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Circular Statistics in Musicology†

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Abstract

An essential aspect of music is structure. Beran (2004) introduced a method of comparing piano plays via circular statistics based on the fact that there is circular structure in music. We expand the application of this method to a pair of two pop songs and discuss the possibility of applying it to detecting musical plagiarism. Circular statistics provides an objective view point comparing the musical works.

Keywords: Circular statistics; statistics in musicology.

1. Introduction

In many diverse scientific fields, the measurements are directions. For instance, a biologist may be measuring the direction of flight of birds or the orientation of an animal, while a meteorologist may be interested in the direction of wind or the ocean current. A set of such observations on directions is referred to as directional data and it is also called circular data in two-dimension. For some systematic accounts of statistical theory and methodology with applications for circular observations (see, Mardia, 1972; Fisher, 1993; Jammalamadaka and SenGupta, 2001). An application of those literatures can be made in music based on the fact that many phenomena in music are circular. The best known examples are repeated rhythmic patterns, the circle of fourths and scales modulo octave in the well-tempered system. By comparing the results of analysis in different piano plays, Beran (2004) suggests that some descriptive statistics can represent certain aspects of musical features. In this paper, we adapt the method presented in Beran (2004) to a pair of two pop songs which are allegedly very similar and discuss the possibility of applying this method to detecting musical plagiarism.

In Section 2, we provide the review of basic principles defined in the sense of circular data. A brief description of the method adapted to musical works and results of the analysis are presented in Section 3. Concluding remarks are given in Section 4.

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2. Review: Circular Statistics

Two-dimensional directions can be represented as angles measured with respect to suitably chosen starting point and positive direction. Since a direction has no magnitude, these can be conveniently represented as points on the circumference of a unit circle centered at the origin or as unit vectors connecting the origin to these points. Such distinctive features make circular analysis substantially different from the standard linear statistical analysis of univariate or multivariate data that one finds in most statistical books. Being aware of the limitations of the standard linear statistical methods, we need to present some basic principles through a new methodology.

Suppose observations are given in angle, φ_i $(i = 1, \dots, n)$. We replace φ_i by

$$x_i = (\cos\varphi_i, \sin\varphi_i)^t,$$

where φ_i is measured anti-clockwise relative to the horizontal axis. The following descriptive statistics can then be defined. Let

$$C = \sum_{i=1}^{n} \cos\varphi_i, \qquad S = \sum_{i=1}^{n} \sin\varphi_i, \qquad R = \sqrt{C^2 + S^2}.$$

And for $p = 1, 2, \dots$, let

$$\begin{aligned} C_p &= \sum_{i=1}^{n} \cos p\varphi_i, & S_p &= \sum_{i=1}^{n} \sin p\varphi_i, & R_p &= \sqrt{C_p^2 + S_p^2}, \\ \bar{C}_p &= \frac{C_p}{n}, & \bar{S}_p &= \frac{S_p}{n}, & \bar{R}_p &= \frac{R_p}{n}. \end{aligned}$$

The mean resultant length of φ_i $(i = 1, \dots, n)$ is equal to

$$\bar{R} = \frac{R}{n}.$$

Note that we have $0 \le \bar{R} \le 1$ and if all angles are identical, then $R = n$ so that $\bar{R} = 1$. In an extreme case with $\varphi_i = 2\pi i/n$ (*i.e.* the angles are scattered uniformly over $[0, 2\pi)$), we have $\bar{R} = 0$. In this sense, $\bar{R}$ measures the amount of concentration around the mean direction. However, $\bar{R}$ is not a perfect measure of concentration, since $\bar{R} = 0$ does not necessarily imply that the data are scattered uniformly. For instance, suppose n is even, $\varphi_{2i+1} = \pi$ and $\varphi_{2i} = 0$. Thus there are two preferred directions, even though $\bar{R} = 0$. An overview of descriptive measures of center and variability is given in Table 2.1.

For a time series φ_t $(t = 1, 2, \cdots)$ of circular data, a sample statistic that measures the association between two circular random variables can be carried over to autocorrelations

$$r_\varphi(k) = \frac{\det\left(n^{-1} \sum_{i=1}^{n-k} x_i x_{i+k}^t\right)}{\det\left(n^{-1} \sum_{i=1}^{n-k} x_i x_i^t\right)}.$$

Table 2.1: Some descriptive statistics for circular data.

Name	Definition	Feature measured
Mean resultant length	$\bar{R}=R/n$	Concentration
Principal direction	a=first eigenvector of $S=\sum_{i=1}^{n} x_i x_i^t$	Center (direction, unit vector)
Concentration	$\hat{\lambda}_1$=first eigenvalue of S	Variability
Circular dispersion	$d_n=\left(1-\sqrt{\bar{C}_2^2+\bar{S}_2^2}\right)/(2\bar{R}^2)$	Variability

3. Applications in Music

If an instrument is tuned according to equal temperament, then from the harmonic perspective, there are only 12 different notes. These can be represented as integers modulo 12 with the set $\mathbb{Z}_{12} = \{0, 1, \ldots, 11\}$.

3.1. Variability and autocorrelation of notes modulo 12

The following analysis in Beran (2004) is done for two compositions, "Das Wohltemperierte Klavier I (Prelude and Fugue No. 5)" written by J. S. Bach and "Fourteen Bagatelles Op. 6, No. 3" written by B. Bartók. Pitch is represented in $\mathbb{Z}_{12}$ with 0 set equal to the note modulo 12 with the highest frequency in the composition. Given a note j in $\mathbb{Z}$, the corresponding circular point is then $x = (x_1, x_2)^t = (\cos(2\pi j/12), \sin(2\pi j/12))^t$. The upper envelope is considered only and $\hat{\lambda}_1$, $\bar{R}$, d and $\log m$ are calculated, where the maximal circular autocorrelation $m = \max_{1\le k\le 10} |r_\varphi(k)|$.

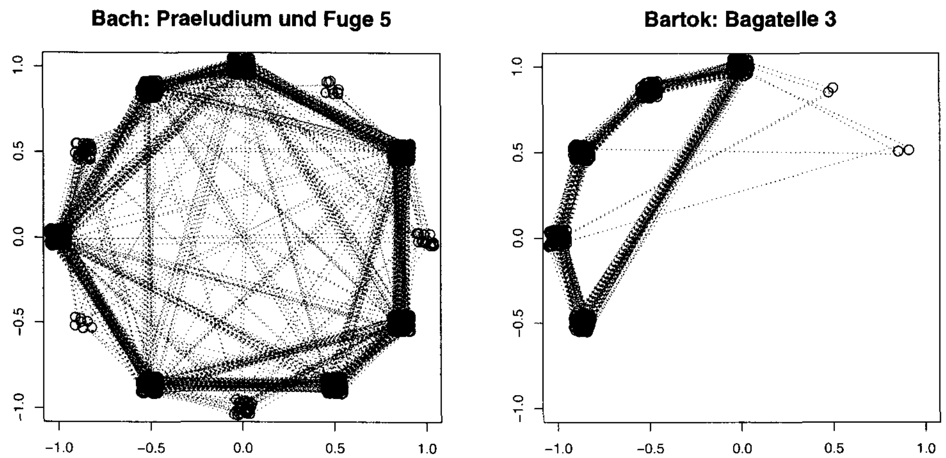

Figure 3.1: Circular representation of compositions written by J. S. Bach (on the left) and B. Bartók (on the right)

Table 3.1: Summary of statistics for notes modulo 12 of the two compositions.

	J. S. Bach (Präludium und Fuge No. 5)	B. Bartók (Bagatelles No. 3)
$\hat{\lambda}_1$	524.7189	209.2528
$\bar{R}$	0.045876	0.736254
d	215.1664	0.744992
$\log m$	−1.888137	−0.063486
$\max_{1\le k\le 10} \lvert r_{\varphi}(k)\rvert$	0.150354 ($k=1$)	0.937488 ($k=5$)

In Figure 3.1, the two compositions are displayed. $\mathbb{Z}_{12}$ is represented by a circle starting on top with 0 and proceeding clockwise as $j \in \mathbb{Z}_{12}$ increases. A composition is represented by pitches $j_1, \dots, j_n \in \mathbb{Z}_{12}$, each pitch is represented by a dot on the circle. In order to visualize how frequent each note is, each point $x_i = (\cos\varphi_i, \sin\varphi_i)^t$, $(i = 1, \dots, n)$ where $\varphi_i = 2\pi j_i/12$ is displaced by slightly adding a random number from a uniform distribution on $[-0.05, 0.05]$ to the angle φ_i. This technique of exploratory data analysis is often referred to as "jittering", see Chambers *et al.* (1983). Moreover, to obtain an impression of the dynamic movement, successive points x_i and x_{i+1} are joined by a line.

For Bach, the main movements take place along the edges which is the feature of a polyphony. The rather curious simple figure for Bartók stems from the continuous repetition of the same chromatic figure in the upper voice.

Table 3.1 shows summary of statistics for notes modulo 12, $\hat{\lambda}_1$, $\bar{R}$, d and $\log m$, comparing Bach and Bartók. Variability measured by d is clearly lower for Bartók.

3.2. Notes on the circle of fourths

The circle (or cycle) of fourths is a mathematical pattern that can be applied to understanding the sounds we hear in Western music. When working out major scales at a piano, the player should memorize the seven notes consists those scales. The representation of the circle of fourths for 12 notes in Figure 3.2 is useful to recognize which notes to play with. For example, say you are wanting to work out the C-major scale. Think of the circle of fourths as a clock. Go to 12 o'clock and find 'C'. Read off the 5 notes that appear to the left of 'C' (so, from 7 o'clock through to 11 o'clock selecting 'B, E, A, D, G') and the note at 1 o'clock ('F') after it. Rearrange them in 'music alphabet' order and you get 'C, D, E, F, G, A, B', the C-major scale.

The same analysis conducted in Section 3.1 is carried out here by ordering notes according to the circle of fourths. The analogous plots are given in Figure 3.3. This specific circular representation makes some symmetries and their harmonic meaning more visible. For Bach, main points and vertices corresponding to the D-major scale is first indicated and each notes are closely distributed in the half of the unit circle. Note that, as we mentioned before, the top point of the circle is set to be the highest frequency in the composition, so the resulting figure looks like following C-major scale. In contrast,

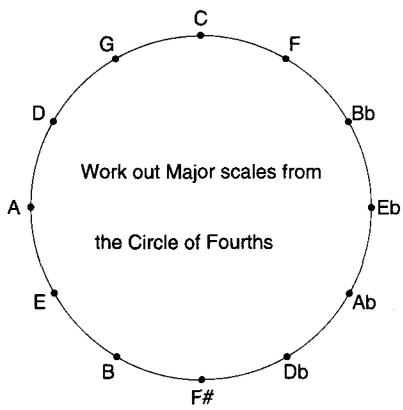

Figure 3.2: Notes represented in the circle of fourths

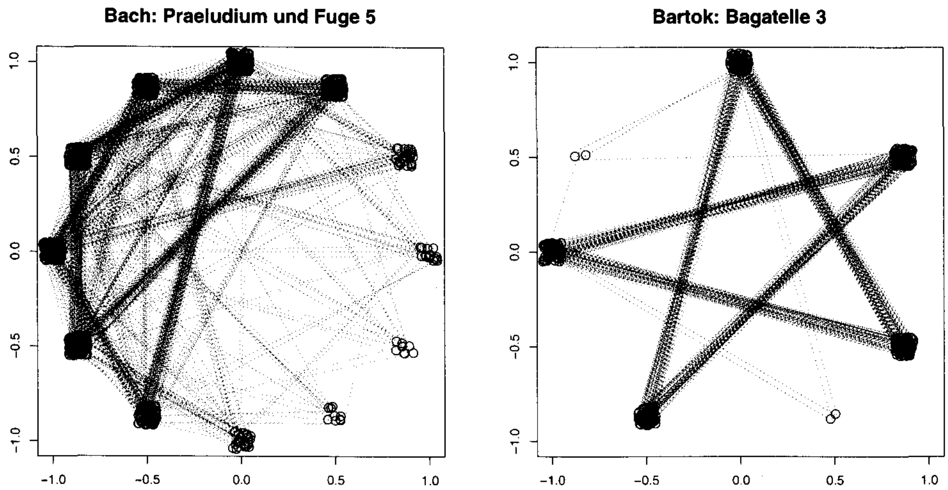

Figure 3.3: Notes ordered according to the circle of fourths of compositions written by J. S. Bach (on the left) and B. Bartók (on the right)

the distances of each notes is outstanding in Bartók, making the figure a star like shape. Summary statistics are in Table 3.2 that we can easily recognize the reverse result of the Table 3.1 in the measure of variability, d.

3.3. Application to the comparison of two pop songs

We adapt the previous method to a pair of two pop songs which is selected from a video clip once have been wide spread through the world wide web. The video clip

Table 3.2: Summary of statistics for notes ordered according to circle of fourths of the two compositions.

	J. S. Bach (*Präludium und Fuge No. 5*)	B. Bartók (*Bagatelles No. 3*)
$\hat{\lambda}_1$	524.7189	209.2528
$\bar{R}$	0.454212	0.050517
d	2.194999	158.2437
$\log m$	-2.735001	-0.041886
$\max_{1\leq k\leq 10}\lvert r_{\varphi}(k)\rvert$	0.063894 ($k=1$)	0.957979 ($k=5$)

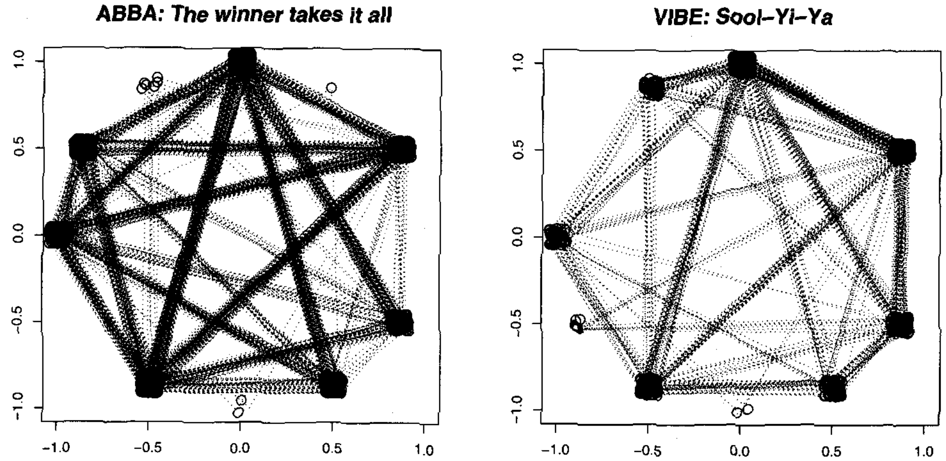

Figure 3.4: Circular representation of pop songs sang by ABBA (on the left) and Vibe (on the right)

contains some pairs of pop songs suspected to be very similar by many Korean music fans. The two pop songs compared in this section are "The Winner Takes It All" sang by ABBA, the well-known Swedish group and "Sool-Yi-Ya" (which means drinking liquor, expressed in Korean alphabets) sang by Vibe, one of the popular Korean groups. We conduct the same analysis as done for the pair of two piano plays in Section 3.1 through 3.2 to this case to verify this method can be used detecting the similarities as well as the differences discussed in those sections. The original note plots analogous to those in Section 3.1 are given in Figure 3.4. Focusing on these plots, one can not easily get an intuition for the musical features of the two pop songs, but summary statistics in Table 3.3 represents the similarity in between the each measurements. The similarity of the two songs is more clear when they are analized by the way given in Section 3.2. The two songs form a contrast in the light of moods, however, the representation of the circles of fourth in Figure 3.5 shows very analogous patterns unlike the results from the two piano plays in the previous section. Table 3.4 indicates this point more clearly.

Table 3.3: Summary of statistics for notes modulo 12 of the two pop songs.

	ABBA (The winner takes it all)	Vibe (Sool-Yi-Ya)
$\hat{\lambda}_1$	609.7389	331.3522
$\bar{R}$	0.155947	0.160526
d	19.67196	15.76255
$\log m$	−3.716118	−2.686786
m	0.023328 ($k = 2$)	0.067100 ($k = 1$)

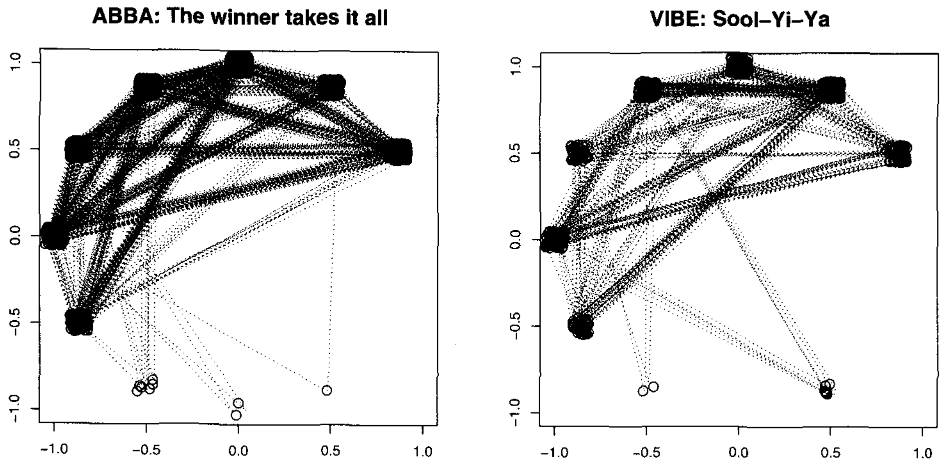

Figure 3.5: Notes ordered according to the circle of fourths of pop songs sang by ABBA (on the left) and Vibe (on the right)

Table 3.4: Summary of statistics for notes ordered according to circle of fourths of the two pop songs.

	ABBA (The winner takes it all)	Vibe (Sool-Yi-Ya)
$\hat{\lambda}_1$	609.7389	331.3522
$\bar{R}$	0.611024	0.587968
d	1.281396	1.174924
$\log m$	−2.715220	−2.286714
m	0.065190 ($k = 5$)	0.100600 ($k = 5$)

So far, each musical data is considered through the entire score. Here, we extract the prominently repeated parts from each of the two pop songs. The starting and ending points of those parts are selected based on the rhythmic patterns of their original sound recordings. The circular representation for the original notes of these parts are given in Figure 3.6 and the summary statistics are in Table 3.5. The size ratio, ABBA/Vibe, of

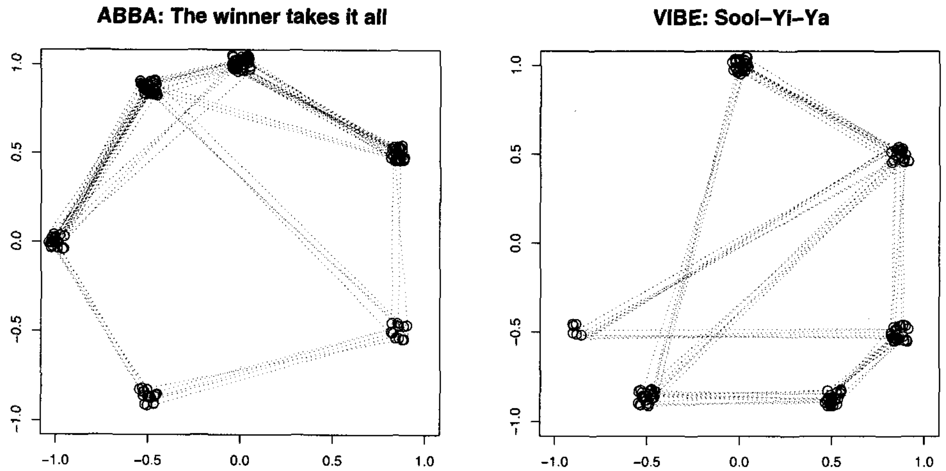

Figure 3.6: Circular representation of the repeated parts of pop songs sang by ABBA (on the left) and Vibe (on the right)

Table 3.5: Summary of statistics for notes modulo 12 of the repeated parts of the two pop songs.

	ABBA (The winner takes it all)	Vibe (Sool-Yi-Ya)
$\hat{\lambda}_1$	84.6053	74.3187
$\bar{R}$	0.426077	0.327878
d	2.249373	3.726901
$\log m$	−0.736188	−1.646398
m	0.477936 ($k = 1$)	0.191743 ($k = 1$)

those parts is approximately 1.153, so that we could conduct the analysis with data of similar lengths. From these features and figures, we can not give any clear comparative statements, but the circle of fourths analysis in Figure 3.7 and Table 3.6 shows some similarities between the two songs. The overall patterns of those parts in Figure 3.7 look alike except on the points at 8 o'clock in ABBA and those at 3 o'clock in Vibe. Although this discrepancy may affect the contrasted moods of the two songs, the point mass of those parts is relatively small, so that the statistics in Table 3.6 show little difference between the two songs. However, we can not conclude one of the two songs is a plagiarism of the other, since the standard measure of similarity is not defined and the selection of the starting and ending points depends on one's subjective view point. In present, we just take the advantage of this method which yields objective results given in figures.

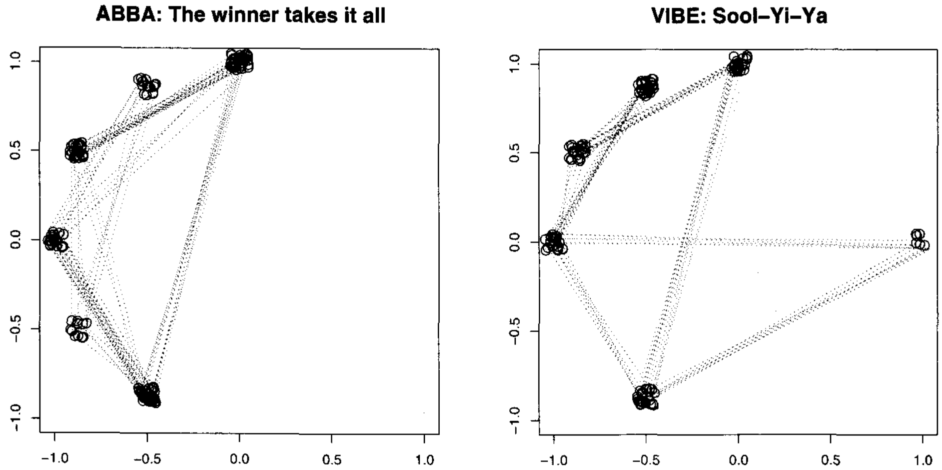

Figure 3.7: Notes ordered according to the circle of fourths of the repeated parts of pop songs sang by ABBA (on the left) and Vibe (on the right)

Table 3.6: Summary of statistics for notes ordered according to circle of fourths of the repeated parts of the two pop songs.

	ABBA (The winner takes it all)	Vibe (Sool-Yi-Ya)
$\hat{\lambda}_1$	84.6053	74.3187
$\bar{R}$	0.560823	0.560645
d	1.298333	1.274665
$\log m$	−0.968056	−1.435608
m	0.378821 ($k = 4$)	0.236971 ($k = 2$)

4. Concluding Remarks

In this paper, we introduced circular statistics and its application in music. Through the comparison of the two pairs of musical plays, we found that circular statistics provides a novel view point for the musical features. In particular, by observing the pop songs case, we can think about the possibility of applying circular statistics to the field of detection of musical plagiarism. There are individual differences in interpretations of the similarity between various musical works due to the subjectivity of each person's judgement. This statistical method, however, suggests a consistent standard of analysis which can be relatively credible and objective. The method will be much improved if the rhythmic patterns with the time when the notes are played can be taken into account in this literature. It remains as a future research.

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