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Aspects of Dependence in Lomax Distribution

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Abstract

In this paper we study some positive dependence concepts, introduced by Capéraà and Genest (1990) and Shaked (1977b), for bivariate lomax distribution. In particular, we obtain some measures of association for this distribution and derive the tail-dependence coefficients by using copula function. We also compare Spearman's ρ_s with Kendall's τ for bivariate lomax distribution.

Keywords: Bivariate lomax distribution; positive quadrant dependence; dependent by total positivity of order two; decreasing failure rate; copula; tail dependence of coefficient.

1. Introduction

In the univariate setup the lomax distribution is being widely used for stochastic modeling of decreasing failure rate life components. In reliability, the component lifetimes are often assumed to be independently distributed random variables. Even if the components are structurally independent, the use of the system in varying environments may make the lifetimes dependent (Shaked, 1977a; Winterbottom, 1984). A bivariate extension of the lomax distribution given in Lindley and Singpurwalla (1986) fails to cover the case of independence. The multivariate lomax distribution is discussed by Nayak (1987) and sums, product, ratios and their moments for the bivariate lomax distribution(BLD) are studied by Nadarjah (2005). According to Lindley and Singpurwalla (1986), we consider a two component system where for a given environment η the component lifetimes X and Y are independently exponentially distributed with failure rates $\eta\lambda_1$ and $\eta\lambda_2$ respectively where λ_1 and λ_2 being the failure rate under the test environment. Let $H(\eta)$ be the distribution function of η then the BLD is obtained by assuming to be a gamma distribution with density function $h(\eta) = b^a/\Gamma(a)\eta^{a-1}e^{\eta b}$, $\eta > 0,\ a, b > 0$ and density of BLD is

$$f(x,y) = \int_0^\infty \eta^2\lambda_1\lambda_2 e^{-\eta\lambda_1 x-\eta\lambda_2 y}h(\eta)d\eta = \frac{\theta_1\theta_2 a(a+1)}{(1+\theta_1 x+\theta_2 y)^{a+2}}, \tag{1.1}$$

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where $\theta_i = \lambda_i/b > 0, \ \ i = 1,2$ and $x, y > 0$.

From (1.1), one can obtained joint and marginal survival functions as follows.

$$\bar{F}(x,y) = (1+\theta_1 x+\theta_2 y)^{-a}, \quad \bar{F}_1(x) = (1+\theta_1 x)^{-a} \text{ and } \bar{F}_2(y) = (1+\theta_2 y)^{-a}. \quad (1.2)$$

This paper is organized as follows. In section 2, we study some concepts of dependence for BLD and obtain some new results. The section 3 is related to failure rate of conditional distributions and some useful properties based on these failure rates. In section 4, we derive copula function and tail dependence coefficient for BLD and we determine some association measures in section 5.

Let (X,Y) be a two dimensional random variable with density function $f(x,y)$, distribution function $F(x,y)$ and reliability function $\bar{F}(x,y) = P(X > x, Y > y)$. Then the following quantities are defined:

1. Total positivity of order two(TP_2) (Karlin, 1968) $f(x,y)$ is TP_2 if $f(x,y) \geq 0$ and for all $x < x'$, $y < y'$,

$$f(x,y)f(x',y') \geq f(x,y')f(x',y).$$

2. Let the conditional distribution of Y given $X(F_{Y|x}(y))$ is continuous and strictly increasing in y. Then $F^{-1}_{Y|x}(y)$ exist and we can construct a cumulative distribution function $F_{x,x'}(u) : [0,1] \to [0,1]$ as $F_{x,x'}(u) = F_{Y|x'} o F^{-1}_{Y|x}(u)$ (Capéraà and Genest, 1990).

3. (i) The random vector (X,Y) or its distribution function is said to be *right corner set increasing*(RCSI) if $P(X > x, Y > y | X > x', Y > y')$ is increasing in x' and y' for all x and y (Harris, 1970).

   (ii) The random variable X is said to be *stochastically increasing* in Y (SI($X|Y$)) if $P(X > x | Y = y)$ is increasing in y for all x.

   (iii) The random variable X is said to be *right tail increasing* in Y (RTI($X|Y$)) if $P(X > x | Y > y)$ is non-decreasing in y for all x and X is called *left tail* decreasing in Y (LTD($X|Y$)) if $P(X \leq x | Y \leq y)$ is non-increasing in y for all x (Barlow and Proschan, 1975, p.142–146).

4. The random variables X and Y are said to be positively quadrant dependent(PQD) if for all x and y, $P(X > x,\ Y > y) \geq P(X > x)P(Y > y)$. Note that the likelihood ratio dependence(LRD) is a TP_2-property for the density $f(x,y)$. Moreover, LRD $\Rightarrow$ PQD (Lehmann, 1966).

5. The bivariate failure rate (Basu, 1971). The failure rate of a random vector (X,Y) having joint density $f(x,y)$ and distribution function $F(x,y)$ is given by

$$r(x,y) = \frac{f(x,y)}{\bar{F}(x,y)}.$$

Johnson and Kotz (1975) defined the hazard gradient as a vector $(r(x|Y > y), r(y|X > x))$, where $r(x|Y > y)$ is the hazard rate of the conditional distribution of X given $Y > y$. Similarly $r(y|X > x)$ is the hazard rate of the conditional distribution of Y given $X > x$.

6. Arnold and Zahedi (1988) defined the vector $(m(x|Y > y), m(y|X > x))$ where $m(y|X > x) = E(Y - y|X > x, Y > y)$ is the mean residual life function of Y with the additional information that $X > x$. In general $m(y|X \in A) = E(Y - y|X \in A, Y > y)$. $m(x|Y \in A)$ is defined similarly.

7. Dependent by total positivity of order two.

   Shaked (1977b) proposes some nested definitions of dependence. Let

$$\psi_{m,n}(x,y) = \int_y^\infty \int_{y_{n-1}}^\infty \cdots \int_{y_1}^\infty \int_x^\infty \int_{x_{m-1}}^\infty \cdots \int_{x_1}^\infty f(x_o, y_o) dx_o dx_1 \cdots dx_{m-1} dy_o dy_1 \cdots dy_{n-1},$$

   for $m, n > 0$ and for $m = 0$, $n = 0$ define $\psi_{0,0}(x,y) = f(x,y)$. For $m, n \geq 0$ the random vector (X, Y) or its distribution function F or its Survival function $\bar{F}$ is said to be *dependent by total positivity of order two with degree* (m, n) (denoted by DTP (m, n)) if $\psi_{m,n}(x, y)$ is TP_2 in x and y $(x, y \in R)$.

Remark 1.1 Let (X, Y) be a random vector with joint distribution function F and joint density function f and suppose that (X, Y) is absolutely continuous, based on definition and proposition 3.3 of Shaked (1977b), the random vector (X, Y) is:

(i) DTP(0,0) or equivalently LRD, when the joint density $f(x, y)$ is TP_2.

(ii) DTP(0,1), when $-\partial/\partial x\bar{F}(x,y)$ is TP_2, similarly DTP(1,0) when $-\partial/\partial y\bar{F}(x,y)$ is TP_2.

(iii) DTP(1,1), when $\bar{F}(x, y)$ is TP_2.

(iv) DTP(0,2) (DTP(2,0)), when the mean residual life function,

$$m(y|X = x) = E[Y - y|X = x, Y > y]\,(m(x|Y = y) = E[X - x|X > x, Y = y]),$$

increasing in $y(x)$ for all $x(y)$.

(v) DTP(1,2) (DTP(2,1)), when $m(y|X > x)$ $(m(x|Y > y))$ is increasing in x (y) for all y (x).

8. Bivariate decreasing failure rate. Brindley and Thompson (1972) proved that if X, Y are non-negative random variables with joint distribution function $F(x, y)$ then $F(x, y)$ is decreasing failure rate(DFR) if $\bar{F}(x + \Delta, y + \Delta)/\bar{F}(x, y)$ is increasing in x and y for each $\Delta > 0$ and all $x, y \geq 0$, such that $\bar{F}(x, y) > 0$.

9. Let X,Y be non-negative random variables, denote the conditional hazard function of X given $Y \in A$ by

$$R(x|Y \in A) = \int_0^x r(t|Y \in A)dt = -\log P(X > x|Y \in A). \tag{1.3}$$

2. Some Concepts of Dependence

In this section, We proved some concepts of dependence for BLD. In fact, we studied dependence structure of BLD according to Capéraà and Genest (1990) and nested definitions of dependence in Shaked (1977b).

Proposition 2.1 (Drouet Mari and Kotz, 2001) The random vector (X, Y) is LRD if and only if

$$\frac{F_{x'x}(t) - F_{x'x}(u)}{F_{x'x}(v) - F_{x'x}(u)} \leq \frac{t-u}{v-u},$$

for all $x < x'$ and $0 < u < t < v < 1$.

Remark 2.1 It is clear that, the above inequality holds if and only if $F_{x'x}(u)$ *convex* on [0,1], for all $x < x'$. By using the convexity of $F_{x'x}(u)$, we have the following proposition.

Proposition 2.2 Let (X, Y) be a random vector with BLD function. Then (X, Y) is LRD and consequently is PQD.

Proof: We can show that

$$F_{Y|x}(y) = 1 - \left(1 + \frac{y}{\sigma(x)}\right)^{-(a+1)}, \quad \sigma(x) = \frac{1 + \theta_1 x}{\theta_2}.$$

The function $F_{Y|x}(y)$ is continuous and strictly increasing in y so that

$$F^{-1}_{Y|x}(u) = \sigma(x)\left[(1-u)^{-\frac{1}{a+1}} - 1\right], \quad 0 \leq u \leq 1$$

and

$$F_{x'x}(u) = 1 - [1 + H(u)]^{-(a+1)}, \quad 0 \leq u \leq 1,$$

where $H(u) = \sigma(x)/\sigma(x')[(1-u)^{-1/(a+1)} - 1]$. Monotonicity of $F_{x'x}(u)$ is equivalent to monotonicity of H on [0,1]. We have

$$H'(u) = \frac{\sigma(x)}{(a+1)\sigma(x')}(1-u)^{-\frac{1}{a+1}-1} = \frac{1}{(a+1)(1-u)}\left(H(u) + \frac{\sigma(x)}{\sigma(x')}\right) > 0.$$

Hence $F_{x'x}(u)$ is increasing in u. Moreover $\sigma(x') - \sigma(x)$ is positive for all $x < x'$ therefore

$$\frac{d^2}{du^2}F_{x'x}(u) = \frac{a+2}{(a+1)(1-u)} \cdot \frac{\sigma(x') - \sigma(x)}{\sigma(x')[1 + H(u)]}F'_{x'x}(u) > 0.$$

This implies that $F_{x'x}(u)$ is convex on [0,1], hence Remark 2.1 complete the proof. □

In the following proposition, we show that the all concepts of dependence in Remark 1.1, is valid for BLD function.

Proposition 2.3 Let (X,Y) be a random vector with BLD function. Then (X,Y) is DTP(n,m) for $(n,m)=(0,0),(0,1),(1,0),(1,1),(0,2),(2,0),(1,2),(2,1)$.

Proof:

(i) It is clear that $x_1y_1+x_2y_2 \geq x_1y_2+x_2y_1$, for all $x_1<x_2$, $y_1<y_2$. This implies that $f(x,y)$ in (1.1) is TP_2, then by Remark 1.1 (i) (X,Y) is DTP(0,0).

(ii) We can show that, the functions $-\partial/\partial x\bar{F}(x,y)=a\theta_1(1+\theta_1x+\theta_2y)^{-(a+1)}$ and $-\partial/\partial y\bar{F}(x,y)=a\theta_2(1+\theta_1x+\theta_2y)^{-(a+1)}$ are TP_2, therefore (X,Y) is DTP(0,1) and DTP(1,0) respectively.

(iii) It is easy to see that, $\bar{F}(x,y)$ is TP_2, this means that (X,Y) is DTP(1,1).

(iv) For DTP(2,0) and DTP(0,2), we note that

$$\begin{aligned} m(x|Y\in A) &= E[X-x|X>x,Y\in A] \\ &= \int_0^\infty \frac{P(X>x+t,Y\in A)}{P(X>x,Y\in A)}dt \\ &= \int_0^\infty \frac{P(X>x+t|Y\in A)}{P(X>x|Y\in A)}dt \\ &= \int_x^\infty \frac{\bar{F}(t|Y\in A)}{\bar{F}(x|Y\in A)}dt. \end{aligned}$$

So we have $m(x|Y=y)=\int_x^\infty \bar{F}(t|y)dt/\bar{F}(x|y)=1/a\{\sigma(y)+x\}$ that is increasing in x for all y. Therefore (X,Y) is DTP(2,0). Similarly, it can shown that $m(y|X=x)$ is increasing in y for all x and this yields DTP(0,2).

(v) For all $x,y>0$, we get $m(y|X>x)=E[Y-y|X>x,Y>y]=\int_y^\infty \bar{F}(x,t)/\bar{F}(x,y)$ $dt=(1+\theta_1x+\theta_2y)/\{\theta_2(a-1)\}$, for $a>1$. It is obvious that $m(y|X>x)$ is increasing in x for all y. Then (X,Y) is DTP(1,2). We prove that (X,Y) is DTP(2,1) analogously. □

Remark 2.2 The distribution function $F(x,y)$ is right corner set increasing (RCSI) if and only if $\bar{F}(x,y)$ is TP_2 (Nelsen, 1999, Theorem 5.2.15). Therefore by Proposition 2.3 (iii), the BLD is RCSI if and only if it is DTP(1,1).

3. Failure Rates of the Conditional Distributions

In this section we study failure rate of the conditional distributions and some useful properties for BLD function.

Lemma 3.1 Let (X,Y) be a random vector with BLD function. Then

(i) For all $y > 0$, the random variable $X|Y > y$ distributed as the pareto random variable with parameters $\sigma(y) = (1 + \theta_2 y)/\theta_1$ and a. And $Y|X > x$ distributed as the pareto random variable with parameters $\sigma(x) = (1 + \theta_1 y)/\theta_2$ and a.

(i) For all $y > 0$, the random variable $X|Y = y$ distributed as the pareto random variable with parameters $\sigma(y)$ and $a + 1$. And for all $x > 0$, $Y|X = x$ distributed as the pareto random variable with parameters $\sigma(x)$ and $a + 1$.

Proof: Lemma 3.1 is proved by the definition of conditional distribution. □

Note that, if T distributed as the pareto random variable with parameters α and β, then $r(t) = \beta/(\alpha + t)$. Using this notion, Lemma 3.1 and Propositions 3.4 and 3.9 in Shaked (1977b) we have the following Proposition.

Proposition 3.1 Let (X, Y) be a random vector with BLD function. Then,

(i) For all x (y), $r(x|Y = y)$ $(r(y|X = x))$ is decreasing in y (x) if and only if (X, Y) is DTP(1,0) (DTP(0,1)).

(ii) For all x (y), $r(x|Y > y)$ $(r(y|X > x))$ is decreasing in y (x) if and only if (X, Y) is DTP(1,1).

(iii) The random variable X is SI in Y if and only if $R(x|Y = y)$ is decreasing in y for all x.

(iv) The random variable X is right tail increasing in Y if and only if $R(x|Y > y)$ is decreasing in y for all x.

Proof:

(i) Since $X|Y = y$ is a Pareto random variable with parameter $\sigma(y)$ and $a + 1$, hence $r(x|Y = y) = (a + 1)/\{x + \sigma(y)\} = \{(a + 1)\theta_1\}/(1 + \theta_1 x + \theta_2 y)$ is decreasing in y for all $x > 0$. By Proposition 3.4 in Shaked (1977b), it is equivalent to DTP(1,0).

(ii) Using Lemma 3.1 (i), $r(x|Y > y) = (a\theta_2)/(1 + \theta_1 x + \theta_2 y)$ is decreasing in y for all $x > 0$. This is equivalent to DTP(1,1).

(iii) By using (1.3), $R(x|Y = y) = -\log P(X > x|Y = y) = \log(1 + (\theta_1 x)/(1 + \theta_2 y))^{a+1}$ is decreasing in y for all $x > 0$, so by Proposition 3.9 in Shaked (1977b) it is equivalent to SI$(X|Y)$.

(iv) It is easy to see that, $R(x|Y > y) = -\log P(X > x|Y > y) = \log(1 + (\theta_1 x)/(1 + \theta_2 y))^a$. Therefore Proposition 3.9 in Shaked (1977b) complete the proof. □

Remark 3.1 Stochastically increasing(SI$(X|Y)$) implies right tail increasing(RTI$(X|Y)$) and left tail decreasing(LTD$(X|Y)$) (Barlow and Proschan, 1975).

In the univariete case the lomax distribution is decreasing failure rate(DFR). We will show this for bivariate case.

Proposition 3.2 The bivatiate lomax distribution is decreasing failure rate.

Proof: By the definition of bivariate-DFR, the proof follows from this fact that $\bar{F}(x+\Delta, y+\Delta)/\bar{F}(x,y) = \{1+(\theta_1+\theta_2)\Delta/(1+\theta_1 x+\theta_2 y)\}^{-a}$ is increasing coordinate-wise, for each $\Delta > 0$ and all $x, y \geq 0$. This complete the proof. □

Remark 3.2 The DFR property is preserved under the scale change, that is for all $c > 0$, $c\,X$ and $c\,Y$ are DFR if and only if X, Y are DFR.

4. Tail Dependence

In this section we obtain the tail dependence of coefficients of BLD via the concept of copula. A function C: $[0,1]^2 \to [0,1]$ is called a bivariate copula if it is restriction to $[0,1]^2$ of a bivariate distribution function whose marginals are given by the uniform distribution on [0,1].

Let (X,Y) be a random vector with distribution function $F(x,y)$ with the marginals $F_1(x)$ and $F_2(y)$, by Sklar's Theorem (Nelsen, 1999, p.41), there exists a function C: $[0,1]^2 \to [0,1]$ such that for all x, $y \in R$ we have

$$F(x,y) = C\left(F_1(x), F_2(y)\right).$$

If F is continuous then C is unique and it can be constructed as follows:

$$C(u,v) = F\left(F_1^{-1}(u), F_2^{-1}(v)\right), \quad \text{for all } u,v \in [0,1],$$

or equivalently

$$\hat{C}(u,v) = \bar{F}(\bar{F}_1^{-1}(u), \bar{F}_2^{-1}(v)), \tag{4.1}$$

where $\hat{C}(u,v) = u+v-1+C(1-u,1-v)$. The function $\hat{C}$ is a copula and we refer to $\hat{C}$ as the survival copula of X and Y.

Proposition 4.1 Let (X,Y) be a random vector with BLD function. Then,

$$C(u,v) = \left[(1-u)^{-\frac{1}{a}} + (1-v)^{-\frac{1}{a}} - 1\right]^{-a} + u + v - 1. \tag{4.2}$$

Proof: From (1.2), we can drive $\bar{F}_1^{-1}(u) = (u^{-1/a}-1)/\theta_1$ and $\bar{F}_2^{-1}(v) = (v^{-1/a}-1)/\theta_2$. Using (1.2) and (4.1) we get $\hat{C}(u,v) = \bar{F}((u^{-1/a}-1)/\theta_1, (v^{-1/a}-1)/\theta_2) = (u^{-1/a} + v^{-1/a}-1)^{-a}$ and hence,

$$C(1-u,1-v) = \left(u^{-\frac{1}{a}} + v^{-\frac{1}{a}} - 1\right)^{-a} - u - v + 1.$$

This complete the proof. □

Let (X,Y) be a random vector with joint distribution function F and marginals F_1 and F_2. The quantity $\lambda_u = \lim_{t\to 1^-} P(F_1(X) > t|F_2(Y) > t)$ is called the upper tail dependence coefficient(UTDC) provided the limit exists. We say that (X,Y) has upper tail dependent if $\lambda_u > 0$ and upper tail independent if $\lambda_u = 0$. Similarly, we defined the lower tail dependence coefficient(LTDC) by $\lambda_l = \lim_{t\to 0^+} P(F_1(X) \le t|F_2(Y) \le t)$. The upper tail dependence coefficient (or lower tail dependence coefficient) can also defined via the notion of copula. If C is the copula of (X,Y), then

$$\lambda_u = \lim_{t\to 1^-} \frac{1-2u+C(u,u)}{1-u} \quad \text{and} \quad \lambda_l = \lim_{t\to 0^+} \frac{C(u,u)}{u}. \tag{4.3}$$

Proposition 4.2 Let (X,Y) be a random vector with BLD function. Then (X,Y) has upper tail dependent and lower tail independent.

Proof: By using (4.3) we have

$$\lambda_u = \lim_{u\to 1^-} \frac{\left[2(1-u)^{-\frac{1}{a}}-1\right]^{-a}}{1-u} = \lim_{u\to 1^-} \left[2-(1-u)^{\frac{1}{a}}\right]^{-a} = 2^{-a} > 0,$$

therefore (X,Y) has upper tail dependent. After some calculation we obtain

$$\lambda_l = 2 - 2 \lim_{u\to 0^+} (1-u)^{-\frac{1}{a}-1} \left[2(1-u)^{-\frac{1}{a}}-1\right]^{-a-1} = 0,$$

hence (X,Y) has lower tail independent. □

Definition 4.1 (Frahm, 2006) Let $(X_1, X_2, \ldots, X_n)$ be a random vector with joint distribution function $F(x_1, x_2, \ldots, x_n)$ and marginal distribution functions $F_1, \ldots, F_n$. Moreover, $F_{min} = \min\{F_1(X_1), \ldots, F_2(X_n)\}$ and $F_{\max} = \max\{F_1(X_1), \ldots, F_2(X_n)\}$. The lower extermal dependence coefficient(LEDC) of $(X_1, X_2, \ldots, X_n)$ is defined as $\varepsilon_l = \lim_{t\to 0^+} P(F_{\max} \le t|F_{\min} \le t)$, whereas the upper extermal dependence coefficient(UEDC) of $(X_1, X_2, \ldots, X_n)$ is defined as $\varepsilon_u = \lim_{t\to 1^-} P(F_{\min} > t|F_{\max} > t)$, provided the corresponding limits exist.

Remark 4.1 By Proposition 1 in Frahm (2006), we can derive ε_l and ε_u via the quantities λ_l and λ_u as follows,

$$\varepsilon_l = \frac{\lambda_l}{2-\lambda_l} \quad \text{and} \quad \varepsilon_u = \frac{\lambda_u}{2-\lambda_u}.$$

Therefore if (X,Y) is distributed as BLD, then Proposition 4.2 implies that $\varepsilon_l = \text{o}$ and $\varepsilon_u = 1/(2^{a+1}+1)$. This means that (X,Y) has UED but not LED.

5. Some Association Measures

In this section, we determine Clayton-Oakes association measure (denoted $\theta(x,y)$), Kendall's τ and Spearman's ρ_s association measures for bivariate lomax distribution. Moreover, we compare ρ_s with τ.

5.1. Clayton-Oakes association measure.

Clayton(1978) and Oakes (1989) defined the following association measure

$$\theta(x,y) = \frac{\bar{F}(x,y)D_{12}\bar{F}(x,y)}{D_1\bar{F}(x,y)D_2\bar{F}(x,y)},$$

where $D_{12}\bar{F}(x,y) = \partial^2/(\partial x \partial y)\bar{F}(x,y)$, $D_1\bar{F}(x,y) = \partial/\partial x\bar{F}(x,y)$ and $D_2\bar{F}(x,y) = \partial/\partial y\bar{F}(x,y)$. The parameter $\theta(x,y)$ measures the degree of association between X and Y, independence being implied by $\theta(x,y) = 1$, positive dependence by $\theta(x,y) > 1$ and negative dependence by $\theta(x,y) < 1$ (Gupta, 2003).

Remark 5.1 Gupta (2003) proved, $\theta = r(x|Y = y)/r(x|Y > y)$. If (X,Y) is a random vector with BLD, then we obtain $\theta(x,y) = (a+1)/a > 1$. This implies that (X,Y) is RCSI, consequently (X,Y) is PQD.

5.2. Kendall's τ and Spearman's ρ_s association measures.

Let (X,Y) be a continuous random vector with copula C. Then Kendall's τ and Spearman's ρ_s are formulized via the copula function C as follows (Nelsen, 1999)

$$\begin{aligned} \tau &= 1 - 4I, \quad \text{here} \quad I = \iint_{[0,1]^2} \frac{\partial C(u,v)}{\partial u}\frac{\partial C(u,v)}{\partial v} dudv \quad \text{and} \\ \rho_s &= 12I' - 3, \text{ where } I' = \iint_{[0,1]^2} C(u,v)dudv. \end{aligned}$$

Theorem 5.1 Let (X,Y) be a random vector with BLD function. Then

(i) $\tau = \dfrac{1}{2a+1}$,

(ii) $\rho_s = \displaystyle\sum_{k=0}^{\infty} \frac{12a^2}{2a+k}\binom{k+a}{k} B(3a, k+1) - 3,$

where $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ is the Beta function.

Proof:

(i) Using (4.2), we have

$$\begin{aligned} I &= \iint_{[0,1]^2} \frac{\partial C(u,v)}{\partial u} \cdot \frac{\partial C(u,v)}{\partial v} dudv \\ &= 1 - \int_0^1\int_0^1 (1-v)^{-\frac{1}{a}-1}\left\{(1-u)^{-\frac{1}{a}} + (1-v)^{-\frac{1}{a}} - 1\right\}^{-a-1} dudv \\ &\quad - \int_0^1\int_0^1 (1-u)^{-\frac{1}{a}-1}\left\{(1-u)^{-\frac{1}{a}} + (1-v)^{-\frac{1}{a}} - 1\right\}^{-a-1} dudv \\ &\quad + \int_0^1\int_0^1 (1-u)^{-\frac{1}{a}-1}(1-v)^{-\frac{1}{a}-1}\left\{(1-u)^{-\frac{1}{a}} + (1-v)^{-\frac{1}{a}} - 1\right\}^{-2a-2} dudv \\ &= 1 - I_1 - I_2 + I_3. \end{aligned}$$

It is easy to see that $I_1 = I_2 = 1/2$. With the substitution $(1-u) = t^{-a}$ and $(1-v) = s^{-a}$ the integral I_3 can be calculated as follows

$$\begin{aligned} I_3 &= \int_0^1 \int_0^1 (1-u)^{-\frac{1}{a}-1}(1-v)^{-\frac{1}{a}-1}\left\{(1-u)^{-\frac{1}{a}} + (1-v)^{-\frac{1}{a}} - 1^{-2a-2}\right\} dudv \\ &= \int_1^\infty \int_1^\infty a^2 t^{-2a-2}\left(1 + \frac{s-1}{t}\right)^{-2a-2} dudv \\ &= \frac{a}{2(2a+1)} \end{aligned}$$

Those prove that $\tau = 1 - 4I = 1/(2a+1)$.

(ii) It is clear to show that

$$I' = \iint_{[0,1]^2} C(u,v)dudv = \int_0^1 \int_0^1 \left\{(1-u)^{-\frac{1}{a}} + (1-v)^{-\frac{1}{a}} - 1\right\}^{-a} dudv.$$

Taking $1 - u = t^{-a}$ and $1 - v = s^{-a}$, we derive

$$I' = \int_1^\infty \int_1^\infty a^2 t^{-a-1} s^{-a-1}(t+s-1)^{-a} dsdt.$$

Using the transformations, $t + s - 1 = w$ and $s - r = 1$, we obtain

$$\begin{aligned} I' &= a^2 \int_0^\infty \int_{r+1}^\infty (r+1)^{-a-1} w^{-2a-1}\left(1 - \frac{r}{w}\right)^{-a-1} dwdr \\ &= a^2 \sum_{k=0}^\infty \frac{1}{2a+k}\binom{k+a}{k} B(3a, k+1). \end{aligned}$$

Therefore

$$\rho_s = \sum_{k=0}^\infty \frac{12a^2}{2a+k}\binom{k+a}{k} B(3a, k+1) - 3, \quad a > 0.$$

□

5.3. The Comparison of ρ_s with τ.

Considering the relation of ρ_s, it is not possible to compare analytically ρ_s and τ. However, with the help of some dependence concepts we are able to give a lower bound and a plausible suggestion for upper bound of ratio ρ_s/τ. In particular, if X and Y are continuous random variables and X is left tail decreasing and right tail increasing in Y, then $0 \le \tau \le \rho_s$. Also if X and Y are PQD then $\rho_s \le 3\tau$ (see, Fredricks and Nelsen, 2007). Therefore by Proposition 3.1 and Remark 3.1, for the bivariate lomax distribution $\rho_s/\tau \ge 1$. We compute ρ_s for some a, then compare it with τ. Tables 5.1 and 5.2 show values of $\rho_s = \rho_s(a) = \sum_{k=0}^\infty (12a^2)/(2a+k)\binom{k+a}{k} B(3a, k+1) - 3$, $\tau = \tau(a) = 1/(2a+1)$

Table 5.1: Values of $\rho_s(a)$, $\tau(a)$ and $\rho_s(a)/\tau(a)$ for some $a \in (0,1)$

	a						
	.001	.011	.021	.031	.041	$\cdots$	.991
$\rho_s(a)$	.99999350	.99924680	.9973865	.9945713	.9909380	$\cdots$	.4810656
$\tau(a)$	.99800399	.97847358	.9596929	.94161959	.92421442	$\cdots$	.33534541
$\rho_s(a)/\tau(a)$	1.001993	1.021230	1.039277	1.056235	1.072195	$\cdots$	1.434538

Table 5.2: Values of $\rho_s(a)$, $\tau(a)$ and $\rho_s(a)/\tau(a)$ for some $a \in (1,100)$

	a						
	1.1	2.1	3.1	4.1	5.1	$\cdots$	99.1
$\rho_s(a)$	.45076680	.28394200	.20656080	.16217260	.13343770	$\cdots$	.00753000
$\tau(a)$	.31250000	.19230700	.13888800	.10869500	.08928570	$\cdots$	.00502008
$\rho_s(a)/\tau(a)$	1.44245	1.4764986	1.4872382	1.4919880	1.49450305	$\cdots$	1.4999810

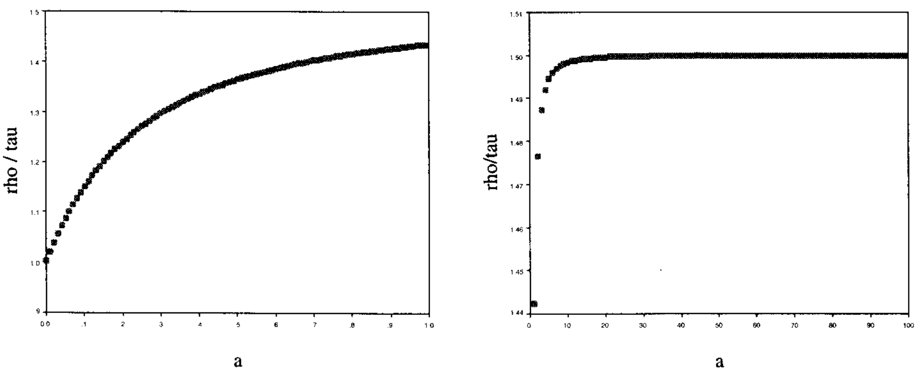

Figure 5.1: $\rho_s(a)/\tau(a)$ for some values of $a \in (0,1)$ (left) and $a \in (1,100)$ (right)

and $\rho_s(a)/\tau(a)$ for some values of a. Tables 5.1, 5.2 and graphs of Figure 5.1 seem to give empirical evidence that $\rho_s/\tau \leq 3/2$. Tables 5.1 and 5.2 provide for $0 < a < 1$ and $1 \leq a < 100$ respectively. Ascending a, gives an empirical evidence that the ratio of ρ_s to τ approach to 3/2.

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