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SUPERQUADRATIC FUNCTIONS AND REFINEMENTS OF SOME CLASSICAL INEQUALITIES

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Abstract. Using known properties of superquadratic functions we obtain a sequence of inequalities for superquadratic functions such as the Converse and the Reverse Jensen type inequalities, the Giaccardi and the Petrović type inequalities and Hermite-Hadamard's inequalities. Especially, when the superquadratic function is convex at the same time, then we get refinements of classical known results for convex functions. Some other properties of superquadratic functions are also given.

1. Introduction

Let I be an interval in $\mathbb{R}$ and $\varphi : I \to \mathbb{R}$ a convex function on I. If $(x_1, \ldots, x_n)$ is any n-tuple in I^n $(n \geq 2)$ and $(p_1, \ldots, p_n)$ is any non-negative n-tuple such that $P_n = \sum_{i=1}^n p_i > 0$, then the well known *Jensen's inequality*

$$\varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^n p_i \varphi(x_i) \tag{1.1}$$

holds.

Under the same conditions as above, for $x_i \in [m, M] \subseteq I$ $(i = 1, \ldots, n)$, where $-\infty < m < M < \infty$, a *converse of Jensen's inequality* gives an upper bound for the right side of (1.1):

$$\frac{1}{P_n}\sum_{i=1}^n p_i \varphi(x_i) \leq \frac{M - \overline{x}}{M - m}\varphi(m) + \frac{\overline{x} - m}{M - m}\varphi(M), \tag{1.2}$$

where $\overline{x} = \frac{1}{P_n}\sum_{i=1}^n p_i x_i$. This result remains valid under a weaker condition on φ, i.e., it is sufficient that either $(\varphi(x) - \varphi(m))/(x - m)$ or $(\varphi(M) - \varphi(x))/(M - x)$ is a non-decreasing function of x on $(m, M]$ and $[m, M)$, respectively (see [5, p.105]).

An integral analogue of the above inequality in the measure space $(\Omega, \mathcal{A}, \mu)$ with $0 < \mu(\Omega) < \infty$ states that if $f : \Omega \to [m, M]$ is a function such that

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$f, \varphi \circ f \in L_1(\mu)$, then

$$\frac{1}{\mu(\Omega)} \int_\Omega (\varphi \circ f) d\mu \leq \frac{M - \overline{f}}{M - m} \varphi(m) + \frac{\overline{f} - m}{M - m} \varphi(M) \tag{1.3}$$

holds for any convex function $\varphi : [m, M] \to \mathbb{R}$, where $\overline{f} = \frac{1}{\mu(\Omega)} \int_\Omega f d\mu$.

The result of the following theorem ([5, p.152]) is a simple consequence of inequality (1.2).

Theorem A. *Let $(p_1, \dots, p_n)$ be a non-negative n-tuple and $(x_1, \dots, x_n)$ be a real n-tuple such that $\sum_{k=1}^n p_k x_k \in [0, a]$,*

$$\sum_{k=1}^n p_k x_k \neq x_0, \qquad (x_i - x_0) \left(\sum_{k=1}^n p_k x_k - x_i \right) \geq 0, \quad i = 1, \dots, n,$$

where $x_0 \in [0, a]$. If $(\varphi(x) - \varphi(x_0))/(x - x_0)$ is a non-decreasing function on $[0, a]$, then

$$\sum_{k=1}^n p_k \varphi(x_k) \leq A \varphi \left(\sum_{k=1}^n p_k x_k \right) + B \left(\sum_{k=1}^n p_k - 1 \right) \varphi(x_0) \tag{1.4}$$

holds, where

$$A = \frac{\sum_{k=1}^n p_k (x_k - x_0)}{\sum_{k=1}^n p_k x_k - x_0}, \qquad B = \frac{\sum_{k=1}^n p_k x_k}{\sum_{k=1}^n p_k x_k - x_0}.$$

The *Giaccardi type inequality* (1.4) can be obtained from (1.2) by setting $m = x_0$ and $M = \sum_{k=1}^n p_k x_k$ (if $x_0 < \sum_{k=1}^n p_k x_k$, or $M = x_0$ and $m = \sum_{k=1}^n p_k x_k$ in the reverse case).

Moreover, for $x_0 = 0$ we have $A = B = 1$ in (1.4), so in that case we get *Jensen-Petrović's inequality*:

$$\sum_{k=1}^n p_k \varphi(x_k) \leq \varphi \left(\sum_{k=1}^n p_k x_k \right) + \left(\sum_{k=1}^n p_k - 1 \right) \varphi(0). \tag{1.5}$$

Furthermore, under the following assumptions on a real n-tuple $(p_1, \dots, p_n)$:

$$p_1 > 0, \qquad p_i \leq 0 \quad (i = 2, \dots, n), \qquad P_n = \sum_{i=1}^n p_i > 0,$$

if $x_i \in I$ $(i = 1, \dots, n)$ and $\frac{1}{P_n} \sum_{i=1}^n p_i x_i \in I$, the *reverse of Jensen's inequality*

$$\varphi \left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i \right) \geq \frac{1}{P_n} \sum_{i=1}^n p_i \varphi(x_i) \tag{1.6}$$

holds for any convex function $\varphi : I \to \mathbb{R}$.

In this paper we refine all the above inequalities and we also refine the following pair of inequalities, that is called Hermite-Hadamard's inequalities:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)\, dt \leq \frac{f(a) + f(b)}{2}, \tag{1.7}$$

which holds for any convex function $f : I \to \mathbb{R}$ and $a, b \in I$.

Now we quote some definitions and state a list of basic properties of superquadratic functions established in [2] and [3].

Definition A ([2, Definition 2.1]). A function $\varphi : [0, \infty) \to \mathbb{R}$ is *superquadratic* provided that for all $x \geq 0$ there exists a constant $C(x) \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi(|y - x|) \geq C(x)(y - x) \tag{1.8}$$

for all $y \geq 0$. We say that φ is *subquadratic* if $-\varphi$ is a superquadratic function.

Jensen's inequality for a superquadratic function is given in the following theorem:

Theorem B ([2, Theorem 2.3]). *The inequality*

$$\varphi\left(\int_\Omega f d\mu\right) \leq \int_\Omega \left(\varphi(f(s)) - \varphi(|f(s) - \textstyle\int_\Omega f d\mu|)\right) d\mu(s) \tag{1.9}$$

holds for all probability measures μ and all non-negative, μ-integrable functions f if and only if φ is superquadratic.

A discrete version of the above theorem is also used in the sequel. It can be obtained by choosing a discrete measure μ on $\{1, \ldots, n\}$, defined by $\mu(i) = p_i/P_n$, $p_i \geq 0$, $P_n = \sum_{i=1}^n p_i > 0$ and the function f defined by $f(i) = x_i$:

Lemma A ([3, Lemma 2.3]). *Suppose that φ is superquadratic. Let $x_i \geq 0$, $i = 1, \ldots, n$ and let $\overline{x} = \frac{1}{P_n} \sum_{i=1}^n p_i x_i$ where $p_i \geq 0$ and $P_n = \sum_{i=1}^n p_i > 0$. Then*

$$\frac{1}{P_n} \sum_{i=1}^n p_i \varphi(x_i) - \varphi(\overline{x}) \geq \frac{1}{P_n} \sum_{i=1}^n p_i \varphi(|x_i - \overline{x}|) . \tag{1.10}$$

Lemma B ([2, Lemma 2.2]). *Let φ be a superquadratic function with $C(x)$ as in Definition A. Then:*

(i) $\varphi(0) \leq 0$,

(ii) *If $\varphi(0) = \varphi'(0) = 0$ then $C(x) = \varphi'(x)$ whenever φ is differentiable at $x > 0$.*

(iii) *If $\varphi \geq 0$, then φ is convex and $\varphi(0) = \varphi'(0) = 0$.*

Lemma C ([2, Lemma 3.2]). *Suppose φ is differentiable and $\varphi(0) = \varphi'(0) = 0$. If φ is superquadratic then $\varphi(x)/x^2$ is non-decreasing on $(0, \infty)$.*

It is interesting to see some examples of superquadratic functions.

Examples. (1) [2] Function $\varphi(x) = x^p$ is superquadratic for $p \geq 2$ and subquadratic for $p \in (0, 2]$.

(2) [1] Let $h : (0,\infty) \to \mathbb{R}$ be a continuous and a non-decreasing function with $\lim_{t\to 0+} th(t) = 0$. Then the function $f : [0,\infty) \to \mathbb{R}$ defined by

$$f(t) = \int_0^t sh(s)ds,$$

is superquadratic.

(3) [1] Let $h : (0,\infty) \to \mathbb{R}$ be a continuously differentiable and a non-decreasing function with

$$\lim_{t\to 0+} th(t) = 0 \quad \text{and} \quad \lim_{t\to 0+} t^2 h'(t) = 0$$

such that the function $t \mapsto th'(t)$ is non-decreasing. Then the function $f : [0,\infty) \to \mathbb{R}$ defined by

$$f(t) = t^2 h(t),$$

is superquadratic.

The rest of this paper is divided into four parts. The first part is devoted to the Converse of Jensen's inequality, the Giaccardi and the Jensen-Petrović inequalities for superquadratic function and its refinements in the case when the considered function is convex and superquadratic simultaneously. In the second part the reversals of Jensen's and Hölder's inequality for superquadratic function are considered. If moreover, we know that φ is superquadratic and $\varphi'(0) = \varphi(0) = 0$, then $\varphi(x)/x^2$ is non-decreasing and such kind of functions are considered in the third part. And finally, the last part is devoted to the Hermite-Hadamard inequalities and its refinements.

2. Converse of Jensen's and related inequalities

The first theorem that we prove here is the *Converse of Jensen's inequality for superquadratic functions.*

Theorem 1. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with $0 < \mu(\Omega) < \infty$ and let $\varphi : [0,\infty) \to \mathbb{R}$ be a superquadratic function. If $f : \Omega \to [m, M] \subseteq [0,\infty)$ is such that f, $\varphi \circ f \in L_1(\mu)$, then we have*

$$\frac{1}{\mu(\Omega)} \int_\Omega (\varphi \circ f) d\mu + \Delta_c \le \frac{M - \overline{f}}{M - m}\varphi(m) + \frac{\overline{f} - m}{M - m}\varphi(M), \tag{2.1}$$

where $\overline{f} = \frac{1}{\mu(\Omega)} \int_\Omega f d\mu$ and

$$\Delta_c = \frac{1}{\mu(\Omega)} \frac{1}{M - m} \int_\Omega \left[(M - f)\varphi(f - m) + (f - m)\varphi(M - f)\right] d\mu.$$

Proof. It can be easily obtained, from Definition A, that for a superquadratic function φ the following inequality

$$\varphi(x) \le \frac{y_2 - x}{y_2 - y_1}(\varphi(y_1) - \varphi(x - y_1)) + \frac{x - y_1}{y_2 - y_1}(\varphi(y_2) - \varphi(y_2 - x)) \tag{2.2}$$

holds for all $x, y_1, y_2 \geq 0$, such that $y_1 < x < y_2$ (see [2]). Setting $x = f(t)$, $y_1 = m$, $y_2 = M$ and for all $t \in \Omega$, we get

$$\begin{aligned}&\varphi(f(t)) + \frac{M - f(t)}{M - m}\varphi(f(t) - m) + \frac{f(t) - m}{M - m}\varphi(M - f(t))\\ \leq\ & \frac{M - f(t)}{M - m}\varphi(m) + \frac{f(t) - m}{M - m}\varphi(M).\end{aligned}$$

Integrating this over Ω we obtain

$$\begin{aligned}&\int_\Omega (\varphi \circ f) d\mu + \frac{1}{M - m}\int_\Omega [(M - f)\varphi(f - m) + (f - m)\varphi(M - f)] d\mu\\ \leq\ & \frac{\mu(\Omega)M - \int_\Omega f d\mu}{M - m}\varphi(m) + \frac{\int_\Omega f d\mu - \mu(\Omega)m}{M - m}\varphi(M),\end{aligned}$$

and after dividing with $\mu(\Omega)$ we get (2.1). □

Remark 1. Under the assumptions of the Theorem 1, and if φ is a non-negative and superquadratic function (and therefore convex) we have both of the results, (1.3) and (2.1). Since the term Δ_c is non-negative in this case, then the inequality (2.1) refines the inequality (1.3).

If μ is a discrete measure we have a discrete version of the previous result i.e., we have the following theorem.

Theorem 2. *Let $\varphi : [0, \infty) \to \mathbb{R}$ be a superquadratic function. Let $(x_1, \ldots, x_n)$ be an n-tuple in $[m, M]^n$ $(0 \leq m < M < \infty)$ and $(p_1, \ldots, p_n)$ be a non-negative n-tuple such that $P_n = \sum_{i=1}^n p_i > 0$. Then*

$$\frac{1}{P_n}\sum_{i=1}^n p_i\varphi(x_i) + \Delta_{dc} \leq \frac{M - \overline{x}}{M - m}\varphi(m) + \frac{\overline{x} - m}{M - m}\varphi(M), \tag{2.3}$$

where $\overline{x} = \frac{1}{P_n}\sum_{i=1}^n p_i x_i$ and

$$\Delta_{dc} = \frac{1}{P_n(M - m)}\sum_{i=1}^n p_i [(M - x_i)\varphi(x_i - m) + (x_i - m)\varphi(M - x_i)].$$

In particular, under the assumptions of the above Theorem, for $m = x_0$ and $M = \sum_{k=1}^n p_k x_k$ (if $x_0 < \sum_{k=1}^n p_k x_k$, or reversely in case $x_0 > \sum_{k=1}^n p_k x_k$) we get the following *Giaccardi type inequality for superquadratic function:*

Corollary 1. *Let $\varphi : [0, \infty) \to \mathbb{R}$ be a superquadratic function and $x_0 \geq 0$. Let $(p_1, \ldots, p_n)$ and $(x_1, \ldots, x_n)$ be non-negative n-tuples such that*

$$\sum_{k=1}^n p_k x_k \neq x_0, \qquad (x_i - x_0)(\sum_{k=1}^n p_k x_k - x_i) \geq 0, \quad i = 1, \ldots, n.$$

Then

$$\sum_{i=1}^n p_i\varphi(x_i) + \Delta_G \leq A\varphi(\sum_{k=1}^n p_k x_k) + B(\sum_{k=1}^n p_k - 1)\varphi(x_0), \tag{2.4}$$

where

$$\Delta_G = \frac{1}{\sum\limits_{k=1}^{n} p_k x_k - x_0} \left(\sum_{i=1}^{n} p_i \left(\sum_{k=1}^{n} p_k x_k - x_i\right)\varphi(|x_i - x_0|) + \sum_{i=1}^{n} p_i (x_i - x_0)\varphi\left(\left|\sum_{k=1}^{n} p_k x_k - x_i\right|\right) \right) \tag{2.5}$$

and A and B are defined in Theorem A.

In the special case of Corollary 1, when $x_0 = 0$, and so $\sum_{k=1}^{n} p_k x_k \geq x_i \geq 0$, we get *Jensen-Petrović's type inequality for superquadratic functions*:

$$\sum_{i=1}^{n} p_i \varphi(x_i) + \frac{1}{\sum_{k=1}^{n} p_k x_k} \sum_{i=1}^{n} p_i \left[\left(\sum_{k=1}^{n} p_k x_k - x_i\right)\varphi(x_i) + x_i \varphi\left(\sum_{k=1}^{n} p_k x_k - x_i\right)\right] \leq \left(\sum_{k=1}^{n} p_k - 1\right)\varphi(0) + \varphi\left(\sum_{k=1}^{n} p_k x_k\right). \tag{2.6}$$

Remark 2. Considering the case when φ is a non-negative superquadratic function, by Lemma B (iii) φ is also convex, so (1.2) is also valid for such function. Since the term Δ_{dc} in (2.3) is non-negative in this case, we conclude that inequality (2.3) refines the inequality (1.2). Similarly, a non-negative superquadratic function φ is convex and therefore

$$(\varphi(x) - \varphi(x_0))/(x - x_0)$$

is an increasing function, so (1.4) is also valid for that function. Under the assumptions of Corollary 1 we have

$$\frac{\sum\limits_{k=1}^{n} p_k x_k - x_i}{\sum\limits_{k=1}^{n} p_k x_k - x_0} \geq 0 \quad \text{and} \quad \frac{x_i - x_0}{\sum\limits_{k=1}^{n} p_k x_k - x_0} \geq 0, \qquad i = 1, \ldots, n,$$

so in this case the term Δ_G in (2.4) is non-negative. Hence, in this special case, inequality (2.4) represents a refinement of inequality (1.4). Finally, in the same way as above, we conclude that if φ is non-negative and superquadratic, then (2.6) refines (1.5).

3. Reversal of Jensen's inequality and Popoviciu's type inequality

In this section we obtain Popoviciu's type inequality. As we know that inequality is a reverse version of Hölder's type inequalities which are given in articles [2] and [6]. For this purpose we need one more general result which is called the Reversal of Jensen's inequality.

First, we will prove a discrete *Reversal of Jensen's inequality* for superquadratic functions.

Theorem 3. *Let* $(p_1, \ldots, p_n)$ *be a real* n*-tuple such that*

$$p_1 > 0, \qquad p_i \leq 0 \quad (i = 2, \ldots, n), \qquad P_n = \sum_{i=1}^{n} p_i > 0.$$

If $x_i \geq 0$ $(i = 1, \ldots, n)$ *and* $\overline{x} = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \geq 0$, *then for a superquadratic function* $\varphi : [0, \infty) \to \mathbb{R}$ *the following inequality*

$$\varphi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \geq \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i) + \varphi(|\overline{x} - x_1|) - \frac{1}{P_n}\sum_{i=2}^{n} p_i \varphi(|x_i - x_1|) \tag{3.1}$$

holds.

Proof. After making the substitutions:

$$p_1 \to P_n, \qquad x_1 \to \overline{x} = \tfrac{1}{P_n}\sum_{i=1}^{n} p_i x_i,$$
$$p_i \to -p_i, \qquad x_i \to x_i, \qquad i = 2, \ldots, n$$

in Lemma A, we get

$$\frac{1}{p_1}\left(P_n \varphi(\overline{x}) - \sum_{i=2}^{n} p_i \varphi(x_i)\right) - \varphi(x_1)$$
$$\geq \frac{1}{p_1}\left(P_n \varphi(|\overline{x} - x_1|) - \sum_{i=2}^{n} p_i \varphi(|x_i - x_1|)\right).$$

Multiplying the above inequality by p_1/P_n we obtain (3.1). □

Remark 3. As in the previous cases, we are considering the case of Theorem 3 when φ is a non-negative superquadratic function. In that case φ is also a convex function, so inequality (1.6) holds too. Since $\varphi \geq 0$, $P_n > 0$ and $p_i \leq 0$ for $i = 2, \ldots, n$, we have

$$\varphi(|\overline{x} - x_1|) - \frac{1}{P_n}\sum_{i=2}^{n} p_i \varphi(|x_i - x_1|) \geq 0.$$

Hence, in Theorem 3 we get the following refinement of the Reversal of Jensen's inequality (1.6):

$$\varphi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \geq \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i) + \varphi(|\overline{x} - x_1|) - \frac{1}{P_n}\sum_{i=2}^{n} p_i \varphi(|x_i - x_1|)$$
$$\geq \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i).$$

Using the previous Theorem we are able to prove an integral version of the Reversal of Jensen's inequality.

Theorem 4. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with $0 < \mu(\Omega) < \infty$ and let $\varphi : [0,\infty) \to \mathbb{R}$ be a superquadratic function. If $p, g : \Omega \to [0,\infty)$ are functions and $a, u \in [0,\infty)$ are real numbers such that $p, pg, p\varphi(g), p\varphi(|\frac{\int_\Omega pgd\mu}{\int_\Omega pd\mu} - g|) \in L_1(\mu)$, $0 < \int_\Omega pd\mu < u$ and $ua - \int_\Omega pgd\mu \geq 0$, then*

$$\varphi\left(\frac{ua - \int_\Omega pgd\mu}{u - \int_\Omega pd\mu}\right) \geq \frac{u\varphi(a) - \int_\Omega p\varphi(g)d\mu}{u - \int_\Omega pd\mu} + \Delta_{RJ}, \tag{3.2}$$

where

$$\begin{aligned}
&\Delta_{RJ}\\
&= \frac{1}{u - \int_\Omega pd\mu}\left[\int_\Omega p\varphi\left(\left|\frac{\int_\Omega pgd\mu}{\int_\Omega pd\mu} - g\right|\right)d\mu + \left(\int_\Omega pd\mu\right)\varphi\left(\left|\frac{\int_\Omega pgd\mu}{\int_\Omega pd\mu} - a\right|\right)\right.\\
&\left.\quad +(u - \int_\Omega pd\mu)\varphi\left(\frac{\int_\Omega pd\mu}{u - \int_\Omega pd\mu}\left|\frac{\int_\Omega pgd\mu}{\int_\Omega pd\mu} - a\right|\right)\right].
\end{aligned}$$

Proof. Putting in Theorem 3 $n = 2$, $p_1 = u$, $p_2 = -\int_\Omega pd\mu$, $x_1 = a$, $x_2 = \frac{\int_\Omega pgd\mu}{\int_\Omega pd\mu}$ we obtain the following

$$\begin{aligned}
\varphi\left(\frac{ua - \int_\Omega pgd\mu}{u - \int_\Omega pd\mu}\right) \geq\ & \frac{u\varphi(a) - (\int_\Omega pd\mu)\varphi(\frac{\int_\Omega pgd\mu}{\int_\Omega pd\mu})}{u - \int_\Omega pd\mu}\\
&+\varphi\left(\frac{\int_\Omega pd\mu}{u - \int_\Omega pd\mu}\left|\frac{\int_\Omega pgd\mu}{\int_\Omega pd\mu} - a\right|\right)\\
&+\frac{\int_\Omega pd\mu}{u - \int_\Omega pd\mu}\varphi\left(\left|\frac{\int_\Omega pgd\mu}{\int_\Omega pd\mu} - a\right|\right).
\end{aligned}$$

Using Jensen's inequality for superquadratic function (Theorem B) we get

$$\left(\int_\Omega pd\mu\right)\varphi\left(\frac{\int_\Omega pgd\mu}{\int_\Omega pd\mu}\right) \leq \int_\Omega p\varphi(g)d\mu - \int_\Omega p\varphi\left(|g - \frac{\int_\Omega pgd\mu}{\int_\Omega pd\mu}|\right)d\mu.$$

Combining these two inequalities we get inequality (3.2). □

Considering the superquadratic function $\varphi(x) = x^p$, $p \geq 2$, and applying result (3.2) we obtain the *Popoviciu type inequality.*

Theorem 5. *Let $(\Omega, \mathcal{A}, \mu)$ be as in Theorem 4. Let $p \geq 2$ and q be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f, g : \Omega \to [0,\infty)$ are functions and f_0, g_0 are positive real numbers such that $f^p, g^q, fg, |g^{q/p}\int_\Omega fgd\mu - f\int_\Omega g^q d\mu|^p \in L_1(\mu)$,*

$g_0^q - \int_\Omega g^q d\mu > 0$, $f_0^p - \int_\Omega f^p d\mu > 0$ *and* $\int_\Omega g^q d\mu > 0$ *then*

$$f_0 g_0 - \int_\Omega fg d\mu \geq \left[\left(f_0^p - \int_\Omega f^p d\mu\right)\left(g_0^q - \int_\Omega g^q d\mu\right)^{p/q} + \Delta_P\right]^{1/p} \tag{3.3}$$

$$\geq \left(f_0^p - \int_\Omega f^p d\mu\right)^{1/p}\left(g_0^q - \int_\Omega g^q d\mu\right)^{1/q},$$

where

$$\Delta_P = \frac{(g_0^q - \int_\Omega g^q d\mu)^{p/q}}{(\int_\Omega g^q d\mu)^p}\int_\Omega |g^{q/p}\int_\Omega fg d\mu - f\int_\Omega g^q d\mu|^p d\mu$$

$$+\left|f_0 g_0^{-q/p}\int_\Omega g^q d\mu - \int_\Omega fg d\mu\right|^p \cdot \left(1 + \frac{(g_0^q - \int_\Omega g^q d\mu)^{p/q}}{(\int_\Omega g^q d\mu)^{p/q}}\right).$$

Proof. By applying the substitutions: $\varphi(x) = x^p$, $p \geq 2$,

$$u \to g_0^q, \quad p \to g^q, \quad a \to g_0^{-q/p} f_0, \quad g \to g^{-q/p} f$$

in (3.2) we obtain inequality (3.3), while the second inequality follows from the positivity of Δ_P. □

As we can see, the result of the previous theorem is, in fact, the refinement of the classical Popoviciu's inequality, [5, p.125].

4. The case $\varphi(0) = \varphi'(0) = 0$

Here we state another versions of the considered inequalities for superquadratic functions. First we give a result which is, in special case, a refinement of the Converse of Jensen's inequality (1.2) for $m = 0$.

Theorem 6. *Let* $\varphi : [0, M] \to \mathbb{R}$ $(M > 0)$ *be a differentiable superquadratic function with* $\varphi(0) = \varphi'(0) = 0$ *and let* $(\Omega, \mathcal{A}, \mu)$ *be a measure space with* $0 < \mu(\Omega) < \infty$. *If* $f : \Omega \to (0, M]$ *is such that* $f, \frac{\varphi \circ f}{f} \in L_1(\mu)$, *then we have*

$$\int_\Omega \frac{\varphi \circ f}{f} d\mu \leq \frac{\varphi(M)}{M^2}\int_\Omega f d\mu. \tag{4.1}$$

Proof. Under the given assumptions on φ and according to Lemma C the function $\varphi(x)/x^2$ is non-decreasing on $(0, M]$. So, for any $t \in \Omega$ we have

$$\frac{\varphi(f(t))}{f(t)^2} \leq \frac{\varphi(M)}{M^2}.$$

Then we obtain

$$\int_\Omega \frac{\varphi \circ f}{f} d\mu = \int_\Omega \frac{\varphi \circ f}{f^2} \cdot f d\mu$$

$$\leq \int_\Omega \frac{\varphi(M)}{M^2} f d\mu = \frac{\varphi(M)}{M^2}\int_\Omega f d\mu.$$

□

Remark 4 (A refinement of the Converse of Jensen's inequality). Especially, if the superquadratic function φ is additionally non-negative, then (by Lemma B) it is also a convex function with $\varphi(0)=\varphi'(0)=0$. In that case, beside (4.1), the Converse of Jensen's inequality (1.3) for $m=0$:

$$\frac{1}{\mu(\Omega)}\int_\Omega \frac{\varphi\circ f}{M}d\mu \le \frac{\overline{f}}{M^2}\varphi(M), \tag{4.2}$$

holds too.

Since $f(t)\le M$ for all $t\in\Omega$ we get

$$\int_\Omega \frac{\varphi\circ f}{M}d\mu \le \int_\Omega \frac{\varphi\circ f}{f}d\mu \le \frac{\varphi(M)}{M^2}\int_\Omega f d\mu.$$

So, inequality (4.1) represents a refinement of the Converse of Jensen's inequality.

Let us point out a corresponding result for a discrete measure.

Theorem 7. *Let $\varphi:[0,M]\to\mathbb{R}$ $(M>0)$ be a differentiable superquadratic function with $\varphi(0)=\varphi'(0)=0$. If $p_i\ge 0$, $x_i\in(0,M]$ $(i=1,\dots,n)$, and $\sum_{i=1}^n p_i>0$, then*

$$\sum_{i=1}^n p_i\frac{\varphi(x_i)}{x_i}\le \frac{\varphi(M)}{M^2}\sum_{i=1}^n p_i x_i$$

holds.

Remark 5 (A refinement of the discrete Converse of Jensen's inequality). Similarly as in the previous remark if the superquadratic function φ is additionally non-negative, then we have a refinement of the Converse of Jensen's inequality (1.2)

$$\sum_{i=1}^n p_i\frac{\varphi(x_i)}{M}\le \sum_{i=1}^n p_i\frac{\varphi(x_i)}{x_i}\le \frac{\varphi(M)}{M^2}\sum_{i=1}^n p_i x_i.$$

Remark 6 (A refinement of Jensen-Petrović's inequality). In a particular case, by setting $M=\sum_{i=1}^n p_i x_i$ in Theorem 7, as we already have $m=x_0=0$, we get one more version of Jensen-Petrović's inequality for superquadratic function:

$$\sum_{i=1}^n p_i\frac{\varphi(x_i)}{x_i}\le \frac{\varphi(\sum_{i=1}^n p_i x_i)}{\sum_{i=1}^n p_i x_i},$$

and multiplying it by $\sum_{i=1}^n p_i x_i\ge 0$, the following equivalent inequality:

$$\sum_{i=1}^n p_i x_i\sum_{i=1}^n p_i\frac{\varphi(x_i)}{x_i}\le \varphi(\sum_{i=1}^n p_i x_i). \tag{4.3}$$

In case when φ is (beside the above assumptions) also a non-negative function, as we mentioned earlier, φ is a convex function too, and we have $\varphi(0)=$

$\varphi'(0) = 0$. So, in that case, Jensen-Petrović's inequality

$$\varphi(\sum_{i=1}^{n} p_i x_i) \geq \sum_{i=1}^{n} p_i \varphi(x_i) \tag{4.4}$$

holds.

Let us consider the inequalities (4.3) and (4.4). They are not comparable for any choice of weights p_i, but if $p_i \geq 1$ $(i = 1, \dots, n)$, then we have

$$\sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i y_i = \sum_{i=1}^{n} p_i^2 x_i y_i + \sum_{\substack{i \neq j \\ i,j=1}}^{n} p_i p_j x_i y_j$$
$$\geq \sum_{i=1}^{n} p_i^2 x_i y_i \geq \sum_{i=1}^{n} p_i x_i y_i,$$

where $y_i = \varphi(x_i)/x_i$

So, using (4.3) we get

$$\varphi(\sum_{i=1}^{n} p_i x_i) \geq \sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i \frac{\varphi(x_i)}{x_i} \geq \sum_{i=1}^{n} p_i \varphi(x_i), \tag{4.5}$$

which is a refinement of the Jensen-Petrović inequality (4.4).

Corollary 2. *Inequalities for sums of order p Let $(x_1, \dots, x_n)$ and $(p_1, \dots, p_n)$ be non-negative n-tuples with $p_i \geq 1$ $(i = 1, \dots, n)$. If $r \geq 2p$, then*

$$(\sum_{i=1}^{n} p_i x_i^r)^{1/r} \leq (\sum_{i=1}^{n} p_i x_i^p \sum_{i=1}^{n} p_i x_i^{r-p})^{1/r} \leq (\sum_{i=1}^{n} p_i x_i^p)^{1/p}.$$

Proof. It is a simple consequence of the inequality (4.5) for a superquadratic function $\varphi(x) = x^{r/p}$. □

It is a refinement of the well-known inequality for sums of order p, [5, p.165], which states that

$$(\sum_{i=1}^{n} p_i x_i^r)^{1/r} \leq (\sum_{i=1}^{n} p_i x_i^p)^{1/p}$$

for $r \geq p$.

5. Hermite-Hadamard's inequality

The following theorem gives to us an analogue of the inequalities in (1.7) for superquadratic functions. As we shall see, in the special case for non-negative superquadratic functions, we obtain the refinements of these inequalities.

Theorem 8. *Let $\varphi : [0,\infty) \to \mathbb{R}$ be an integrable superquadratic function, $0 \le a < b$. Then*

$$\begin{aligned}
&\varphi\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b \varphi\left(\left|t-\frac{a+b}{2}\right|\right)dt \\
&\le \frac{1}{b-a}\int_a^b \varphi(t)dt \qquad (5.1)\\
&\le \frac{\varphi(a)+\varphi(b)}{2} \\
&\quad -\frac{1}{(b-a)^2}\int_a^b \left[(b-t)\varphi(t-a)+(t-a)\varphi(b-t)\right]dt. \qquad (5.2)
\end{aligned}$$

Proof. First we prove (5.1), the left Hermite-Hadamard inequality for superquadratic functions. Since the function φ is superquadratic, then for $f(t) = t$ and measure μ on $\Omega = [a,b]$, defined by $d\mu = \frac{1}{b-a}dt$, inequality (1.9) shows that

$$\varphi\left(\frac{1}{b-a}\int_a^b tdt\right) \le \frac{1}{b-a}\int_a^b \varphi(t)dt - \frac{1}{b-a}\int_a^b \varphi\left(\left|t-\frac{1}{b-a}\int_a^b sds\right|\right)dt$$

holds. Hence, we get (5.1).

Also, as φ is superquadratic, by setting $x = t$, $y_1 = a$ and $y_2 = b$ in (2.2) we get that

$$\varphi(t) \le \frac{b-t}{b-a}(\varphi(a)-\varphi(t-a)) + \frac{t-a}{b-a}(\varphi(b)-\varphi(b-t))$$

holds for all $t, a, b \ge 0$ such that $a < t < b$. After integrating this expression over the segment $[a,b]$:

$$\begin{aligned}
\int_a^b \varphi(t)dt \;&\le\; \frac{1}{b-a}\int_a^b \left[b\varphi(a)-a\varphi(b)+t(\varphi(b)-\varphi(a))\right]dt \\
&\quad -\frac{1}{b-a}\int_a^b \left[(b-t)\varphi(t-a)+(t-a)\varphi(b-t)\right]dt
\end{aligned}$$

and dividing with $(b-a)$ we get (5.2). □

Remark 7. If φ is superquadratic and non-negative, then by Lemma B, φ is also a convex function. In this case the second term on the left side in (5.1) and the last term in (5.2) are non-negative, so we have the following refinements of Hermite-Hadamard's inequality (1.7):

$$\varphi\left(\frac{a+b}{2}\right) \le \varphi\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b \varphi\left(\left|t-\frac{a+b}{2}\right|\right)dt \le \frac{1}{b-a}\int_a^b \varphi(t)dt$$

and

$$\begin{aligned} & \frac{1}{b-a}\int_a^b \varphi(t)dt \\ \leq \quad & \frac{\varphi(a)+\varphi(b)}{2} - \frac{1}{(b-a)^2}\int_a^b ((b-t)\varphi(t-a)+(t-a)\varphi(b-t))dt \\ \leq \quad & \frac{\varphi(a)+\varphi(b)}{2}. \end{aligned}$$

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