J. Korean Math. Soc. **45** (2008), No. 2, pp. 425–433

THE STRUCTURE OF SEMIPERFECT RINGS

JUNCHEOL HAN

ABSTRACT. Let R be a ring with identity 1_R and let $U(R)$ denote the group of all units of R. A ring R is called *locally finite* if every finite subset in it generates a finite semigroup multiplicatively. In this paper, some results are obtained as follows: (1) for any semilocal (hence semiperfect) ring R, $U(R)$ is a finite (resp. locally finite) group if and only if R is a finite (resp. locally finite) ring; $U(R)$ is a locally finite group if and only if $U(M_n(R))$ is a locally finite group where $M_n(R)$ is the full matrix ring of $n \times n$ matrices over R for any positive integer n; in addition, if $2 = 1_R + 1_R$ is a unit in R, then $U(R)$ is an abelian group if and only if R is a commutative ring; (2) for any semiperfect ring R, if $E(R)$, the set of all idempotents in R, is commuting, then $R/J \cong \oplus_{i=1}^m D_i$ where each D_i is a division ring for some positive integer m and $|E(R)| = 2^m$; in addition, if $2 = 1_R + 1_R$ is a unit in R, then every idempotent is central.

1. Introduction and basic definitions

Let R be a ring with identity 1_R, J be the Jacobson radical of R, $U(R)$ be the group of all units of R and $X(R)$ be the set of all nonzero nonunits of R. Recall that any group G is called *locally finite* if every finitely generated subgroup of G is finite. In ring case, a ring R is called *locally finite* if every finite subset in it generates a finite semigroup multiplicatively (refer [5]). In [5], Lee and Kim have shown that (1) The direct limit of locally finite rings is locally finite [5, Proposition 2.1]; (2) R is a locally finite ring if and only if each finite subset of R generates a finite subring (not necessarily with identity) [5, Theorem 2.2]; (3) if R/I and I are both locally finite for some proper ideal I in R then so is R [5, Theorem 2.2]; (4) a ring R is locally finite if and only if the $n \times n$ full matrix ring over R is locally finite for any positive integer n [5, Corollary 2.3]. Of course, any finite ring is locally finite but the converse is not true by the following example:

Example 1. Let S be a finite ring and let $R_n = M_{2^n}(S)$ be the full matrix ring of $2^n \times 2^n$ matrices over S for any positive integer n. Consider an inclusion

Received August 10, 2006; Revised November 27, 2006.
2000 *Mathematics Subject Classification.* Primary 16L30; Secondary 16U60.
Key words and phrases. locally finite group, locally finite ring, semilocal ring, semiperfect ring, Burnside problem for matrix group, commuting idempotents.
This work was supported for two years by Pusan National University Research Grant.

from R_n to R_{n+1} defined by $A \hookrightarrow \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$. Then the direct limit of R_n is equal to $\bigcup_{n=1}^{\infty} R_n$ and so is locally finite by [5, Proposition 2.1] but is not finite.

A ring R is called *semilocal* if $R/J(R)$ is left artinian where $J(R)$ (or simply J) is the Jacobson radical of R, and hence by the Wedderburn-Artin Structure Theorem for semisimple artinian ring, $R/J \cong \oplus_{i=1}^{m} M_i(D_i)$ where $M_i(D_i)$ is the full matrix ring of all $n_i \times n_i$ matrices over a division ring D_i for each $i = 1, 2, \ldots, m$ and for some positive integer m. A ring R is called *semiperfect* if R is semilocal, and every idempotent in R/J can be lifted to R. In [1], Cohen and Koh have shown that for any compact ring R with identity 1_R, $U(R)$ is a finite group if and only if R is a finite ring; in addition, if $2 = 1_R + 1_R$ is a unit in R, then $U(R)$ is an abelian group if and only if R is a commutative ring. In [7], Nicholson has shown that if R is a semiperfect ring such that $U(R)$ is finite and abelian, then R is finite. In section 2, we will show that for any semilocal (hence semiperfect) ring R with identity 1_R, $U(R)$ is a finite group if and only if R is a finite ring; $U(R)$ is a locally finite group if and only if R is a locally finite ring; in addition, if $2 = 1_R + 1_R$ is a unit in R, then $U(R)$ is an abelian group if and only if R is a commutative ring; $U(R)$ is a locally finite group if and only if $U(M_n(R))$ is a locally finite group where $M_n(R)$ is the full matrix ring of $n \times n$ matrices over R for any positive integer n; if $U(R)$ is a finitely generated abelian group, then R/J is finite. In group theory, the Burnside problem for matrix groups has been considering and Burnside has shown that a torsion group of matrices over a field is locally finite. We will answer partially Burnside problem for matrix group as follows; if F is a locally finite field, $U(M_n(F))$ is a locally finite group for any positive integer n. It is also shown that if R is a semiperfect ring R such that $gx = xg$ for all $g \in U(R)$ and all $x \in X$, then $R/J \cong \oplus_{i=1}^{m} F_i$, where F_i is a field for each $i = 1, 2, \ldots, m$ and for some positive integer m.

In section 3, it is shown that if $E(R)$, the set of all idempotents in a semiperfect R, is commuting, then $R/J \cong \oplus_{i=1}^{m} D_i$, where D_i is a division ring for each $i = 1, 2, \ldots, m$ and for some positive integer m and $|E(R)| = |E(R/J)| = 2^m$; in addition, if $2 = 1_R + 1_R$ is a unit in R, then every idempotent is central.

2. Semilocal ring in which $U(R)$ is finite (resp. locally finite) or abelian (resp. finitely generated abelian)

We begin with the following lemma:

Lemma 2.1. *Let R be a ring with identity 1_R. Then $g \in U(R)$ if and only if $g + J \in U(R/J)$.*

Proof. ($\Rightarrow$) Clear.
($\Leftarrow$) Suppose that $\bar{g} = g + J \in U(R/J)$. Then there exists $\bar{h} = h + J \in R/J$ such that $\bar{g}\bar{h} = \bar{h}\bar{g} = \bar{1}_R$, where $\bar{1}_R$ is the identity of $U(R/J)$. Hence $1_R - gh \in J$. By the definition of J, $1_R + J \subseteq U(R)$ and then gh and $hg \in U(R)$. Hence $g \in U(R)$. $\square$

In general, for a ring R it is not true that

(∗) $U(R)$ is a finite (resp. locally finite) group if and only if R is a finite (resp. locally finite) ring by noting that the group of units in $\mathbb{Z}$, the ring of all integers, is finite (resp. locally finite) but $\mathbb{Z}$ is not finite (resp. locally finite).

On the other hand, the statement (∗) is true for a semilocal ring as follows:

Theorem 2.2. *Let R be a semilocal ring. Then $U(R)$ is a finite group if and only if R is a finite ring.*

Proof. (⇒) Suppose that $U(R)$ is a finite group. Since R is a semilocal ring, by the Wedderburn-Artin Structure Theorem for semisimple artinian ring, $R/J \cong \oplus_{i=1}^{m} M_i(D_i)$, where $M_i(D_i)$ is the full matrix ring of all $n_i \times n_i$ matrices over a division ring D_i for each $i = 1, 2, \ldots, m$ and for some positive integer m. Since $U(R)$ is a finite group, clearly $U(R/J)$ is a finite group by Lemma 2.1. Then D_i is finite for each $i = 1, 2, \ldots, n$. Indeed, suppose that D_i is infinite for some i. For simplicity of notation, we can assume that $R/J = \oplus_{i=1}^{n} M_i(D_i)$. Consider $U(R/J)_i^* = \oplus_{i=1}^{n} H_i$, where $H_j = \{e_j\}$, (e_j is the identity of $M_j(D_j)$) for $j \neq i$ and $H_i = \{(a_{ij}) \in M_i(D_i) : a_{11} \in D_i \setminus \{0_i\}, a_{ss} = 1_i (n_i \geq s \geq 2), a_{st} = 0_i (n_i \geq s, t \geq 2, s \neq t)\}$, where 0_i(resp. 1_i) is zero (resp. identity) of D_i. Then $U(R/J)_i^*$ is a subgroup of $U(R/J)$ and $|U(R/J)_i^*| = |D_i \setminus \{0_i\}|$ is infinite, which contradicts to the fact $U(R/J)$ is a finite group. Hence D_i is finite for each $i = 1, 2, \ldots, n$, and so R/J is finite. Since $1_R + J \subseteq U(R)$ and $U(R)$ is a finite group, J is finite. Hence $|R| = |J| \cdot |R/J|$ is finite.
(⇐) Clear. □

Lemma 2.3. *Let R be a ring. If $U(R)$ is a locally finite group, then $U(R/J)$ is a locally finite group.*

Proof. Clear. □

In [5], the following theorem has been proved:

Theorem 2.4. *Let R be a ring. Then R is a locally finite ring if and only if R/J and J are locally finite rings.*

Proof. Refer [5, Theorem 2.2]. □

In [5], the following corollary has also been proved:

Corollary 2.5. *Let $R = M_n(S)$ be the full matrix ring of all $n \times n$ matrices over a ring S for any positive integer n. Then S is a locally finite ring if and only if R is a locally finite ring.*

Proof. Refer [5, Corollary 2.3]. □

We can have the following question:

Question 1. For any ring R with identity and for any positive integer n, is $U(R)$ a locally finite group if and only if $U(M_n(R))$, the group of all nonsingular matrices of $M_n(R)$, is a locally finite group?

The answer to the above question is negative by the following example.

Example 2. Let $\mathbb{Z}$ be the ring of all integers. Then $U(\mathbb{Z}) = \{1, -1\}$ is a locally finite group. But $U(M_2(\mathbb{Z}))$ is not a locally finite group. Indeed, consider a cyclic subgroup $H = \langle \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \rangle$ generated by $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ for some $a (\neq 0) \in \mathbb{Z}$. Then $H = \{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)^k = \left(\begin{smallmatrix} 1 & ka \\ 0 & 1 \end{smallmatrix}\right) : \forall k \in \mathbb{Z}\}$ is infinite. Hence $U(M_2(\mathbb{Z}))$ is not a locally finite group.

On the other hand, the above question may be affirmative for any division ring by the following argument:

Proposition 2.6. *Let $R = M_n(D)$ be the full matrix ring of all $n \times n$ matrices over a division ring D for any positive integer n. Then the following are equivalent:*

(1) *$U(D) = D \setminus \{0\}$ is a locally finite group;*
(2) *D is a locally finite ring;*
(3) *R is a locally finite ring;*
(4) *$U(R)$ is a locally finite group.*

Proof. (1) $\Leftrightarrow$ (2). Clear.
(2) $\Leftrightarrow$ (3). It follows from Corollary 2.5.
(3) $\Rightarrow$ (4). Clear.
(4) $\Rightarrow$ (3). Suppose that $U(R)$ is a locally finite group. In order to show that R is a locally finite ring, it is enough to show that D is a locally finite ring by Corollary 2.5. Assume that D is not a locally finite ring. Then $D \setminus \{0\}$ is not a locally finite group. Thus there exists a finite subset $\{a_1, a_2, \ldots, a_s\}$ of $D \setminus \{0\}$ such that the subgroup $\langle a_1, a_2, \ldots, a_s \rangle$ generated by $\{a_1, a_2, \ldots, a_s\}$ of $D \setminus \{0\}$ is not a finite subgroup of $D \setminus \{0\}$. Consider a subgroup $\langle A_1, A_2, \ldots, A_s \rangle$ of $U(R)$ generated by $\{A_1, A_2, \ldots, A_s\}$, where $A_k = (p(k)_{ij}) \in R$ with $p(k)_{11} = a_k$, $p(k)_{ii} = 1$ for all $i \geq 2$ and $p(k)_{ij} = 0$ for all $i, j \geq 1$ $(i \neq j)$ for all $k = 1, \ldots, s$. Since $U(R)$ is a locally finite group, $\langle A_1, A_2, \ldots, A_s \rangle$ is a finite subgroup of $U(R)$ and $\langle A_1, A_2, \ldots, A_s \rangle$ is isomorphic to $\langle a_1, a_2, \ldots, a_s \rangle$ as groups, which is a contradiction. Therefore R is a locally finite ring. □

Corollary 2.7. *Let R be a semilocal ring. If $U(R)$ is a locally finite group, then R/J is a locally finite ring.*

Proof. Suppose that $U(R)$ is a locally finite group. Since R is a semilocal ring, by the Wedderburn-Artin Structure Theorem for semisimple artinian ring, $R/J \cong \oplus_{i=1}^m M_i(D_i)$, where $M_i(D_i)$ is the full matrix ring of all $n_i \times n_i$ matrices over a division ring D_i for each $i = 1, 2, \ldots, m$ and for some positive integer m. Since a direct sum of locally finite rings is locally finite, it is enough to show that $M_i(D_i)$ is a locally finite ring for each i. Since $U(R)$ is a locally finite group, $U(R/J)$ is a locally finite group by Lemma 2.3, and then $U(M_i(D_i))$ is a locally finite group for each i. Hence $M_i(D_i)$ is a locally finite ring for each i by Proposition 2.6, and so R/J is a locally finite ring. □

Remark 1. Note that any locally finite group is torsion but the converse is not true. In group theory, the Burnside problem for matrix groups has been considered. In [4, Theorem 2.3.5], Burnside has shown that a torsion group of matrices over a field is locally finite. By Proposition 2.6, we have answered partially the Burnside problem for matrix group as follows; for any locally finite field F, $U(M_n(F))$, the group of $n \times n$ invertible matrices over a field F, is locally finite.

Lemma 2.8. *Let R be a semilocal ring. If $U(R)$ is an abelian group, then $R/J \cong \oplus_{i=1}^{m} F_i$, where F_i is a field for each $i = 1, 2, \ldots, m$ and for some positive integer m.*

Proof. Since R is a semilocal ring, by the Wedderburn-Artin Structure Theorem for semisimple artinian ring, $R/J \cong \oplus_{i=1}^{m} M_i(D_i)$, where $M_i(D_i)$ is the full matrix ring of all $n_i \times n_i$ matrices over a division ring D_i for each $i = 1, 2, \ldots, m$ and for some positive integer m. Since $U(R)$ is an abelian group, $U(R/J)$ is also an abelian group. Since $R/J \cong \oplus_{i=1}^{m} M_i(D_i)$ and $U(R/J)$ is an abelian group, $U(M_i(D_i))$ is an abelian group for each $i = 1, 2, \ldots, m$, and so D_i must be a field for each $i = 1, 2, \ldots, m$. Hence we have the result. □

Theorem 2.9. *Let R be a semilocal ring such that $2 = 1_R + 1_R$ is a unit in R. Then $U(R)$ is an abelian group if and only if R is a commutative ring.*

Proof. ($\Rightarrow$) Suppose that $U(R)$ is an abelian group. Then by Lemma 2.8, $R/J \cong \oplus_{i=1}^{m} F_i$, where F_i is a field for each $i = 1, 2, \ldots, m$ and for some positive integer m. Since $2 = 1_R + 1_R$ is a unit in R, R is a commutative ring by [3, Lemma 4].
($\Leftarrow$) It is clear. □

Note that in Theorem 2.9, the condition that $2 = 1_R + 1_R$ is a unit in R is essential by the following example:

Example 3. Let $R = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) : a, b, c \in \mathbb{Z}_2\}$, where $\mathbb{Z}_2$ is the ring of integers modulo 2. Then R is a noncommutative semilocal ring with identity but $U(R) = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\}$ is an abelian group.

Proposition 2.10. *Let R be a semilocal ring with identity 1_R. If $U(R)$ is a finitely generated abelian group, then R/J is finite.*

Proof. Since $U(R)$ is an abelian group, then by Lemma 2.8, $R/J \cong \oplus_{i=1}^{m} F_i$, where F_i is a field for each $i = 1, 2, \ldots, m$ and for some positive integer m. Since $U(R)$ is finitely generated, the group of units of each field F_i is finitely generated (if and only if F_i is finite in the proof of Theorem 2 in [3]). Hence R/J is finite. □

Corollary 2.11. *Let R be a semilocal ring with identity 1_R. If $U(R)$ is a finitely generated abelian group, then J is finite if and only if R is finite.*

Proof. If follows from the Proposition 2.10. □

Remark 2. In [7, Proposition 3], Nicholson has shown that for a semiperfect ring R with a finitely generated abelian group $U(R)$, if J is a nil ideal in R (in this case, R is a semiperfect ring if and only if R is a semilocal ring), then R is finite.

Proposition 2.12. *Let R be a semilocal ring with identity 1_R. If $gx = xg$ for all $g \in U(R)$ and all $x \in X$, then $R/J \cong \oplus_{i=1}^{m} F_i$, where F_i is a field for each $i = 1, 2, \dots, m$ and for some positive integer m.*

Proof. By The Weddedrburn-Artin Structure Theorem for semisimple artinian ring, $R/J \cong \oplus_{i=1}^{m} M_i(D_i)$, where $M_i(D_i)$ is the full matrix ring of all $n_i \times n_i$ matrices over a division ring D_i for each $i = 1, 2, \dots, m$ and for some positive integer m. Let $\bar{U}$ (resp. $\bar{X}$) be the group of all units (resp. the set of all nonzero, nonunits) in R/J. Since $gx = xg$ for all $g \in U(R)$ and all $x \in X$,

$$(**) \qquad \bar{g}\bar{x} = \bar{x}\bar{g} \text{ for all } \bar{g} \in \bar{U} \text{ and all } \bar{x} \in \bar{X}.$$

We can easily check that if $n_i \geq 2$ for some i, then $M_i(D_i)$ does not satisfy (**). Hence $R/J \cong \oplus_{i=1}^{m} D_i$. Next, we will show that D_i is a field for all i. Assume that D_i is not a field for some i. Then there exists $a, b \in D_i$ such that $ab \neq ba$. Choose $\bar{a} = (a_1, \dots, a_i, \dots, a_m)$ and $\bar{b} = (b_1, \dots, b_i, \dots, b_m)$ with $a_i = a, a_j = 0_j$ for all $j \neq i$ and $b_i = b, b_j = 1_j$, where 0_j (resp. 1_j) is the zero (resp. the identity) of D_j. Then $\bar{a} \in \bar{X}$ and $\bar{b} \in \bar{U}$ and $\bar{a}\bar{b} = (0_1, \dots, ab, \dots, 0_m) \neq (0_1, \dots, ba, \dots, 0_m) = \bar{b}\bar{a}$, which contradicts to (**). Hence we have the result. □

3. Commuting idempotents in a semiperfect ring

Recall that an element $e \in R$ is called an *idempotent* if $e^2 = e$ and an element $g \in R$ is called an *involution* if $g^2 = 1$. Let $E(R)$ (resp. $V(R)$) be the set of all idempotents (resp. involutions) in R. Note that if $2 = 1_R + 1_R$ is a unit in R, then the mapping $e \to 1_R - 2e$ is a bijection from $E(R)$ to $V(R)$.

We begin this section with the following lemma:

Lemma 3.1. *Let R be a semiperfect ring with identity 1_R such that $2 = 1_R + 1_R$ is a unit in R. Then every involution in R/J can be lifted to R.*

Proof. Let $\bar{v} = v + J$ be an arbitrary involution of R/J, i.e., $v^2 + J = 1_R + J$. Let $e = \frac{1_R - v}{2}$. Then $\bar{e} = e + J$ is an idempotent of R/J. Since R is semiperfect, then $\bar{e}$ can be lifted to R, i.e., there is an idempotent $f \in R$ such that $f + J = e + J$. Let $v_0 = 1_R - 2f$. Then v_0 is an involution of R and $v_0 + J = (1_R - 2e) + J = v + J$. Hence $\bar{v} = v + J$ in R/J can be lifted to R. □

Lemma 3.2. *Let R be a ring with identity 1_R and let $V(R)$ be the set of all involutions of R. If $V(R)$ is finite, then $\langle V(R)\rangle$, the group generated by $V(R)$, is finite.*

Proof. Let $m = |V(R)|$. If $m = 1$ or 2, then clearly $\langle V(R)\rangle$ is finite. Suppose that $m \geq 3$. For all $g \in \langle V(R)\rangle$, consider $l(g)$, the length of g, which is the smallest positive integer k such that $g = v_1v_2\cdots v_k$ for some $v_1, v_2, \ldots, v_k \in V(R)$. We will show that $\langle V(R)\rangle = V(R)^{m-1}$. Assume that there exists $g \in \langle V(R)\rangle$ such that $l(g) \geq m$. Let $n = l(g)$, i.e., $g = v_1v_2\cdots v_n$ for some $v_1, v_2, \ldots, v_n \in V(R) \setminus \{1_R\}$. Since $n \geq m = |V(R)|$ and $v_1, v_2, \ldots, v_n \in V(R) \setminus \{1_R\}$, there exist $i, j \in \mathbb{Z}^+$ such that $v_i = v_{i+j}$ $(n \geq i + j > i \geq 1)$. Let $h = v_iv_{i+1}\cdots v_{i+j} \in \langle V(R)\rangle$. Then $h \in \langle V(R)\rangle$ and $l(h) = j + 1$ since $l(g) = n$. On the other hand, since $h = (v_iv_{i+1}v_i)(v_iv_{i+2}v_i)\cdots(v_iv_{i+j-1}v_{i+j}) = (v_iv_{i+1}v_i)(v_iv_{i+2}v_i)\cdots(v_iv_{i+j-1}v_i)$ and $v_iv_{i+1}v_i$, $v_iv_{i+2}v_i, \ldots, v_iv_{i+j-1}v_i \in V(R)$, $j - 1 \geq l(h)$, a contradiction. Hence for all $g \in \langle V(R)\rangle$, $m - 1 \geq l(g)$, and so $\langle V(R)\rangle = V(R)^{m-1}$. Consequently, $\langle V(R)\rangle$ is finite. □

Theorem 3.3. *Let R be a semiperfect ring with identity 1_R and let $E(R)$ be the set of all idempotents of R. If $E(R)$ is commuting, i.e., $e_1e_2 = e_2e_1$ for all $e_1, e_2 \in E(R)$, then $R/J \cong \oplus_{i=1}^m D_i$, where D_i is a division ring for all $i = 1, 2, \ldots, m$ and for some positive integer m and $|E(R)| = |E(R/J)| = 2^m$.*

Proof. Since $E(R)$ is commuting, $E(R/J)$, the set of all idempotents of R/J, is also commuting. Indeed, for all $e_1 + J, e_2 + J \in E(R/J)$, there exist $f_1, f_2 \in E(R)$ such that $f_1 + J = e_1 + J, f_2 + J = e_2 + J$ since R is semiperfect. Thus $(e_1 + J)(e_2 + J) = (f_1 + J)(f_2 + J) = f_1f_2 + J = f_2f_1 + J = (f_2 + J)(f_1 + J) = (e_2 + J)(e_1 + J)$. By the Wedderburn-Artin Structure Theorem for semisimple artinian ring, $R/J \cong \oplus_{i=1}^m M_i(D_i)$, where $M_i(D_i)$ is the full matrix ring of all $n_i \times n_i$ matrices over a division ring D_i for each $i = 1, 2, \ldots, m$ and for some positive integer m. We will show that $n_i = 1$ for each $i = 1, 2, \ldots, m$. Assume that $n_i \geq 2$ for some i. For all $a, b (a \neq b) \in D_i$, consider two idempotents

$$A = \begin{pmatrix} 1 & a & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & b & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \in M_i(D_i).$$

Then $A, B \in E(M_i(D_i))$ and $AB = B \neq A = BA$, a contradiction to the fact $E(R/J)$ is commuting. Hence $R/J \cong \oplus_{i=1}^m D_i$, where D_i is a division ring for each $i = 1, 2, \ldots, m$ and for some positive integer m and so $|E(R/J)| = 2^m$. Since every idempotents of R/J can be lifted to R, $|E(R)| \geq 2^m$. Next, we will show that $|E(R)| = |E(R/J)|$. Suppose that there exist two idempotents e and f of R $(e \neq f)$ such that $e + J = f + J$. Then $e - f \in J$. Since $ef = fe$, $(e - f)^4 = (e - f)^2$, and so $(e - f)^2 \in E(R) \cap J = (0)$. Hence $(e - f)^2 = 0$, which implies that $e = f$, a contradiction. Therefore, $|E(R)| = |E(R/J)| = 2^m$ for some positive integer m. □

Remark 3. From Theorem 3.3, we note that if R is a semiperfect ring with identity 1_R such that $E(R)$ is commuting, then R/J is a finite product of division rings, and hence R is a basic ring by [6, Proposition 25.10].

Corollary 3.4. *Let R be a semiperfect ring with identity 1_R such that $2 = 1_R + 1_R$ is a unit in R, and $V(R)$ be the set of all involutions of R. If $V(R)$ is commuting, i.e., $v_1v_2 = v_2v_1$ for all $v_1, v_2 \in V(R)$, then $R/J \cong \oplus_{i=1}^m D_i$, where D_i is a division ring for all $i = 1, 2, \ldots, m$ and for some positive integer m and $|V(R)| = |V(R/J)| = 2^m$.*

Proof. Since 2 is a unit in R and $V(R)$ is commuting, there exists a bijection from $E(R)$ to $V(R)$ and $E(R)$ is commuting. Hence the result follows from Lemma 3.1 and Theorem 3.3. □

Corollary 3.5. *Let R be a semiperfect ring with identity 1_R such that $2 = 1_R + 1_R$ is a unit in R and $V(R)$ is commuting. If $U(R)$ is a simple group, then R is a finite commutative ring.*

Proof. Since $2 = 1_R + 1_R$ is a unit in R and $V(R)$ is commuting, $V(R)$ is a finite abelian group by Corollary 3.4. Since $U(R)$ is a simple group and $V(R)(\neq (\{1_R\})$ is a normal subgroup of G, $V(R) = U(R)$. Hence R is a finite commutative ring by Theorem 2.2 and Theorem 2.9. □

Corollary 3.6. *Let R be a semiperfect ring with identity 1_R such that $2 = 1_R + 1_R$ is a unit in R and $V(R)$ is commuting. If $U(R)$ is a simple group and J is a nil ideal of R, then $J = (0)$, and so $R \cong \oplus_{i=1}^m F_i$, where F_i is a finite field for all $i = 1, 2, \ldots, m$ and for some positive integer m by Corollary 3.5.*

Proof. Assume that there exists $j(\neq 0) \in J$. Since J is nil ideal, $j^n = 0$ and $j^{n-1} \neq 0$ for some positive integer n. Since $U(R)$ is a simple group and $1_R + J(\neq \{1_R\})$ is a normal subgroup of $U(R)$, $U(R) = 1_R + J = V(R)$. Thus $(1_R + j)^2 = 1_R$, and so $2j = -j^2$. Since $2 = 1_R + 1_R$ is a unit in R, we have $j^{n-1} = 0$, a contradiction. Hence $J = (0)$, and so $R \cong \oplus_{i=1}^m F_i$, where F_i is a finite field for all $i = 1, 2, \ldots, m$ and for some positive integer m. □

Remark 4. From Corollary 3.6, we note that for a left artinian ring R with identity 1_R such that $2 = 1_R + 1_R$ is a unit in R and $V(R)$ is commuting if $U(R)$ is a simple group, then R is a finite semisimple artinian ring.

Corollary 3.7. *Let R be a semiperfect ring with identity 1_R such that $2 = 1_R + 1_R$ is a unit in R. If $E(R)$ is commuting, then every idempotent of R is central.*

Proof. Since $E(R)$ is commuting, $E(R) = E(R)^2$, where $E(R)^2 = \{ab|\forall a, b \in E(R)\}$. By [2, Lemma 2.1 and Proposition 2.2], $ea = ae$ for all elements $e \in E(R)$ and $a \in J$. Since $E(R)$ is commuting, by Theorem 2.3, $R/J \cong \oplus_{i=1}^m D_i$, where D_i is a division ring for all $i = 1, 2, \ldots, m$ and for some positive integer m, and so every idempotent of R/J is central and then every idempotent of R is central by [2, Lemma 2.3]. □

Acknowledgements. The author thanks Professor J. Park at Pusan National University for reading this paper and for kind comments.

References

[1] J. Cohen and K. Koh, *The group of units in a compact ring*, J. Pure Appl. Algebra **54** (1988), no. 2-3, 167–179.

[2] ———, *The structure of compact rings*, J. Pure Appl. Algebra **77** (1992), no. 2, 117–129.

[3] K. E. Eldridge and I. Fischer, D.C.C. *rings with a cyclic group of units*, Duke Math. J. **34** (1967), 243–248.

[4] I. N. Herstein, *Noncommutative Rings*, The Mathematical Association of America, John Wisley and Sons, Inc., 1976.

[5] C. Huh, N. K. Kim, and Y. Lee, *Examples of strongly π-regular rings*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 195–210.

[6] T. Y. Lam, *A First Course in Noncommutative Rings*, Graduate Texts in Mathematics, 131. Springer-Verlag, New York, 1991.

[7] W. K. Nicholson, *Semiperfect rings with abelian group of units*, Pacific J. Math. **49** (1973), 191–198.

Department of Mathematics Education
Pusan National University
Pusan 609-735, Korea
E-mail address: jchan@pusan.ac.kr