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CHOW STABILITY CRITERION IN TERMS OF LOG CANONICAL THRESHOLD

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Abstract. In this paper, we provide a criterion for Chow stability in terms of log canonical threshold of the Chow form in the Grassmannian.

1. Introduction

The Geometric Invariant Theory (GIT for short) is one of the most useful methods to construct a moduli space or a compactified moduli space of algebraic varieties if one knows the effective criteria for stability and semi-stability.

The special linear group $SL(n+1)$ acts on $V_{d,n+1} = \mathrm{Sym}^d(V)$, which is the vector space of homogeneous polynomials of degree d in $\mathbb{C}[x_0, \ldots, x_n]$. The Hilbert-Mumford numerical criterion [11] provides a simple way to decide the stability and the semi-stability of $f \in V_{d,n+1}$ by the position of nonzero monomials of f in a n-dimensional Newton polyhedron. For a higher codimension case, the stability is defined by the Chow form. Let X be a subvariety of dimension r and of degree d in $\mathbb{P}^n$. Consider the set $Z(X)$ of all the $(n-r-1)$-dimensional projective subspaces L in $\mathbb{P}^n$ that intersects X. This is a subvariety in the Grassmannian $G(n-r, n+1)$ which parameterizes all the $(n-r-1)$-dimensional projective subspaces in $\mathbb{P}^n$. The subvariety $Z(X)$ is a hypersurface of degree d in $G(n-r, n+1)$. Let $\mathcal{B} = \oplus_{d=0}^{\infty} \mathcal{B}_d$ be the coordinate ring of $G(n-r, n+1)$ in the Plüker embedding. Then $Z(X)$ is defined by the vanishing of some element $R_X \in \mathcal{B}_d$ which is unique up to a constant factor. This element is called the Chow form of X. A variety X is called Chow semi-stable (resp. Chow stable) if its Chow form is semi-stable (resp. stable) for the natural $SL(n+1)$-action. Mumford [11] provides a way to decide Chow stability or Chow semi-stability by giving the weighted flag in $H^0(X, \mathcal{O}_X(1))$. Contrary to hypersurfaces in $\mathbb{P}^n$, there is no simple way to decide Chow stability.

There is an expectation of the restriction of singularities by the notion of stability. A natural question arises, to give a criterion for stability in terms of the nature of the singularities. There are various ways to measure how singularities of a variety are. Let Y be a nonsingular variety and D an effective

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$\mathbb{Q}$-Cartier divisor of Y. The invariant of the singularities of the pair (Y, D), called the log canonical threshold of Y along D, is an important topic to study the classification of higher dimensional algebraic varieties. It received a lot of attention recently.

The aim of this paper is to provide a criterion for Chow stability of X in $\mathbb{P}^n$ including log canonical threshold of the Chow form $Z(X)$ in the Grassmannian $G = G(n-r, n+1)$. We prove the following

Theorem. *Let X be a nondegenerate r-dimensional variety of degree d in $\mathbb{P}^n$. Let $(G, Z(X))$ be a pair as above. Then we have the following criterion for Chow stability of X: If $\operatorname{lct}(G, Z(X)) \geq \frac{n+1}{d}$ (resp. $>$) then X is Chow semistable (resp. stable).*

This result is a generalization of its in [7]. There are two main ingredients of the proof. The first one is that the criterion for stability of hypersurfaces and the determination of the log canonical threshold involve the Newton polyhedron in the same way. On the stability side the criterion is due to Hilbert. On the side of the log canonical threshold, the required statement is made at least in the paper [14]. The second one is the Cayley's trick. It tells that the Chow form in the Grassmannian can be interpreted as the dual variety in the projective space embedded by the Segre embedding of $X \times \mathbb{P}^r$ (cf. [3], Chapter 3). The main advantage of the log canonical threshold condition over Chow stability is that it is a local analytic condition on the singularities and so perhaps more tractable if the Chow form can be computed. The proof of our theorem is not complicated, but our main statement is not in the literature and our main contribution is the interpretation of Chow stability via the log canonical threshold of the Chow form. One can ask how closely the two conditions are related, i.e., to what extent the converse of the theorem holds. In the case of plane curves they are very closely related, and the case of hypersurfaces is rather similar. However, for $X \subset \mathbb{P}^n$ of codimension greater than one which is not union of subvarieties of degree 1, the two conditions seem to be not closely related.

We work throughout over the complex number field $\mathbb{C}$.

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2. Stability criterion for hypersurfaces

Let $\mathbb{P}(V^*) = \mathbb{P}^n$. The special linear group $SL(n+1)$ acts on $V_{d,n+1} = \operatorname{Sym}^d(V)$ which is the vector space of homogeneous polynomials of degree d in

$\mathbb{C}[z_0, \dots, z_n]$ by

$$A \cdot F :\ = F \circ A \text{ for } A \in SL(n+1) \text{ and } F \in V_{d,n+1}.$$

Recall the Geometric invariant theory ([11], [12]). Let $F \in V_{d,n+1}$. F is

- semi-stable if $0 \notin \overline{O^{SL(n+1)}(F)}$,
- stable if the orbit $O^{SL(n+1)}(F)$ is closed and the stabilizer $\mathrm{Stab}^{SL(n+1)}(F)$ is finite.

Each point $F \in V_{d,n+1}$ defines a hypersurface of degree d in $\mathbb{P}^n$. There is a simple way to decide the stability of F by using the Hilbert-Mumford criterion ([11], [12]). This approach was first devised by Hilbert (Lecture II.5, [5]) in terms of ternary null forms, and used by Mumford and others to classify various hypersurfaces of fixed degrees in projective spaces. We illustrate the case $n = 2$. The technique for determining stability is essentially same for any n.

Represent F as below by a triangle of coefficients, T. We can coordinate this triangle by 3 coordinates i_x, i_y, i_z (the exponents of x, y and z respectively) with $i_x + i_y + i_z = d$. The condition that a line L with equation $ai_x + bi_y + ci_z = 0$, $(a, b, c) \neq (0, 0, 0)$, should pass through the center is just $a+b+c = 0$; if L also passes through a point with integral coordinates then a, b and c can be chosen integral. Let λ be a one parameter subgroup of $SL(3)$. Then λ can always be diagonalized in a suitable basis: $\lambda(t) = \begin{bmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{bmatrix}$, where $a + b + c = 0$.

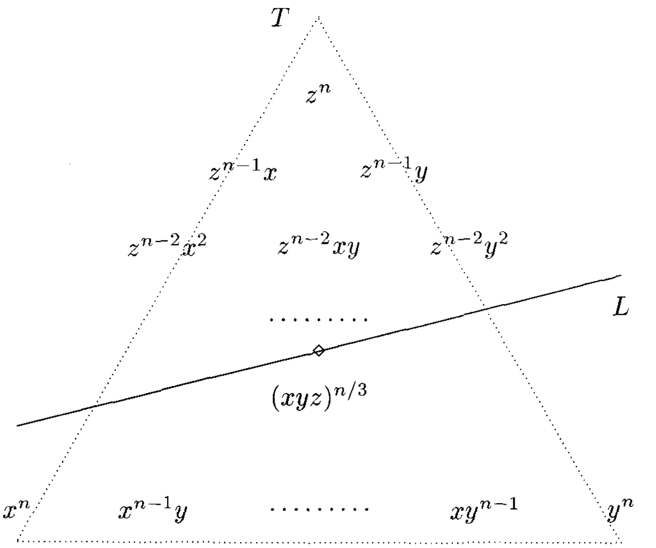

Fig 1. Triangle

Let $F = \sum_{i_x+i_y+i_z=d} \alpha_{i_x i_y i_z} x^{i_x} y^{i_y} z^{i_z}$ in these coordinates. Then

$$\lambda(t)F = \sum_{i_x+i_y+i_z=d} \alpha_{i_x i_y i_z} t^{ai_x+bi_y+ci_z} x^{i_x} y^{i_y} z^{i_z}.$$

By the Hilbert-Mumford numerical criterion, F is stable(resp. semi-stable) if an only if, for all coordinates and all L, F has non-zero coordinates on both sides of L (resp. F has non-zero coordinates on both sides of L or has non-zero coefficients on L).

The Hilbert-Mumford numerical criterion for a hypersurface in $\mathbb{P}^n$ can be checked by assigning the weights to the coordinates. In the paper [9], Kollár develops in a very similar direction. Let p be a point in a hypersurface $X : F = 0$ in $\mathbb{P}^n$. By a linear coordinate change we may assume that $p = (1, 0, \ldots, 0)$. Let $f(x_1, \ldots, x_n) = F(1, z_1, \ldots, z_n)$. We define $I_p(\mathbb{C}^n, X)$ to be the infimum of $\frac{\sum_{i=1}^n w(x_i)}{w(f)}$ for all the positive rational weights w and for all linear coordinate changes which fixes the point p. The value $w(f)$ is the lowest weight of monomial occurring in f. We set

$$I(\mathbb{P}^n, X) = \inf_{p \in X} I_p(\mathbb{C}^n, X).$$

The following lemma is reinterpretation of the Hilbert-Mumford numerical criterion.

Lemma 2.1. *Let X be a hypersurface of degree d in $\mathbb{P}^n$. Then we have the following criterion for stability of X : $I(\mathbb{P}^n, X) \geq \frac{n+1}{d}$ (resp. $>$) if and only if X is semi-stable (resp. stable).*

Proof. Assume that X is unstable. Then by the Hilbert-Mumford numerical criterion, we have coordinates $z_0, \ldots, z_n$ and weights $w(z_i) = k_i$ such that

(1) $k_0 \leq k_1 \leq \cdots \leq k_n$,

(2) $k_0 + \cdots + k_n = 0$,

(3) $k_0 i_0 + \cdots + k_n i_n > 0$ for every monomial $z_0^{i_0} \cdots z_n^{i_n}$ in F.

Let $f(x_1, \ldots, x_n) = F(1, x_1, \ldots, x_n)$. The proof is obtained if we prove the following:

$$\frac{\sum_{j=1}^n w(x_j)}{w(f)} < \frac{n+1}{d}.$$

Note that $w(x_j) = k_j - k_0 \geq 0$. For every monomial $x_1^{i_1} \cdots x_n^{i_n}$ in f,

$$\begin{aligned}
& \textstyle\sum_{j=1}^n w(x_j) i_j - \frac{d}{n+1} \sum_{j=1}^n w(x_j) \\
= \; & \textstyle\sum_{j=1}^n (k_j - k_0) i_j - \frac{d}{n+1} \sum_{j=1}^n (k_j - k_0) \\
= \; & -k_0(i_1 + \cdots + i_n) + (k_1 i_1 + \cdots + k_n i_n) - d(k_1 + \cdots + k_n) \\
= \; & -k_0(i_1 + \cdots + i_n - d) + (k_1 i_1 + \cdots + k_n i_n) \\
= \; & k_0 i_0 + \cdots + k_n i_n > 0.
\end{aligned}$$

Therefore there is a point $p \in X$ with $z_j(p) = 0, j = 1, \ldots, n$, $z_0(p) = 1$ and $I_p(\mathbb{C}^n, X) < \frac{n+1}{d}$. Conversely if there is a point $p \in X$ with $I_p(\mathbb{P}^n, X) < \frac{n+1}{d}$ then one can find coordinates $z_0, \ldots, z_n$ and weights $w(z_i) = k_i$ such that

(1) $k_0 \leq k_1 \leq \cdots \leq k_n$,
(2) $k_0 + \cdots + k_n = 0$,
(3) $k_0 i_0 + \cdots + k_n i_n > 0$ for every monomial $z_0^{i_0} \cdots z_n^{i_n}$ in F. Hence X is unstable. □

There is an expectation of the restriction of singularities by the notion of stability. A natural question arises, to give a criterion for stability in terms of the nature of the singularities. There are various ways to measure how singularities of a variety are. Let Y be a nonsingular variety and D an effective $\mathbb{Q}$-Cartier divisor of Y. The invariant of the singularities of the pair (Y, D), called the log canonical threshold of Y along D, is an important topic to study the classification of higher dimensional algebraic varieties. The notion of discrepancy is the fundamental measure of the singularities of (Y, D) (cf. [8], [10]).

Definition. Let (Y, D) be a pair as above and let $p \in D$. The log canonical threshold of (Y, D) at the point p is defined by $\operatorname{lct}_p(Y, D) = \sup\{c \in \mathbb{Q}_+ \mid (Y, cD)$ is log canonical in a neighborhood of $p\}$. And define $\operatorname{lct}(Y, D) = \inf\{\operatorname{lct}_y(Y, D) \mid y \in Y\}$.

The log canonical threshold of the pair can be computed by using a log resolution of the pair or by assigning the weights to the variables. Let $\pi : W \to Y$ be a proper birational morphism. Write

$$K_W = \pi^* K_Y + \sum a_i E_i, \quad \text{and} \quad \pi^* D = \sum b_i E_i.$$

Then

$$\operatorname{lct}_p(Y, D) \leq \min_{p \in \pi(E_i)} \left\{ \frac{a_i + 1}{b_i} \right\}.$$

Equality holds if $\sum E_i$ is a divisor with normal crossing only. In particular, $\operatorname{lct}_p(Y, D) \in \mathbb{Q}$.

In general, it is hard to construct a log resolution explicitly. An efficient way of computation of log canonical threshold is in the weighted case:

Lemma 2.2 ([8]). *Let f be a holomorphic function near $0 \in \mathbb{C}^n$ and $D = (f = 0)$. Assign positive integer weights $w(x_i)$ to the variables x_i, and let $w(f)$ be the weighted multiplicity of f. Then*

$$\operatorname{lct}_0(\mathbb{C}^n, D) \leq \min \left\{ 1, \frac{\sum w(x_i)}{w(f)} \right\}.$$

And the equality holds if the weighted homogeneous leading term

$$f_w(x_1^{w(x_1)}, \ldots, x_n^{w(x_n)}) = 0 \subset \mathbb{P}^{n-1}$$

is smooth or has an isolated critical point at the origin.

The following lemma is basically due to Lemma 2.2.

Lemma 2.3. *Let f be a polynomial function near $0 \in \mathbb{C}^n$ and $D = (f = 0)$. Assume that there are assigned positive integer weights $w(x_i)$ to the variables x_i satisfying that the weighted homogeneous leading term $f_w(x_1^{w(x_1)}, \dots, x_n^{w(x_n)}) = 0 \subset \mathbb{P}^{n-1}$ is smooth or has an isolated critical point at the origin. Let $w(f)$ be the weighted multiplicity of f. Then*

$$I_0(\mathbb{C}^n, D) = \frac{\sum_{i=1}^n w(x_i)}{w(f)}.$$

Example 2.4. Let $f = y^2 - x^4$ and D the zero set of the polynomial f in $\mathbb{C}^2$.

(1) By blowing up twice, we have a log resolution $\pi : W \to \mathbb{C}^2$ and

$$K_W = \pi^* K_{\mathbb{C}^2} + E_1 + 2E_2, \;\; \pi^* D = {\pi_*}^{-1} D + 2E_1 + 4E_2.$$

Hence we have $\mathrm{lct}_0(\mathbb{C}^2, D) = \min\{\frac{0+1}{1}, \frac{1+1}{2}, \frac{2+1}{4}\} = \frac{3}{4}$.

(2) Assign the weights $w(x) = 1$ and $w(y) = 2$, then $w(f) = 4$. It implies that $\mathrm{lct}_0(\mathbb{C}^2, D) = \frac{w(x)+w(y)}{w(f)} = \frac{3}{4}$.

By Lemma 2.2, we have $\mathrm{lct}(\mathbb{P}^n, D) \le I(\mathbb{P}^n, D)$. Then by Lemma 2.1, we have the following:

Proposition 2.5. *Let D be a hypersurface of degree d in $\mathbb{P}^n$. Then we have the following criterion: If $\mathrm{lct}(\mathbb{P}^n, D) \ge \frac{n+1}{d}$ (resp. $>$) then D is semi-stable (resp. stable).*

One can ask if the converse holds in Proposition 2.5. It is easy to find the example. However, in the case of plane curves they are very closely related (cf. [4], [7]). The case of hypersurfaces is rather similar. Roughly speaking, the stability depends on all linear coordinate changes, but the log canonical thresholds depends on all analytic coordinate changes. In the paper [7], Proposition 2.5 for the case of plane curves is already observed to find some relations between Hacking's compact moduli space of plane curves [4] and the GIT compactification of moduli space of plane curves.

Example 2.6. Let $D = 3C$, where C is a nonsingular plane conic $(xz + y^2)$. Then $(\mathbb{P}^2, D)$ is semi-stable but $\mathrm{lct}(\mathbb{P}^2, D) = \frac{1}{3}$. Let $\bar{f}(x, y) = f(x, y, 1) = x + y^2$ and assign the weights $w(x) = 2, w(y) = 1$. Then $\bar{f}_w = (x^2 + y^2)^3 = 0$ in $\mathbb{P}^1$ does not give distinct points.

Remark. The condition $\mathrm{lct}(\mathbb{P}^n, D) > \frac{n+1}{d}$ can be expressed in other way. Note that

$$\mathrm{lct}(\mathbb{C}^{n+1}, \mathrm{Cone}(D)) = \min\left\{\frac{n+1}{d}, \mathrm{lct}(\mathbb{P}^n, D)\right\}.$$

Therefore the following are equivariant:

(i) $\mathrm{lct}(\mathbb{P}^n, D) > \frac{n+1}{d}$.

(ii) $(\mathbb{C}^{n+1}, \mathrm{Cone}(D))$ has the worst singularity at 0; the non log terminal locus of the pair $(\mathbb{C}^{n+1}, t\,\mathrm{Cone}(D)) = \{0\}$ for $t = \mathrm{lct}(\mathbb{C}^{n+1}, \mathrm{Cone}(D))$.

3. Chow stability criterion

Let X be a subvariety of dimension r and of degree d in $\mathbb{P}^n$. Consider the set $Z(X)$ of all $(n-r-1)$-dimensional projective subspaces L in $\mathbb{P}^n$ that intersects X. This is a subvariety in the Grassmannian $G(n-r, n+1)$ which parameterizes all the $(n-r-1)$-dimensional projective subspaces in $\mathbb{P}^n$. The subvariety $Z(X)$ is a hypersurface of degree d in $G(n-r, n+1)$. Let $\mathcal{B} = \oplus_{d=0}^{\infty} \mathcal{B}_d$ be the coordinate ring of $G(n-r, n+1)$ in the Plüker embedding. The subvariety $Z(X)$ is defined by the vanishing of some element $R_X \in \mathcal{B}_d$ which is unique up to a constant factor. This element is called the Chow form of X.

Let $u = (u_i) \in (\mathbb{P}^n)^*$, and let H_u be the hyperplane $\sum_{i=0}^{n} u_i z_i = 0$ where z_i, $i = 0, \ldots, n$ are coordinates of $\mathbb{P}^n$. Then

$$[X \cap H_u^{(1)} \cap \cdots \cap H_u^{(r+1)} \neq \emptyset] \Leftrightarrow [R_X(u_i^{(1)}, \ldots, u_i^{(r+1)}) = 0].$$

The coordinate ring $\oplus_{d=0}^{\infty} \mathcal{B}_d$ = subring of $\mathbb{C}[\ldots, U_i^{(j)}, \ldots]$ is generated by the Plücker coordinates $P_{i_1, \ldots, i_{r+1}}$ = determinant of $(r+1) \times (r+1)$ maximal minors of $(U_{i_\ell}^{(j)})$, $i_1 < \cdots < i_r$.

A variety X is called Chow semi-stable (resp. Chow stable) if its Chow form is semi-stable (resp. stable) for the natural $SL(n+1)$-action. Contrary to hypersurfaces in $\mathbb{P}^n$, there is no simple way to decide Chow stability. Proposition 2.5 can be generalized to the pair of Grassmannian variety and Chow form. Let X be a r-dimensional nondegenerate variety of degree d in $\mathbb{P}^n$. The Chow form R_X determines a hypersurface $Z(X)$ in the Grassmannian variety $G = G(n-r, n+1)$.

Proof of Theorem. We consider the product $\tilde{X} = X \times \mathbb{P}^r$ as a subvariety of $\mathbb{P}(\mathbb{C}^{n+1} \otimes (\mathbb{C}^{r+1})^*)$ via the Segre embedding. And we identify $\mathbb{C}^{n+1} \otimes (\mathbb{C}^{r+1})^*$ with the space $\mathrm{Mat}(r+1, n+1)$ of $(r+1) \times (n+1)$-matrices and consider the projection

$$\mathrm{Mat}(r+1, n+1) \supset S(r+1, n+1) \xrightarrow{p} G = G(n-r, n+1)$$

where $S(r+1, n+1)$ is the subset of $\mathrm{Mat}(r+1, n+1)$ with full rank. By this identification, the equation of dual variety $\tilde{X}^\vee$ in $\mathbb{P}^{(n+1)(r+1)-1}$ is the same as the equation $\tilde{R}_X$ lifted by R_X. It implies that

$$\tilde{X}^\vee = \text{projectivization of the closure of } p^{-1}(Z(X)) \text{ ([3], Chapter 3).}$$

Assume that $Z(X)$ is not Chow semi-stable in G. By the functorial properties [12], $p^{-1}(Z(X))$ is not semi-stable in $S(r+1, n+1)$. It implies that $\tilde{X}^\vee$ is not semi-stable in $\mathbb{P}^{(n+1)(r+1)-1}$. By the proof of Proposition 2.5,

$$\inf_{y \in U} \mathrm{lct}_y(\mathbb{P}^{(n+1)(r+1)-1}, \tilde{X}^\vee) < \frac{(n+1)(r+1)}{\deg \tilde{X}^\vee} = \frac{n+1}{d}$$

where U is the projectivization of $S(r+1,n+1)$ in $\mathbb{P}^{(n+1)(r+1)-1}$. And

$$\begin{aligned}\inf_{y\in U}\operatorname{lct}_y(\mathbb{P}^{(n+1)(r+1)-1},\tilde{X}^\vee) &= \operatorname{lct}(S(r+1,n+1),p^{-1}(Z(X)))\\ &= \operatorname{lct}(G,Z(X))\end{aligned}$$

because $S(r+1,n+1)$ is a $GL(r+1)$-bundle over G. □

Example 3.1. Let $X = p_1\cup\cdots\cup p_d$ be d points in $\mathbb{P}^n$. Let $Z(X)$ be the Chow form of X in $G(n,n+1)=(\mathbb{P}^n)^*$. Then X is Chow semi-stable (resp. Chow stable) if and only if for every proper linear subspace W of $\mathbb{P}^n$ (cf. [2])

$$\#\{i|p_i\in W\}\le\frac{d}{n+1}(\dim W+1)\ \ (\text{resp. } <).$$

By the following easy lemma, this is the same condition as

$$\operatorname{lct}((\mathbb{P}^n)^*,Z(X))\ge\frac{n+1}{d}\ \ (\text{resp. } >).$$

Lemma 3.2. *Let Y be a nonsingular variety of dimension m. Let D be a union of nonsingular divisors $D_1,\dots,D_d$ of Y. Assume that the scheme theoretic intersection Z of $D_1,\dots,D_d$ is a nonsingular variety of dimension k, and that $D_1,\dots,D_d$ meet transversely at Z. Then $\operatorname{lct}(Y,D)=\frac{m-k}{d}$.*

Proof. The proof is obtained by blowing-up of Z in Y. □

Example 3.3. Let $X=\ell_1\cup\cdots\cup\ell_d$ be d lines in $\mathbb{P}^3$. Then X is Chow semi-stable if and only if it satisfies the following (cf. [2]):

(1) no more than $\frac{d}{2}$ lines intersects at one point,
(2) no more than $\frac{d}{2}$ lines coincides and no more than $m-2t$ lines intersects a line which is repeated t times,
(3) no more than $\frac{d}{2}$ lines are coplanar.

If $\operatorname{lct}(G(2,4),Z(X))\ge\frac{4}{d}$ then X is Chow semi-stable. However, the conditions (1), (2), (3) do not imply $\operatorname{lct}(G(2,4),Z(X))\ge\frac{4}{d}$. If we translate the conditions (1) and (3) into the conditions in Chow form, then we have the following:

(1) no more than $\frac{d}{2}$ hyperplanes meets at quadric surface induced by the intersection of G with two hyperplanes (the set of lines through at one point in $\mathbb{P}^3$),
(3) no more than $\frac{d}{2}$ hyperplanes meets at $\mathbb{P}^2$ (the set of lines in the coplane).

These imply that $\operatorname{lct}(G(2,4),Z(X))\ge\frac{4-2}{\frac{d}{2}}=\frac{4}{d}$ by Lemma 3.2. But the second condition gives $\operatorname{lct}(G(2,4),Z(X))\ge\frac{2}{d}$.

Let X be a nonsingular variety of dimension r in $\mathbb{P}^n$. Assume that the degree of X is d and X is nondegenerate. Furthermore, we assume that the dual variety $X^\vee$ of X in $(\mathbb{P}^n)^*$ is a hypersurface. If $X^\vee$ is not a hypersurface then X is ruled in projective spaces (cf. [3], Chapter I). Let $(G,Z(X))$ be a pair of Grassmannian variety and Chow form as before. Let $\tilde{X}=X\times\mathbb{P}^r$ in $\mathbb{P}^{nr+n+r}$ via the Segre embedding (cf. the proof of Theorem). We have the inequality $\mathrm{lct}((\mathbb{P}^n)^*, X^\vee) \le \mathrm{lct}((\mathbb{P}^{nr+n+r})^*, \tilde{X}^\vee)$ by $X^\vee = \tilde{X}^\vee \cap (\mathbb{P}^n)^*$ from the construction and by Lemma 3.4.

Lemma 3.4 (cf. [1], [13]). *Let (Y, Z) be a pair. If $H \subset Y$ is a smooth irreducible divisor and $p \in H$ then $\mathrm{lct}_p(Y, Z) \ge \mathrm{lct}_p(H, Z \cap H)$.*

And by the Cayley's trick in the proof of Theorem, we have the inequality $\mathrm{lct}((\mathbb{P}^{nr+n+r})^*, \tilde{X}^\vee) \le \mathrm{lct}(G, Z(X))$. Therefore we have the following:

Proposition 3.5. *Let $X, \tilde{X}, G, Z(X)$ be varieties as above. Then we have the following inequality: $\mathrm{lct}((\mathbb{P}^n)^*, X^\vee) \le \mathrm{lct}(G, Z(X))$.*

Example 3.6. Let X be a rational normal curve of degree d in $\mathbb{P}^d$. Then the dual variety $X^\vee$ in $(\mathbb{P}^d)^*$ is the classical discriminant (cf. [3], Chapter I). Let $f(x) = \sum_{i=0}^d a_i x^{d-i}$. The classical discriminant $\triangle(f) = R(f, f')$ vanishes when $f(x)$ has multiple root, i.e., $f(x)$ has a multiple root if and only if $(a_0, \ldots, a_d) \in X^\vee \subset (\mathbb{P}^d)^*$.

By the definition of $\triangle(f)$, it has at worst singularity when $f(x)$ has a d-multiple root. Let $p = (1, 0, \ldots, 0)$. The discriminant $\triangle(f) = \triangle(a_0, \ldots, a_d)$ is a homogeneous polynomial in the a_i of degree $2d - 2$. In addition, it satisfies the quasi-homogeneity condition (cf. [3], Chapter 12):

$$\triangle(\lambda^0 a_0, \lambda^1 a_1, \ldots, \lambda^d a_d) = \lambda^{d(d-1)} \triangle(a_0, a_1, \ldots, a_d).$$

Assign the weights $w(a_i) = i$. Then by Lemma 2.2,

$$\mathrm{lct}((\mathbb{P}^d)^*, X^\vee) = \mathrm{lct}_p((\mathbb{P}^d)^*, X^\vee) \le \frac{1 + \cdots + d}{d(d-1)} = \frac{1}{2}\frac{d+1}{d-1}.$$

Example 3.7. Let X be a rational normal curve of degree d in $\mathbb{P}^d$. Consider the product $\tilde{X} = X \times \mathbb{P}^1$ as a subvariety of $\mathbb{P}^{2d+1}$ via the Segre embedding. Then the dual variety $(\tilde{X})^\vee$ in $(\mathbb{P}^{2d+1})^*$ is the classical resultant (cf. [3], Chapter 3). Let $f(x) = \sum_{i=0}^d a_i x^{d-i}$, $g(x) = \sum_{i=0}^d b_i x^{d-i}$. The classical resultant $R(f, g)$ vanishes when f and g has a d-multiple common root.

By the definition of $R(f, g)$ it has at worst singularity when f and g have a d-multiple common root. Let $p = (1, 0, \ldots, 0, 1, 0, \ldots, 0)$. The classical resultant $R(f, g)$ is homogeneous of degree d in the a_i and in the b_i. In addition, it satisfies the following quasi-homogeneity (cf. [3], Chapter 12):

$$R(\lambda^0 a_0, \ldots, \lambda^d a_d, \lambda^0 b_0, \ldots, \lambda^d b_d) = \lambda^{d^2} R(a_0, \ldots, a_d, b_0, \ldots, b_d).$$

We assign the weights $w(a_i) = i$, $w(b_i) = i$. Then

$$\begin{aligned}\mathrm{lct}(G, Z(X)) &= \mathrm{lct}_p(G, Z(X)) \\ &\le \min\left\{1, \tfrac{1+\cdots+d+1+\cdots+d}{d^2}\right\} \\ &= \min\left\{1, \tfrac{d+1}{d}\right\}.\end{aligned}$$

Example 3.8. Let X be the image of the map $\mathbb{P}^1 \to \mathbb{P}^{d-2}$ defined by

$$(s,t) \to (s^d, s^{d-2}t^2, s^{d-3}t^3, \ldots, s^3t^{d-3}, s^2t^{d-2}, t^d).$$

Then X is a rational curve with two cusps, and it is obtained by the projection of a rational normal curve of degree d in $\mathbb{P}^d$. By the similar computation in Example 3.7 and by assigning the weights $w(a_i) = i$, $w(b_i) = i$, we have the inequality $\mathrm{lct}(G, Z(X)) \leq \frac{d-1}{d}$.

In Example 3.7, Chow semi-stability implies $I(G, Z(X)) = \frac{d+1}{d}$. In Example 3.8, we expect that $I(G, Z(X)) = \frac{d-1}{d}$. In [6] we prove that X is Chow semi-stable when X is the image of the map $\mathbb{P}^1 \to \mathbb{P}^4$ defined by $(s,t) \to (s^6, s^4t^2, s^3t^3, s^2t^4, t^6)$. Therefore $I(G, Z(X)) = \frac{5}{6}$ if $d = 6$.

Example 3.9. Let x, y, z, w be a coordinate of $\mathbb{P}^3$. Let $d \geq 5$ and let C be a curve represented by the divisor class $(d-1, 1)$ in a nonsingular quadric surface $Q : xw - yz = 0$ in $\mathbb{P}^3$. The image of the map $\mathbb{P}^1 \to \mathbb{P}^3$ defined by $(s,t) \to (s^d, s^{d-1}t, st^{d-1}, t^d)$ is a $(d-1, 1)$-curve in Q. There is a one-dimensional family of $d-1$ secant lines L in Q, and there is no k $(3 \leq k \leq d-2)$ secant line in Q. Since the dimension of $G = G(2,4)$ is four and there is a one-dimensional family of $d-1$ secant lines, $\mathrm{lct}(G, Z(C)) \leq \frac{3}{d-1}$; For any line $L \in G$, $L \cap C$ is finite points. Therefore the multiplicity of $Z(C)$ is determined $\mathrm{mult}_L(Z(C)) = \sum \mathrm{mult}(C \cap L, x_i)$ where $x_i \in L \cap C$. Let $m = \max_{L \in Z(C)} \mathrm{mult}_L(Z(C))$. Consider a subscheme $Y = \{L \in Z(C) \mid \mathrm{mult}_L(Z(C)) = m\}$. By upper semi-continuity of $\mathrm{mult}_L(Z(C))$, Y is a finite union of subvarieties Y_i of $Z(C)$. Let $\ell = \max\{\dim(Y_i)\}$. Then it is easily obtained that $\mathrm{lct}(G, Z(C)) \leq \frac{\dim G - \ell}{m}$.

By the adjunction formula, the genus of C is zero. So C is linearly semi-stable, and it is Chow semi-stable [11]. In this case $I(G, Z(C)) = \frac{4}{d}$; Let z_0, z_1, z_2, z_3 be coordinates of $\mathbb{P}^3$. By assign the weights $w(z_0) = 0, w(z_1) = 1, w(z_2) = d-1, w(z_3) = d$, we have the inequality

$$I(G, Z(C)) \leq \frac{1 + d - 1 + d + 1 + d - 1 + d}{d^2} = \frac{4}{d}.$$

And Chow semi-stability implies that $I(G, Z(C)) = \frac{4}{d}$.

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