\{f\in\overrightarrow{F_g}|f\geq g\}$$

for each $g\in\overrightarrow{F}$. Then, it is easily to see that for each $g\in\overrightarrow{F}$, $\overrightarrow{F_g^*}$ is a fuzzy invex subset at g with respect to the mapping η.

Lemma 3.2 If $f,g\in\overrightarrow{F}$ with $g\geq f$ and $f-g\sim g$, then we have

$$\int f-g d\mu=(C)\int f d\mu-(C)\int g d\mu.$$

Proof. We put $h=f-g\sim g$. Then we have

$$f=g+h \text{ and } h\sim g.$$

By Theorem 2.3(3),

$$\begin{aligned}\int f d\mu &=(C)\int g+h\, d\mu\\ &=(C)\int g\, d\mu+(C)\int h\, d\mu.\end{aligned}$$

Thus,

$$\begin{aligned}(C)\int f-g d\mu &=(C)\int h\, d\mu\\ &=(C)\int f d\mu-(C)\int g d\mu.\end{aligned}$$

Secondly, we investigate some characterizations in Choquet integrals as follows: fuzzy preinvexity, fuzzy quasi-preinvexity, and fuzzy logarithmic preinvexity.

Theorem 3.3 Let $\overrightarrow{F_g^*}$ be as in the same above set. Assume that for all $f\in\overrightarrow{F_g^*}$,
(i) $\eta(f,g)$ is comonotonic to g (ii) $\eta(f,g)\leq f-g$, and (iii) $f-g\sim g$. If $\Phi_C^*:\overrightarrow{F_g^*}\to R^+$ is defined by

$$\Phi_C^*(f)=(C)\int f d\mu,$$

then Φ_C^* is a fuzzy preinvex at g with respect to η.

Proof. For all $f\in\overrightarrow{F_g^*}$, it is easily to see that $f-g\in\overrightarrow{F_g^*}$ and

$$\Phi_C^*(f-g)=\Phi_C^*(f)-\Phi_C^*(g).$$

For all $t\in[0,1]$, by the assumption and Theorem 2.4 (1), (3) and Lemma 3.2, we have

$$\begin{aligned}&\Phi_C^*(g+t\eta(f,g))\\ &=(C)\int (g+t\eta(f,g))d\mu\\ &=(C)\int g d\mu+t(C)\int \eta(f,g)d\mu\\ &\leq(C)\int g d\mu+t(C)\int f-g\, d\mu\\ &=(C)\int g d\mu+t(C)\int f d\mu-t(C)\int g d\mu\\ &=(1-t)(C)\int g d\mu+t(C)\int f d\mu\\ &=(1-t)\Phi_C^*(g)+t\,\Phi_C^*(f).\end{aligned}$$

Therefore Φ_C^* is a fuzzy preinvex at g with respect to η.

Theorem 3.4 If a functional $\Phi_C^*:\overrightarrow{F_g^*}\to R^+$ is defined by

$$\Phi_C^*(h)=(C)\int h\, d\mu,$$

and if $\ln\Phi_C^*$ is a fuzzy preinvexity at $g\in\overrightarrow{F}$ with respect to η, then Φ_C^* is a fuzzy logarithmic preinvexity at $g\in\overrightarrow{F}$ with respect to η.

Proof. For any $t\in(0,1)$, $f\in\overrightarrow{F_g^*}$, by the assumption of $\ln\Phi_C^*$,

$$\begin{aligned}&\ln\Phi_C^*(g+t\eta(f,g))\\ &\leq(1-t)\ln\Phi_C^*(g)+t\ln\Phi_C^*(f)\\ &=\ln\Phi_C^*(g)^{(1-t)}+\ln\Phi_C^*(f)^{t}\\ &=\ln\Phi_C^*(g)^{(1-t)}\Phi_C^*(f)^{t}.\end{aligned}$$

Thus, we have

$$\Phi_C^*(g+t\eta(f,g))\leq\Phi_C^*(g)^{(1-t)}\Phi_C^*(f)^{t}.$$

Remark 3.5 By Definition 2.6 and Definition 2.7, it is easily to see that fuzzy preinvexity $\Rightarrow$ fuzzy quasi preinvexity. Thus, under the same assumtion in Theorem 3.3, we obtain that Φ_C^* is a fuzzy quasi preinvex at g with respect to η.

Finally, we introduce Jensen's inequality in real and complex analysis([7]) and discuss Jensen's type inequality in Choquet integrals.

Theorem 3.6 (Jensen's inequality) ([7]) Let ν be a positive measure on a σ-algebra $\mathfrak{I}$ in a set R^- so that $\nu(R^-)=1$. If $f:R^-\to R$ is a function such that $\int_{R^-} f d\nu<\infty$, if $a<f(\alpha)<b$ for all $\alpha\in R^-$, and if ζ is convex on (a,b), then

$$\zeta\left(\int_{R^-} f d\nu\right)\leq\int_{R^-}(\zeta\circ f)d\nu.$$

Theorem 3.7 Assume that μ is a finite fuzzy measure with

$$\begin{aligned}&\ln(\mu\{x\in X|h(x)>\alpha\})\\ &\leq\mu(\{x\in X|\ln h(x)>\alpha\})\end{aligned}$$

and that $\{\|h\|_\infty\,|\,h\in\overrightarrow{F_g^*}\}=L$ is finite. If a functional $\Phi_C^*:\overrightarrow{F_g^*}\to R^+$ is defined by

$$\Phi^*_C(h) = (C)\int h\, d\mu,$$

Φ^*_C satisfies the following inequality;

$$\ln \Phi^*_C(h) \le \Phi^*_C(\ln(h)).$$

Proof. If m is Riemann measure and we put

$$a=0, b=L \quad \nu = \frac{m}{\mu(X)},$$

$$f(\alpha) = \mu(\{x \in X \mid h(x) > \alpha\}),$$

and $\zeta(x) = \ln(x)$ (=logarithmic function), then we obtain $\nu(R^+)=1$ and $\zeta = \ln$ is convex on (a,b). Thus by Theorem 3.4 and the hypothesis, we have

$$\begin{aligned}
\ln \Phi^*_C(h) &= \ln((C)\int_X h\, d\mu) \\
&= \ln \zeta\, (\int_{R^+} f(\alpha)\, dm(\alpha)) \\
&\le \int_{R^+} \ln \circ f(\alpha) dm(\alpha) \\
&= (C)\int_{R^+} \ln \circ \mu(x \in X \mid h(x) > \alpha) d\alpha \\
&\le \int_{R^+} \mu(x \in X \mid \quad \ln h(x) > \alpha) d\alpha \\
&= \Phi^*_C(\ln(h)).
\end{aligned}$$

4. References

[1] G. Choquet, Theory of capacities, *Annales de I'Institut Fourier* Vol.5, pp.131-295, 1953.

[2] T. Murofushi and M. Sugeno, *An interpretation of fuzzy measures and the Choquet integral as an integral with respect to a fuzzy measure,* Fuzzy Sets and Systems Vol.29, pp.201-227, 1989.

[3] T. Murofushi and M. Sugeno, *A theory of fuzzy measures: representations, the Choquet integral, and nulls,* J. Math. Anal. and Appl. Vol.159, pp.532-549, 1991.

[4] Y. Narukawa, T. Murofushi and M. Sugeno, *Extension and representation of comonotonically additive functionals*, Fuzzy Sets and Systems Vol.121, pp. 217-226, 2001.

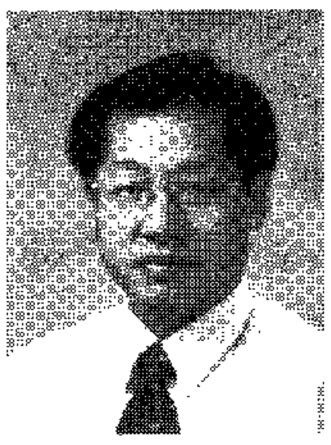
[5] M. A. Noor, *Fuzzy preinvex functions,* Fuzzy Sets and Systems Vol.64, pp.95-104, 1994.

[6] M. A. Noor and K. I. Noor, *Some characterizations of strongly preinvex functions*, J. Math. Appl. Vol.316, pp.697-706, 2006.

[7] W. Rudin, *Real and complex analysis,* McGraw Hill Book Company, New York, 1966.

[8] D. Schmeidler, *Subjective probability and expected utility without additivity,* Econometrica Vol.57, pp.571-587, 1986.

저 자 소 개



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