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THE PRODUCT OF ANALYTIC FUNCTIONALS IN $\mathcal{Z}'$

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ABSTRACT. Current studies on products of analytic functionals have been based on applying convolution products in $\mathcal{D}'$ and the Fourier exchange formula. There are very few results directly computed from the ultra-distribution space $\mathcal{Z}'$. The goal of this paper is to introduce a definition for the product of analytic functionals and construct a new multiplier space $F(\mathcal{N}_m)$ for $\delta^{(m)}(s)$ in a one or multiple dimension space, where $\mathcal{N}_m$ may contain functions without compact support. Several examples of the products are presented using the Cauchy integral formula and the multiplier space, including the fractional derivative of the delta function $\delta^{(\alpha)}(s)$ for $\alpha > 0$.

1. Introduction

Physicists have long been using the singular function $\delta(x)$, although it can not be properly defined within the structure of classical function theory. In elementary particle physics, one finds the need to evaluate δ^2 when calculating the transition rates of certain particle interactions [15]. Schwartz [24] established the theory of distributions by treating singular functions as linearly continuous functionals on the testing function space whose elements have compact support. Although they are of great importance to quantum field theory, it is difficult to define products, convolutions and compositions of distributions in general. The sequential method [6]-[14], [17], [19]-[23] and complex analysis approach [1]-[5], [7], [16]-[21], including non-standard analysis [18], have been the main tools in dealing with those non-linear operations of distributions in the Schwartz space $\mathcal{D}'$ with many results. However, little progress has been made towards obtaining products of analytic functionals in $\mathcal{Z}'$ (or $\mathcal{Z}(R^m)$) directly without using convolution results in $\mathcal{D}'$ (or $\mathcal{D}'(R^m)$) and the Fourier transform as a bridge. As outlined in the abstract, we will work on the space $\mathcal{Z}'$ and initiate a move

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towards computing the product of analytic functionals without the help of the exchange formula.

To make this paper as self-contained as possible, we state Paley-Wiener-Schwartz theorem in the following, which will be used a couple of times throughout the article.

Theorem 1.1. *An entire function $f(s)$ on $\mathcal{C}^m$ is the Fourier transform of distribution $\lambda(x)$ with compact support if and only if for all $s \in \mathcal{C}^m$,*

$$|f(s)| \leq Ce^{b|\operatorname{Im} s|}(1 + |s|)^q$$

for some constants C, q and b. The distribution $\lambda(x)$ will in fact be supported in the closed ball of center zero and radius b.

2. Analytic functionals in $\mathcal{Z}'$

Let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. We say a sequence $\phi_1(x), \phi_2(x), \ldots, \phi_\nu(x), \ldots$ of test functions converges to zero in $\mathcal{D}$ if all of these functions vanish outside a certain fixed bounded region K, which is independent of ν, and converge uniformly to zero (in the usual sense) together with their derivatives of any order.

As in [16], we define the Fourier transform of a function ϕ in $\mathcal{D}$ by

$$\psi(s) = F(\phi)(s) = \tilde{\phi}(s) = \int_{-\infty}^{\infty} \phi(x) e^{ixs}\, dx.$$

Here $s = \sigma + i\tau$ is a complex variable and it is well known that $\psi(s)$ is an entire analytic function with the following property for $q = 0, 1, 2, \ldots$

$$|s^q \psi(s)| \leq C_q e^{a|\operatorname{Im} s|} \tag{1}$$

for some constants C_q and a depending on $\psi(s)$. The set of all entire analytic functions with property (1) is indeed the space

$$\mathcal{Z} = F(\mathcal{D}) = \{\psi \mid \exists\, \phi \in \mathcal{D} \text{ and } F(\phi) = \psi\}.$$

The definition of convergence in $\mathcal{Z}$ can be carried over from $\mathcal{D}$. That is, a sequence of functions $\psi_\nu(s)$ converges to zero in $\mathcal{Z}$ if the sequence of their inverse images $\phi_\nu(s)$ converges to zero in $\mathcal{D}$. In other words, the sequence $\psi_\nu(s)$ converges to zero in $\mathcal{Z}$ if for each function in this sequence we have

$$|s^q \psi_\nu(s)| \leq C_q e^{a|\operatorname{Im} s|}$$

with C_q and a independent of ν, and if the functions converge to zero uniformly on every interval of the (real) σ axis.

The Fourier transform $\tilde{f}$ of a distribution f in $\mathcal{D}'$ is an ultradistribution in $\mathcal{Z}'$, i.e., a linear and continuous functional on $\mathcal{Z}$. It is defined by Parseval's equation

$$(\tilde{f},\, \tilde{\phi}) = 2\pi(f,\, \phi).$$

Clearly we have

$$\mathcal{Z}' = F(\mathcal{D}') = \{F(f) \mid f \in \mathcal{D}'\}.$$

The exchange formula is the equality

$$F(f * g) = F(f) \cdot F(g).$$

It was proven in [25] that the exchange formula holds for all convolution products of distributions f and g, provided f and g both have compact support.

We shall call a functional g on $\mathcal{Z}$ analytic if it can be written in the form

$$(g,\ \psi) = \int_\Gamma \bar{g}(s)\psi(s)\,ds,$$

where $g(s)$ is a function and Γ is some contour in the complex plane $\mathcal{C}$. Thus, the delta function given by $(\delta(s - s_0),\ \psi(s)) = \psi(s_0)$, where $s_0 \in \mathcal{C}$, is an analytic functional, since

$$\psi(s_0) = \frac{1}{2\pi i}\oint_\Gamma \frac{\psi(s)}{s - s_0}\,ds,$$

where Γ is any contour enclosing s_0 in counterclockwise. We denote

$$\delta(s - s_0) = \left\{\frac{1}{2\pi i(s - s_0)},\ \Gamma\right\}.$$

Similarly, we have

$$\delta^{(m)}(s - s_0) = \left\{\frac{(-1)^m m!}{2\pi i(s - s_0)^{m+1}},\ \Gamma\right\}.$$

Following the standard notation we let $\mathcal{E}'$ be the space of distributions with compact support. Obviously, we have $\mathcal{D} \subset \mathcal{E}' \subset \mathcal{D}'$ and $\mathcal{Z} = F(\mathcal{D}) \subset F(\mathcal{E}') \subset F(\mathcal{D}') = \mathcal{Z}'$.

Define a multiplier space of $\mathcal{Z}'$ as

$$\mathcal{M} = \left\{h(s) \mid h \text{ is entire and } |h(s)| \le Ce^{b|\operatorname{Im} s|}(1 + |s|)^q\right\}$$

for some b, q and C. By Paley-Wiener-Schwartz theorem stated in the introduction, we imply that $\mathcal{M} = F(\mathcal{E}')$ and $\mathcal{Z} \subset \mathcal{M} \subset \mathcal{Z}'$. For any $g \in \mathcal{Z}'$ and $h(s) \in \mathcal{M}$, the product $\bar{h}(s)g(s)$ is well defined by

$$(\bar{h}(s)g(s),\ \psi) = (g(s),\ h(s)\psi)$$

because $h(s)\psi(s) \in \mathcal{Z}$. It follows that

$$\bar{h}(s)\,\delta^{(m)}(s) = \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}h^{(m-j)}(0)\delta^{(j)}(s). \tag{2}$$

In particular,

$$\sin s\,\delta^{(m)}(s) = \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sin[(m - j)\frac{\pi}{2}]\delta^{(j)}(s),$$

where $\sin s = \frac{1}{2i}(e^{is} - e^{-is}) \in \mathcal{M}$.

Choosing a fixed function $\omega(s) \in \mathcal{M}$, we can construct two different analytic functionals for $d > 0$,

$$(f_+(s),\ \psi(s)) = \int_{-\infty+di}^{\infty+di} \frac{\omega(s)\psi(s)}{s^{n+1}}\, ds$$

and

$$(f_-(s),\ \psi(s)) = \int_{-\infty-di}^{\infty-di} \frac{\omega(s)\psi(s)}{s^{n+1}}\, ds,$$

where n is any non-negative integer. Those two integrals are clearly convergent since $\omega(s)\psi(s) \in \mathcal{Z}$.

The difference between them can be simplified into the form

$$(f_+(s) - f_-(s),\ \psi(s)) = \oint_{|s|=1} \frac{\omega(s)\psi(s)}{s^{n+1}}\, ds$$

in which the integral is taken clockwise along the boundary of $|s| = 1$. By Cauchy's integral theorem

$$(f_+(s) - f_-(s),\ \psi(s)) = -\frac{2\pi i}{n!} \sum_{k=0}^{n} \binom{n}{k} \omega^{(n-k)}(0)\psi^{(k)}(0).$$

Therefore

$$f_+(s) - f_-(s) = \frac{2\pi i}{n!} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \omega^{(n-k)}(0)\delta^{(k)}(s).$$

Finally we have the following product in $\mathcal{Z}'$ from equation (2)

$$\begin{aligned} &\bar{h}(s)\ (f_+(s) - f_-(s)) \\ &= \frac{2\pi i}{n!} \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{j+1} \binom{n}{k} \binom{k}{j} \omega^{(n-k)}(0) h^{(k-j)}(0) \delta^{(j)}(s). \end{aligned}$$

It is well known that every functional in $\mathcal{D}'$ which concentrates on a point is a finite sum of linear combinations of the delta function and its derivatives. However, this property does not hold in general for functionals in $\mathcal{Z}'$. We are going to build up a fractional derivative of the delta function $\delta^{(\alpha)}(s)$ for $\alpha > 0$, which belongs to $\mathcal{Z}'$ and can only be expressed in terms of infinite sums of the delta function and its derivatives. Using the Cauchy-type fractional derivative, we define

$$(\delta^{(\alpha)}(s),\ \psi(s)) = (\cos \alpha\pi + i \sin \alpha\pi) \frac{\Gamma(\alpha+1)}{2\pi i} \oint_{|s|=1} \frac{\psi(s)}{s^{\alpha+1}}\, ds \tag{3}$$

in counterclockwise along $|s| = 1$. Here we choose the fixed analytic branch $\ln 1 = 0$, $-\pi < \arg s \le \pi$ and $k = 0$, such that

$$\frac{1}{s^{\alpha+1}} = s^{-(\alpha+1)} = e^{-(\alpha+1)\ln s} \cdot e^{-(\alpha+1)2k\pi i}$$

is an analytic single-valued function.

Assume that $\alpha \neq 0, 1, 2, \ldots$ and let $\psi(s) = \sum_{n=0}^{\infty} a_n s^n$ be the Taylor series which converges uniformly on $|s| = 1$. Equation (3) therefore yields

$$\begin{aligned}
&(\delta^{(\alpha)}(s),\ \psi(s)) \\
= \ &(\cos\alpha\pi + i\sin\alpha\pi)\frac{\Gamma(\alpha+1)}{2\pi i}\sum_{n=0}^{\infty} a_n \oint_{|s|=1} \frac{s^n}{s^{\alpha+1}}\, ds \\
= \ &(\cos\alpha\pi + i\sin\alpha\pi)\frac{\Gamma(\alpha+1)}{2\pi i}\sum_{n=0}^{\infty} a_n \left(\int_{C_1} s^{n-\alpha-1} ds + \int_{C_2} s^{n-\alpha-1} ds\right),
\end{aligned}$$

where C_1 and C_2 are the parts of $|s| = 1$ in the upper and lower σ axis respectively.

We compute directly

$$\int_{C_1} s^{n-\alpha-1} ds = \left.\frac{s^{n-\alpha}}{n-\alpha}\right|_0^{\pi} = \frac{1}{n-\alpha}(\cos(n-\alpha)\pi + i\sin(n-\alpha)\pi - 1)$$

and

$$\int_{C_2} s^{n-\alpha-1} ds = \left.\frac{s^{n-\alpha}}{n-\alpha}\right|_{-\pi}^{0} = \frac{1}{n-\alpha}(1 - \cos(n-\alpha)\pi + i\sin(n-\alpha)\pi).$$

Adding the two terms we get

$$\int_{C_1} s^{n-\alpha-1} ds + \int_{C_2} s^{n-\alpha-1} ds = \frac{2i\sin(n-\alpha)\pi}{n-\alpha} = \frac{2i(-1)^n \sin\alpha\pi}{\alpha-n},$$

so that

$$(\delta^{(\alpha)}(s),\ \psi(s)) = (\cos\alpha\pi + i\sin\alpha\pi)\frac{\Gamma(\alpha+1)}{\pi}\sin\alpha\pi \sum_{n=0}^{\infty} \frac{(-1)^n a_n}{\alpha-n}.$$

Since the Taylor series coefficient

$$a_n = \frac{\psi^{(n)}(0)}{n!} = \frac{(-1)^n}{n!}(\delta^{(n)}(s), \psi(s)),$$

we arrive at

$$\delta^{(\alpha)}(s) = (\cos\alpha\pi + i\sin\alpha\pi)\sin\alpha\pi\frac{\Gamma(\alpha+1)}{\pi}\sum_{n=0}^{\infty}\frac{\delta^{(n)}(s)}{(\alpha-n)n!} \tag{4}$$

for all $\alpha > 0$. Clearly we have

$$\delta^{(\frac{1}{2})}(s) = \frac{i}{2\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\delta^{(n)}(s)}{(\frac{1}{2}-n)n!}.$$

In particular, we let $\alpha \to n$ and note that all terms on the right-hand side of equation (4) disappear except the nth term. Thus,

$$\lim_{\alpha\to n}(\cos\alpha\pi + i\sin\alpha\pi)\sin\alpha\pi\frac{\Gamma(\alpha+1)}{\pi}\sum_{n=0}^{\infty}\frac{\delta^{(n)}(s)}{(\alpha-n)n!} = \delta^{(n)}(s),$$

which indicates that it is an extension of the normal derivative $\delta^{(n)}(s)$.

It remains to be shown that $\delta^{(\alpha)}(s)$ is linear and continuous on $\mathcal{Z}$. Obviously it is linear since

$$(\delta^{(\alpha)}(s),\ \eta\psi_1 + \xi\psi_2) = \eta\,(\delta^{(\alpha)}(s),\ \psi_1) + \xi\,(\delta^{(\alpha)}(s),\ \psi_2).$$

Let $\{\psi_m\}$ be a sequence converging to zero in $\mathcal{Z}$. Then

$$\psi_m(s) = \int_{-\infty}^{\infty} \phi_m(x) e^{ixs}\, dx,$$

where $\{\phi_m\}$ converges to zero in $\mathcal{D}$, which means $\forall\ \epsilon > 0$, $\exists\ N = N(\epsilon)$ such that $|\phi_m(x)| < \epsilon$ for $m > N$. Assume $\operatorname{supp}\phi_m \subset K$, we infer that for $m > N$

$$|\psi_m(s)| \leq \int_K \epsilon |e^{ixs}| dx \leq M(\rho_0),$$

where $|s| = \rho_0 > 1$. It follows from the fact that $\psi(s)$ is an entire function that

$$\left|\frac{\psi_m^{(n)}(0)}{n!}\right| \leq \frac{M(\rho_0)}{\rho_0^n},$$

which implies that

$$(\cos\alpha\pi + i\sin\alpha\pi)\sin\alpha\pi\frac{\Gamma(\alpha+1)}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\psi_m^{(n)}(0)}{n!(\alpha-n)}$$

uniformly converges with respect to m. Hence

$$\begin{aligned}
&\lim_{m\to\infty}(\delta^{(\alpha)}(s),\ \psi_m(s)) \\
= \ &\lim_{m\to\infty}(\cos\alpha\pi + i\sin\alpha\pi)\sin\alpha\pi\frac{\Gamma(\alpha+1)}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\psi_m^{(n)}(0)}{n!(\alpha-n)} \\
= \ &(\cos\alpha\pi + i\sin\alpha\pi)\sin\alpha\pi\frac{\Gamma(\alpha+1)}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\lim_{m\to\infty}\psi_m^{(n)}(0)}{n!(\alpha-n)} = 0
\end{aligned}$$

since the sequence with its derivatives converges to zero uniformly on every interval of the (real) σ axis.

To end this section, we would like to point out that the following identity is clearly satisfied

$$\psi^{(\alpha)}(0) = \sin\alpha\pi\frac{\Gamma(\alpha+1)}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\psi^{(n)}(0)}{n!(\alpha-n)}$$

for all $\psi \in \mathcal{Z}$.

3. A new multiplier space for $\delta^{(m)}(s)$

Let $\tau(x)$ be an infinitely differentiable function satisfying the following conditions:

$$(i)\ \ \tau(x) = 1 \ \text{ for } \ |x| \leq 1, \qquad (ii)\ \ \tau(x) = 0 \ \text{ for } \ |x| \geq 2.$$

We construct the sequence $\tau_n(x) = \tau(x/n)$ in $\mathcal{D}$ and $\delta_n(s) = \dfrac{1}{2\pi}F(\tau_n)$ in $\mathcal{Z}$. Putting $\psi = \tilde{\phi} = F(\phi)$, we have from Parseval's equation

$$(\tau_n,\ \phi) = \frac{1}{2\pi}(F(\tau_n),\ F(\phi)) = (\delta_n,\ \psi).$$

Thus,

$$\lim_{n\to\infty}(\delta_n,\ \psi) = \lim_{n\to\infty}(\tau_n,\ \phi) = \int_{-\infty}^{\infty}\phi(x)dx = (1,\ \phi)$$

for all $\phi \in \mathcal{D}$. Using $F(1) = 2\pi\delta$, we get $\lim_{n\to\infty}(\delta_n,\ \psi) = (\delta,\ \psi)$ for all $\psi \in \mathcal{Z}$. Hence $\{\delta_n\}$ is a delta sequence in $\mathcal{Z}$. It follows from Parseval's equation that

$$(\delta_n(s),\ \psi(s+v)) = \frac{1}{2\pi}(F(\tau_n),\ F(e^{ixv}\phi)) = (\tau_n(x),\ e^{ixv}\phi) = F(\tau_n\phi).$$

Obviously, $\tau_n(x)\phi(x)$ converges to $\phi(x)$ in $\mathcal{D}$, which implies that $(\delta_n(s),\ \psi(s+v))$ converges to $\psi(v)$ in $\mathcal{Z}$. This gives us

$$\lim_{n\to\infty}(\tilde{f} * \delta_n,\ \psi) = \lim_{n\to\infty}(\tilde{f}(v),\ (\delta_n(s),\ \psi(s+v))) = (\tilde{f},\ \psi)$$

for arbitrary $\psi \in \mathcal{Z}$ and it follows that $\{\tilde{f} * \delta_n\}$ is a sequence converging to $\tilde{f}$ in $\mathcal{Z}'$

Now we are ready to give a new definition for the product of analytic functionals in $\mathcal{Z}'$ using the delta sequence $\delta_n(s)$.

Definition 3.1. Let $h(s)$ be a continuous function and let $g(s) \in \mathcal{Z}'$. Then the product $\bar{h}(s) \cdot g(s)$ is defined as

$$(\bar{h}(s) \cdot g(s),\ \psi) = \lim_{n\to\infty}(g(s),\ (h(s) * \delta_n)\psi)$$

provided the limit exists.

In particular, if $h(s) \in \mathcal{M}$ and $g(s) \in \mathcal{Z}'$, then $\bar{h}(s) \cdot g(s) = \bar{h}(s)g(s)$, which is defined in section 2. By Definition 3.1, we only need to show that $(h(s) * \delta_n)\psi$ converges to $h(s)\psi$ in $\mathcal{Z}$. Assume that $h(s) = F(\lambda(x))$, where $\lambda(x)$ is a distribution with compact support, and $\psi = F(\phi)$ for some $\phi \in \mathcal{D}$. Then $(h(s) * \delta_n)\psi = F((\tau_n\lambda) * \phi)$, and it is not hard to prove that $(\tau_n\lambda) * \phi$ converges to $\lambda * \phi$ in $\mathcal{D}$. Indeed, we easily see that $\lambda * \phi$ in $\mathcal{D}$ since both λ and ϕ have compact support and let γ be the function in $\mathcal{D}$ and $\gamma = 1$ on suppλ. Then

$$((\tau_n\lambda - \lambda) * \phi^{(m)})(x) = (\lambda(y),\ (\tau_n(y) - 1)\phi^{(m)}(x-y)\gamma(y)) \to 0$$

uniformly on every compact subset of R. This implies that $(h(s) * \delta_n)\psi$ converges to $h(s)\psi$ in $\mathcal{Z}$.

Let m be any non-negative integer and we define the space $\mathcal{N}_m$ with the normal addition and scalar multiplication as

$$\mathcal{N}_m = \left\{\phi(x)\ \mid\ F(\phi(x)) \text{ is entire and } \left|\int_{-\infty}^{\infty} x^j\phi(x)dx\right| < \infty\right\}$$

for $j = 0, 1, 2, \ldots, m$. Obviously $\mathcal{N}_m$ is not empty since $e^{-x^2} \in \mathcal{N}_m$ for all $m \geq 0$ because

$$F(e^{-x^2}) = \int_{-\infty}^{\infty} e^{ixs - x^2} dx = e^{-\frac{1}{4}s^2}\sqrt{\pi}$$

is an entire function and

$$\int_{-\infty}^{\infty} x^m e^{-x^2} dx = \frac{1}{2}\Gamma(\frac{1}{2}m + \frac{1}{2})(1 + (-1)^m). \tag{5}$$

Furthermore, we have $\mathcal{N}_0 \supset \mathcal{N}_1 \supset \cdots \supset \mathcal{N}_m \supset \cdots$ and $\mathcal{Z} = F(\mathcal{D}) \subset F(\mathcal{N}_m) \neq \mathcal{M}$ as $\mathcal{D} \subset \mathcal{N}_m$ for all $m \geq 0$, which clearly contains functions without compact support.

Theorem 3.1. *Let $h(s) = F(\phi) \in F(\mathcal{N}_m)$ and m be a non-negative integer. Then the product $\bar{h}(s)$ and $\delta^{(m)}(s)$ exists in $\mathcal{Z}'$ and*

$$\begin{aligned} \bar{h}(s) \cdot \delta^{(m)}(s) = & \sum_{j=0}^{m} (-1)^{m-j} (\cos(m-j)\frac{\pi}{2} \\ & + i \sin(m-j)\frac{\pi}{2}) \binom{m}{j} \int_{-\infty}^{\infty} x^{m-j}\phi(x) dx\, \delta^{(j)}(s). \end{aligned}$$

Proof. By Definition 3.1

$$(\bar{h}(s) \cdot \delta^{(m)}(s),\ \psi(s)) = \lim_{n\to\infty} (\delta^{(m)}(s),\ (h(s) * \delta_n)\psi(s)).$$

It follows that

$$h(s) * \delta_n = (\delta_n(s), h(s+v)) = \int_{-\infty}^{\infty} \tau_n(x) e^{ixv}\phi(x) dx.$$

Since $\phi(x)$ is locally integrable and $\tau_n(x)\phi(x) \in \mathcal{E}'$, we claim from Paley-Wiener-Schwartz theorem that

$$\int_{-\infty}^{\infty} \tau_n(x) e^{ixv}\phi(x) dx = F(\tau_n\phi) \in \mathcal{M},$$

which implies that $(h(s) * \delta_n)\psi \in \mathcal{Z}$. Hence

$$\begin{aligned} & (\delta^{(m)}(s),\ (h(s) * \delta_n)\psi(s)) \\ = & (-1)^m \sum_{j=0}^{m} \binom{m}{j} (h(s) * \delta_n)^{(m-j)}(0)\psi^{(j)}(0) \end{aligned}$$

$$= (-1)^m \sum_{j=0}^{m} \binom{m}{j} \int_{-\infty}^{\infty} \tau_n(x)(ix)^{m-j}\phi(x)\,dx\,\psi^{(j)}(0)$$

$$= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \int_{-\infty}^{\infty} \tau_n(x)(ix)^{m-j}\phi(x)dx\,(\delta^{(j)}(s),\ \psi(s)) \quad \text{and}$$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \tau_n(x)(ix)^{m-j}\phi(x)dx$$

$$= C_{m,j} \int_{-\infty}^{\infty} x^{m-j}\phi(x)dx,$$

where we define

$$C_{m,j} = i^{m-j} = \cos(m-j)\frac{\pi}{2} + i\sin(m-j)\frac{\pi}{2}.$$

This completes the proof of Theorem 3.1. □

We would like to mention that Theorem 3.1 provides a powerful method for computing the product $\bar{h}(s) \cdot \delta^{(m)}(s)$ when it is difficult to evaluate the Fourier transform $h(s) = F(\phi)$ for $\phi \in \mathcal{N}_m$. As an example, let us consider $\phi(x) = x^k e^{-x^2} \in \mathcal{N}_m$ for some positive integer k. Using identity (5) and Theorem 3.1, it is possible to derive the following product, although we have trouble obtaining the Fourier transform $h(s) = F(x^k e^{-x^2})$

$$\bar{h}(s) \cdot \delta^{(m)}(s)$$

$$= \frac{1}{2}\sum_{j=0}^{m} (-1)^{m-j} C_{m,j} \binom{m}{j} \Gamma(\frac{1}{2}m - \frac{1}{2}j + \frac{1}{2}k + \frac{1}{2})(1 + (-1)^{m-j+k})\,\delta^{(j)}(s).$$

Remark. Since any ultradistribution in $\mathcal{Z}'$ is not only infinitely differentiable, but also expandable or analyzable in the sense that for every $g \in \mathcal{Z}'$

$$g(s+h) = \sum_{q=0}^{\infty} g^{(q)}(s)\frac{h^q}{q!},$$

where the series on the right converges in $\mathcal{Z}'$, and $g(s+h)$ is the ultradistribution obtained from $g(s)$ by translation through h. Therefore we can easily compute other products, such as $\bar{h}(s) \cdot \delta^{(m)}(s+1)$ and $\bar{h}(s) \cdot \delta^{(m)}(s-1)$ by Theorem 3.1.

4. The case of several variables

The Fourier transform of a function $\phi(x_1, x_2, \ldots, x_m) \in \mathcal{D}(R^m)$ is defined by

$$\psi(s) = \psi(s_1, s_2, \ldots, s_m)$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(x_1, x_2, \ldots, x_m) \exp[i(x_1 s_1 + \cdots + x_m s_m)]dx_1 \cdots dx_m$$

or, more briefly, by

$$\psi(s) = \int_{R^m} \phi(x) e^{i(x,s)} dx,$$

where $(x,s) = x_1 s_1 + \cdots + x_m s_m$.

The new function $\psi(s)$, defined in $\mathcal{C}^m$, the space of m complex dimensions, is continuous and analytic in each of its variables s_k. If $\phi(x)$ vanishes for $|x_k| > a_k$, $k = 1, 2, \ldots, m$, then $\psi(s)$ satisfies the inequality

$$|s_1^{q_1} \cdots s_m^{q_m} \psi(s_1, \ldots, s_m)| \leq C_q \exp(a_1 |\operatorname{Im} s_1| + \cdots + a_m |\operatorname{Im} s_m|). \tag{6}$$

Conversely, every entire function $\psi(s_1, s_2, \ldots, s_m)$ satisfying (6) is the Fourier transform of some $\phi(x_1, x_2, \ldots, x_m) \in \mathcal{D}(R^m)$ which vanishes for $|x_k| > a_k$, $k = 1, 2, \ldots, m$.

The space of all entire functions $\psi(s)$ satisfying inequality (6) with the natural definitions of the linear operations will be called $\mathcal{Z}(R^m)$, i.e.,

$$\mathcal{Z}(R^m) = F(\mathcal{D}(R^m)) = \{\psi(s) \mid \exists\, \phi \in \mathcal{D}(R^m) \text{ and } F(\phi) = \psi\}.$$

We define the convergence in $\mathcal{Z}(R^m)$ as follows. A sequence $\{\psi_\nu(s)\}$ is said to converge to zero in $\mathcal{Z}(R^m)$ if the sequence of the inverse Fourier transforms converges to zero in $\mathcal{D}(R^m)$.

Let $\tau(x) = \tau(x_1, x_2, \ldots, x_m)$ be an infinitely differentiable function satisfying the following conditions:

$$(i)\ \tau(x) = 1 \text{ for } |x| = \sqrt{x_1^2 + \cdots + x_m^2} \leq 1, \qquad (ii)\ \tau(x) = 0 \text{ for } |x| \geq 2.$$

We build up the sequence $\tau_n(x) = \tau(\frac{x_1}{n}, \frac{x_2}{n}, \ldots, \frac{x_m}{n}) \in \mathcal{D}(R^m)$ and clearly $\tau_n(x)\phi(x)$ converges to $\phi(x)$ in $\mathcal{D}(R^m)$, which implies that $F(\tau_n(x)\phi(x))$ converges to $F(\phi(x)) = \psi(s)$ in $\mathcal{Z}(R^m)$. Define $\delta_n(s) = \frac{1}{2\pi} F(\tau_n)$ in $\mathcal{Z}(R^m)$, we have

$$(\delta_n(s), \psi(s+v)) = (\psi * \delta_n)(v) = F(\tau_n(x)\phi(x)) \to F(\phi(x)) = \psi(s),$$

therefore $(\delta_n(s), \psi(s+v))$ converges to $\psi(v)$ in $\mathcal{Z}(R^m)$. This gives us

$$\lim_{n\to\infty} (\tilde{f} * \delta_n,\ \psi) = \lim_{n\to\infty} (\tilde{f}(v),\ (\delta_n(s),\ \psi(s+v))) = (\tilde{f},\ \psi)$$

for arbitrary $\psi \in \mathcal{Z}(R^m)$ and it follows that $\{\tilde{f} * \delta_n\}$ is a sequence converging to $\tilde{f}$ in $\mathcal{Z}'(R^m)$

Definition 4.1. Let $h(s)$ be a continuous function and let $g(s) \in \mathcal{Z}'(R^m)$. Then the product $\bar{h}(s) \cdot g(s)$ is defined as

$$(\bar{h}(s) \cdot g(s),\ \psi) = \lim_{n\to\infty} (g(s),\ (h(s) * \delta_n)\psi)$$

provided the limit exists.

Assume $\mathcal{N}$ is the set of all nonnegative integers and

$$\mathcal{N}^m = \{j = (j_1, j_2, \ldots, j_m) \mid j_i \in \mathcal{N}, \ 1 \le i \le m\}.$$

Let $|j| = j_1 + j_2 + \cdots + j_m$ and $x^j = x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m}$. We define the space $\mathcal{N}_p(R^m)$ with the normal addition and scalar multiplication as

$$\mathcal{N}_p(R^m) = \left\{ \phi(x) \mid F(\phi(x)) \text{ is entire on } \mathcal{C}^m \text{ and } \left| \int_{R^m} x^j \phi(x) dx \right| < \infty \right\}$$

for $|j| = 0, 1, 2, \ldots, |p|$. Obviously $\mathcal{N}_p(R^m)$ is not empty since $e^{-r^2} \in \mathcal{N}_p(R^m)$ for all $p \in \mathcal{N}^m$ because

$$F(e^{-r^2}) = \int_{R^m} e^{ixs - x^2} dx = e^{-\frac{1}{4}s^2} \pi^{\frac{m}{2}}$$

is an entire function on $\mathcal{C}^m$ and

$$\int_{R^m} x^p e^{-r^2} dx = \frac{1}{2}\Gamma(\frac{1}{2}p_1 + \frac{1}{2})(1 + (-1)^{p_1}) \cdots \frac{1}{2}\Gamma(\frac{1}{2}p_m + \frac{1}{2})(1 + (-1)^{p_m}).$$

is finite.

One can easily extend Theorem 3.1 to the case of several variables and we leave this for interested readers.

References

[1] M. A. Aguirre T, *A convolution product of $(2j)$th derivative of Dirac's delta in r and multiplicative distributional product between r^{-k} and $\nabla(\Delta^j \delta)$*, Int. J. Math. Math. Sci. (2003), no. 13, 789–799.

[2] ———, *The expansion in series (of Taylor types) of $(k-1)$ derivative of Dirac's delta in $m^2 + P$*, Integral Transforms Spec. Funct. **14** (2003), no. 2, 117–127.

[3] ———, *The series expansion of $\delta^{(k)}(r-c)$*, Math. Notae **35** (1991), 53–61.

[4] ———, *A generalization of convolution product of the distributional families related to the diamond operator*, Thai J. Math. **2** (2004), 97–106.

[5] ———, *The expansion of $\delta^{(k-1)}(m^2 + P)$*, Integral Transform. Spec. Funct. **8** (1999), no. 1-2, 139–148.

[6] P. Antosik, J. Mikusinski, and R. Sikorski, *Theory of Distributions. The Sequential Approach*, Elsevier Scientific Publishing Co., Amsterdam; PWN—Polish Scientific Publishers, Warsaw, 1973.

[7] H. J. Bremermann, *Distributions, Complex Variables, and Fourier Transforms*, Addison-Wesley Publishing Co., Inc., Reading, Mass.-London 1965.

[8] L. Z. Cheng and C. K. Li, *A commutative neutrix product of distributions on R^m*, Math. Nachr. **151** (1991), 345–355.

[9] B. Fisher, *The product of distributions*, Quart. J. Math. Oxford Ser. (2) **22** (1971), 291–298.

[10] ———, *A noncommutative neutrix product of distributions*, Math. Nachr. **108** (1982), 117–127.

[11] B. Fisher and K. Taş, *The convolution of functions and distributions*, J. Math. Anal. Appl. **306** (2005), no. 1, 364–374.

[12] ———, *On the composition of the distributions x_+^λ and x_+^μ*, J. Math. Anal. Appl. **318** (2006), no. 1, 102–111.

[13] ———, *On the non-commutative neutrix product of the distributions $x^r \ln^p |x|$ and x^{-s}*, Integral Transforms Spec. Funct. **16** (2005), no. 2, 131–138.

[14] ———, *On the commutative product of distributions*, J. Korean Math. Soc. **43** (2006), no. 2, 271–281.
[15] S. Gasiorowicz, *Elementary Particle Physics*, J. Wiley and Sons, Inc., N. Y. 1966.
[16] I. M. Gel'fand and G. E. Shilov, *Generalized functions*, Vol. I, Academic Press, 1964.
[17] A. Kilicman and B. Fisher, *Further results on the noncommutative neutrix product of distributions*, Serdica **19** (1993), no. 2-3, 145–152.
[18] B. H. Li, *Non-standard analysis and multiplication of distributions*, Sci. Sinica **21** (1978), no. 5, 561–585.
[19] C. K. Li, *The product of* r^{-k} *and* $\nabla\delta$ *on* R^m, Int. J. Math. Math. Sci. **24** (2000), no. 6, 361–369.
[20] ———, *The products on the unit sphere and even-dimension spaces*, J. Math. Anal. Appl. **305** (2005), no. 1, 97–106.
[21] ———, *An approach for distributional products on* R^m, Integral Transforms Spec. Funct. **16** (2005), no. 2, 139–151.
[22] C. K. Li and B. Fisher, *Example of the neutrix product of distributions on* R^m, Rad. Mat. **6** (1990), no. 1, 129–137.
[23] C. K. Li and E. L. Koh, *The neutrix convolution product in* $Z'(m)$ *and the exchange formula*, Internat. J. Math. Math. Sci. **21** (1998), no. 4, 695–700.
[24] L. Schwartz, *Théorie des distributions à valeurs vectorielles, I, II*, Ann. Inst. Fourier, Grenoble **7,8** (1957/58), 1–141. (1–209).
[25] F. Treves, *Topological vector spaces, distributions and kernels*, Academic Press, 1970.

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