

## EXISTENCE OF PERIODIC SOLUTIONS OF A HIGHER ORDER DIFFERENCE SYSTEM

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ABSTRACT. By using critical point theorem, we study a higher order difference system, and obtain some new sufficient conditions ensuring the existence of periodic solutions for such a system.

### 1. Introduction

In the last decade, there has been much progress on the qualitative properties of difference equations, which included results on stability and attractivity [6, 12, 17, 19], oscillation and other topics [1, 11, 15]. However, results on periodic solutions of difference equations are very scarce in the literature, see [2, 5, 20]. On the other hand, there have been many approaches to study periodic solutions of differential equations, such as critical point theory (which includes the minimax theory, the Kaplan-Yorke method and Morse theory), fixed point theory, coincidence theory, and so on, see for example [7, 9, 10, 13, 14]. Among these approaches, critical point theory is an important tool to deal with such problems. The main idea of these papers is constructing suitable variational structure, such that the critical points of the functional correspond to the periodic solutions of the differential equations. It is natural for us to think that critical point theory may be applied to prove the existence of periodic solutions of difference equations. However, there are, at present, only a few papers dealing with this problem, see, for example, [7, 8, 21, 22]. Nevertheless these papers consider only the second order difference equations except [22] which discuss the subquadratic discrete Hamiltonian system.

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In this paper, we first consider the following nonlinear higher order difference system

$$(1.1) \quad \sum_{i=0}^k a_i (X_{n-i} + X_{n+i}) + f(n, X_n) = 0, \quad n \in \mathbf{Z}, \quad k \in \mathbf{N},$$

where  $\mathbf{N}$  and  $\mathbf{Z}$  are the sets of all positive integers and integers respectively,  $m \in \mathbf{N}$ ,  $f = (f_1, f_2, \dots, f_m)^T \in C(\mathbf{R} \times \mathbf{R}^m, \mathbf{R}^m)$ ,  $\mathbf{R}$  is the set of all real numbers, and there exists a positive integer  $M$  such that for any  $(t, Z) \in \mathbf{R} \times \mathbf{R}^m$ ,  $f(t + M, Z) = f(t, Z)$ .

Lately, we study the nonlinear higher order difference equation

$$(1.2) \quad \sum_{i=1}^k r_i \Delta^{2i} x_{n-i} + f(n, x_n) = 0, \quad n \in \mathbf{Z}, \quad k \in \mathbf{N},$$

where  $\Delta$  is the forward difference operator defined by  $\Delta x_n = x_{n+1} - x_n$ , and  $\Delta^s = \Delta(\Delta^{s-1})$  for  $s = 2, 3, \dots$ ;  $f \in C(\mathbf{R} \times \mathbf{R}, \mathbf{R})$ , and there exists a positive integer  $M$  such that for any  $(t, z) \in \mathbf{R} \times \mathbf{R}$ ,  $f(t + M, z) = f(t, z)$ . (1.2) can be seen a special form of system (1.1) with  $m = 1$ .

Let  $p$  be a positive integer, as usual, a solution  $\{X_n\}$  of (1.1) is said to be periodic of period  $p$  if  $X_{p+i} = X_i$ ,  $i \in \mathbf{Z}$ .

The main purpose of this paper is to study the existence of periodic solutions of a higher order difference system (1.1) and equation (1.2), our results improve the corresponding ones in [7].

Throughout this paper, for  $a, b \in \mathbf{Z}$ , we define  $\mathbf{Z}(a) := \{a, a+1, \dots\}$ ,  $\mathbf{Z}(a, b) := \{a, a+1, \dots, b\}$  when  $a \leq b$ . On the other hand, we suppose that there exists a continuously differentiable function  $F(t, Z) \in C^1(\mathbf{R} \times \mathbf{R}^m, \mathbf{R})$ , such that  $\nabla_z F(t, Z) = f(t, Z)$  for any  $(t, Z) \in \mathbf{R} \times \mathbf{R}^m$ , where  $\nabla_z F(t, Z)$  denotes the gradient of  $F(t, Z)$  in  $Z$ . Moreover, for all  $n \in \mathbf{N}$ ,  $|\cdot|$  will denote the Euclidean norm in  $\mathbf{R}^n$  defined by

$$|X| = \left( \sum_{i=1}^n X_i^2 \right)^{\frac{1}{2}} \quad \text{for all } X = (X_1, X_2, \dots, X_n) \in \mathbf{R}^n.$$

For the existence of  $M$ -periodic solutions of system (1.1), we obtain the following results.

**Theorem 1.1.** *Suppose that  $M \geq 2k + 1$ ,  $a_0 + \sum_{s=1}^k |a_s| \leq 0$ , there exists*

*$i \in \mathbf{Z}(0, M - 1)$  such that  $\sum_{s=0}^k \cos \frac{2s\pi}{M} i = 0$ , and  $F(t, Z)$  satisfies*

- (f<sub>1</sub>) *there exists a positive integer  $M$ , such that  $F(t + M, Z) = F(t, Z)$ ,  $\forall (t, Z) \in \mathbf{R} \times \mathbf{R}^m$ , and  $F(t, Z) \geq 0$ ;*
- (f<sub>2</sub>)  *$F(t, Z) = o(|Z|^2)$ , if  $Z \rightarrow 0$ ;*

(f<sub>3</sub>) there are constants  $\rho > 0, \gamma > 0, \beta \in (-a_0 + \sum_{s=1}^k |a_s|, +\infty)$ , such that

$$F(n, Z) \geq \beta|Z|^2 - \gamma, \quad n \in \mathbf{N}, |Z| \geq \rho.$$

Then system (1.1) possesses at least three  $M$ -periodic solutions.

**Theorem 1.2.** Suppose that  $M \geq 2k+1, a_0 + \sum_{s=1}^k |a_s| < 0$ , and  $F(t, Z)$  satisfies (f<sub>1</sub>) to (f<sub>3</sub>). Then system (1.1) possesses at least one  $M$ -periodic nontrivial solution.

**Theorem 1.3.** Suppose  $M \geq 2k + 1, a_0 + \sum_{s=1}^k |a_s| < 0$ , and  $F(t, Z)$  satisfies (f<sub>1</sub>) and

(f<sub>4</sub>) there are constants  $\delta > 0$  and  $\alpha_1 \in (0, -a_0 - \sum_{s=1}^k |a_s|)$ , such that

$$F(n, Z) \leq \alpha_1|Z|^2, \quad n \in \mathbf{N}, Z \in \mathbf{R}^m, |Z| \leq \delta;$$

(f<sub>5</sub>) there are constants  $R_1 > 0, \alpha_2 > 2$ , such that

$$Zf(n, Z) \geq \alpha_2 F(n, Z) > 0, \quad |Z| \geq R_1;$$

(f<sub>6</sub>)  $F(t, Z) = -F(t, -Z), \quad \forall (t, Z) \in \mathbf{R} \times \mathbf{R}^m$ .

Then system (1.1) possesses infinite  $M$ -periodic solutions.

**Theorem 1.4.** Suppose that  $M \geq 2k + 1$ , there exist  $i, j \in \mathbf{Z}(0, M - 1)$  such that  $\sum_{s=0}^k a_s \cos \frac{2s\pi}{M} i \leq 0 \leq \sum_{s=0}^k a_s \cos \frac{2s\pi}{M} j$ , and  $F(t, Z)$  satisfies (f<sub>1</sub>), (f<sub>2</sub>), (f<sub>3</sub>). Then system (1.1) possesses at least one  $M$ -periodic nontrivial solution.

For the existence of  $M$ -periodic solutions of equation (1.2), we have

**Theorem 1.5.** Suppose that  $M \geq 2k + 1, r_{2t+1} \geq 4r_{2t+2}$  when  $M$  is even, or  $r_{2t+1} \geq 2(1 + \cos \frac{\pi}{M})r_{2t+2}$  when  $M$  is odd,  $t = 0, 1, \dots, [\frac{k}{2}] - 1$  and  $r_k > 0$ , and that  $f(t, z)$  satisfies

- (g<sub>1</sub>)  $f(t, z) \in C(\mathbf{R} \times \mathbf{R}, \mathbf{R})$ , there exists a positive integer  $M$  such that  $f(t + M, z) = f(t, z), \quad \forall (t, z) \in \mathbf{R} \times \mathbf{R}$ ;
- (g<sub>2</sub>)  $\int_0^z f(t, s)ds \geq 0, \forall z \in \mathbf{R}$ , and if  $z \rightarrow 0, f(t, z) = o(z)$ ;
- (g<sub>3</sub>) There are constants  $R > 0, \beta' > 2$  such that

$$zf(t, z) \geq \beta' \int_0^z f(t, s)ds > 0, \quad \forall |z| \geq R.$$

Then equation (1.2) possesses at least three  $M$ -periodic solutions.

*Remark 1.1.* The result of [7] is the special case of Theorem 1.5 with  $k = 1, r_1 = 1$ .

## 2. Variational structure

To apply critical point theory to study the existence of periodic solutions of system (1.1) and equation (1.2), we shall construct suitable variational structure. At first, we shall state some basic notations and lemmas which will be used in the proofs of our main results.

Let  $\mathbf{S}$  be the set of sequences

$$X = (\dots, X_{-n}, X_{-n+1}, \dots, X_{-1}, X_0, X_1, X_2, \dots, X_n, \dots) = \{X_n\}_{n=-\infty}^{+\infty},$$

where  $X_n = (X_{n,1}, X_{n,2}, \dots, X_{n,m})^T \in \mathbf{R}^m$ ,  $m$  is a given positive integer.

For any  $X, Y \in \mathbf{S}$ ,  $a, b \in \mathbf{R}$ ,  $aX + bY$  is defined by

$$aX + bY := \{aX_n + bY_n\}_{n=-\infty}^{+\infty},$$

then  $\mathbf{S}$  is a vector space.

For any given positive integer  $M$ ,  $E_M$  is defined as a subspace of  $\mathbf{S}$  by

$$E_M = \{X = \{X_n\} \in \mathbf{S} | X_{n+M} = X_n, n \in \mathbf{Z}\}.$$

$E_M$  can be equipped with inner product  $\langle \cdot, \cdot \rangle_{E_M}$  and norm  $\|\cdot\|_{E_M}$  as follows:

$$(2.1) \quad \langle X, Y \rangle_M = \sum_{j=1}^M X_j \cdot Y_j, \quad \forall X = \{X_n\} \in E_M, Y = \{Y_n\} \in E_M,$$

and

$$(2.2) \quad \|X\|_{E_M} = \left( \sum_{j=1}^M X_j^2 \right)^{\frac{1}{2}}, \quad \forall X \in E_M,$$

where  $|\cdot|$  denotes the Euclidean norm in  $\mathbf{R}^m$ , and  $X_n \cdot Y_n$  denotes the usual scalar product in  $\mathbf{R}^m$ .

Define a linear map  $L : E_M \rightarrow \mathbf{R}^{mM}$  by

$$(2.3) \quad LX = (X_{1,1}, \dots, X_{M,1}, X_{1,2}, \dots, X_{M,2}, X_{1,m}, \dots, X_{M,m})^T,$$

where  $X = \{X_n\}$ ,  $X_i = (X_{i,1}, X_{i,2}, \dots, X_{i,m})^T$ ,  $i \in \mathbf{Z}(1, M)$ . It is easy to see that the map  $L$  defined in (2.3) is a linear homeomorphism with  $\|X\|_{E_M} = |LX|$ ,  $(E_M, \langle \cdot, \cdot \rangle_{E_M})$  is a finite dimensional Hilbert space, which can be identified with  $\mathbf{R}^{mM}$ .

For system (1.1), we consider the functional  $I$  defined on  $E_M$  by

$$(2.4) \quad I(X) = -\frac{1}{2} \sum_{n=1}^M \sum_{i=0}^k a_i (X_{n-i} + X_{n+i}) \cdot X_n - \sum_{n=1}^M F(n, X_n),$$

where  $X_{n+M} = X_n$ ,  $n \in \mathbf{Z}$ ,  $\forall X \in E_M$ .

Since  $E_M$  is linearly homeomorphic to  $\mathbf{R}^{mM}$ , by the continuity of  $f(t, Z)$ ,  $I$  can be viewed as continuously differentiable functional defined on a finite

dimensional Hilbert space. That is  $I \in C^1(E_M, \mathbf{R})$ . Furthermore,  $I'(X) = 0$  if and only if

$$\frac{\partial I(X)}{\partial X_{n,l}} = 0, \quad l \in \mathbf{Z}(1, m), n \in \mathbf{Z}(1, M).$$

If we define  $X_0 := X_M$ , then

$$\frac{\partial I(X)}{\partial X_{n,l}} = - \left[ \sum_{i=0}^k a_i (X_{n-i,l} + X_{n+i,l}) X_{n,l} + f_l(n, X_n) \right],$$

$$l \in \mathbf{Z}(1, m), n \in \mathbf{Z}(1, M).$$

Therefore,  $X \in E_M$  is a critical point of  $I$ , that is,  $I'(X) = 0$  if and only if

$$\sum_{i=0}^k a_i (X_{n-i,l} + X_{n+i,l}) + f_l(n, X_n) = 0,$$

$$\forall l \in \mathbf{Z}(1, m), n \in \mathbf{Z}(1, M).$$

That is,

$$\sum_{i=0}^k a_i (X_{n-i} + X_{n+i}) + f(n, X_n) = 0, \quad n \in \mathbf{Z}(1, M).$$

On the other hand,  $\{X_n\}$  is  $M$ -periodic in  $n$ , and  $f(t, Z)$  is  $M$ -periodic in  $t$ , hence,  $X \in E_M$  is a critical point of  $I$  if and only if

$$\sum_{i=0}^k a_i (X_{n-i} + X_{n+i}) + f(n, X_n) = 0$$

for any  $n \in \mathbf{Z}$ . Thus, we reduce the problem of finding  $M$ -periodic solutions of (1.1) to that of seeking critical points of the functional  $I$  in  $E_M$ .

Due to the identification of  $E_M$  with  $\mathbf{R}^{mM}$ ,  $I(X)$  is rewritten as

$$(2.5) \quad I(X) = \frac{1}{2} (LX)^T A (LX) - \sum_{n=1}^M F(n, X_n),$$

where  $X = X_n \in E_M$ ,  $X_i = (X_{i,1}, X_{i,2}, \dots, X_{i,m})^T, i \in \mathbf{Z}(1, M)$ .

$$A = \begin{pmatrix} B & & & 0 \\ & B & & \\ & & \ddots & \\ 0 & & & B \end{pmatrix}_{mM \times mM},$$

$$-B = \begin{pmatrix} 2a_0 & a_1 & a_2 & \cdots & a_{k-1} & a_k & 0 & 0 & \cdots & 0 & a_k & a_{k-1} & \cdots & a_2 & a_1 \\ a_1 & 2a_0 & a_1 & \cdots & a_{k-2} & a_{k-1} & a_k & 0 & \cdots & 0 & 0 & a_k & \cdots & a_3 & a_2 \\ a_2 & a_1 & 2a_0 & \cdots & a_{k-3} & a_{k-2} & a_{k-1} & a_k & \cdots & 0 & 0 & 0 & \cdots & a_4 & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \cdots & 0 & 0 & 0 & 0 & \cdots & a_k & a_{k-1} & a_{k-2} & \cdots & 2a_0 & a_1 \\ a_1 & a_2 & a_3 & \cdots & a_k & 0 & 0 & 0 & \cdots & a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_1 & 2a_0 \end{pmatrix}_{M \times M}$$

Assume that the eigenvalues of  $B$  are  $\lambda_0, \lambda_1, \dots, \lambda_{M-1}$  respectively, and  $B$  is a circulant matrix [3] denoted by

$$B \stackrel{def}{=} \text{Circ}\{-2a_0, -a_1, -a_2, \dots, -a_k, 0, \dots, 0, -a_k, -a_{k-1}, \dots, -a_2, -a_1\}.$$

By [3], the eigenvalues of  $B$  are

$$\begin{aligned} \lambda_j &= -2a_0 - \sum_{s=1}^k a_s \{\exp i \frac{2j\pi}{M}\}^s - \sum_{s=1}^k a_s \{\exp i \frac{2j\pi}{M}\}^{M-s} \\ (2.6) \quad &= -2 \sum_{s=0}^k a_s \cos(\frac{2js\pi}{M}), \quad j = 0, 1, \dots, M-1. \end{aligned}$$

Apparently, the matrix  $A$  has the same eigenvalues as  $B$ . By (2.6), it is clear that

$$(2.7) \quad -2a_0 - 2 \sum_{s=1}^k |a_s| \leq \lambda_j \leq -2a_0 + 2 \sum_{s=1}^k |a_s|, \quad (j = 0, 1, 2, \dots, M-1).$$

Let

$$\begin{aligned} \lambda_{\max} &= \max\{\lambda_j | \lambda_j \neq 0, j = 0, 1, \dots, M-1\}, \\ \lambda_{\min} &= \min\{\lambda_j | \lambda_j \neq 0, j = 0, 1, \dots, M-1\}. \end{aligned}$$

Thus we have the following cases:

**Case 1.**  $a_0 + \sum_{s=1}^k |a_s| \leq 0$ , and there exists  $i \in \mathbf{N}(0, M-1)$  such that  $\sum_{s=0}^k a_s \cos \frac{2s\pi}{M} i = 0$ . It follows that  $\lambda_j \geq 0 (j = 0, 1, 2, \dots, M-1)$ , and hence the matrix  $A$  is semi-positive definite.

**Case 2.**  $a_0 + \sum_{s=1}^k |a_s| < 0$ . This implies that  $\lambda_j > 0 (j = 0, 1, 2, \dots, M-1)$ , and hence the matrix  $A$  is positive definite.

**Case 3.** There exist  $i, j \in \mathbf{N}(0, M - 1)$  such that  $\sum_{s=0}^k a_s \cos \frac{2s\pi}{M} i \leq 0 \leq \sum_{s=0}^k a_s \cos \frac{2s\pi}{M} j$ . For this case, the matrix  $A$  may have positive eigenvalue, negative eigenvalue, or zero eigenvalue.

For equation (1.2), according to the previous discussion, we can also construct a suitable variable structure and the functional  $J$  defined on  $E_M$  with  $m = 1$ ,

$$(2.8) \quad J(x) = -\frac{1}{2} \sum_{n=1}^M \left( \sum_{i=1}^k r_i \Delta^{2i} x_{n-i} \right) x_n - \sum_{n=1}^M F(n, x_n), \quad \forall x \in E_M,$$

where  $F(t, z) = \int_0^z f(t, s) ds$ .

In view of  $x_{n+M} = x_n, \forall x \in E_M, n \in \mathbf{Z}$ , (2.8) can be rewritten as

$$\begin{aligned} J(x) &= -\frac{1}{2} \sum_{n=1}^M \left( \sum_{i=1}^k r_i \Delta^{2i} x_{n-i} \right) x_n - \sum_{n=1}^M F(n, x_n) \\ &= -\frac{1}{2} \sum_{n=1}^M \sum_{i=1}^k \sum_{j=0}^{2i} r_i (-1)^j C_{2i}^j x_{n+i-j} x_n - \sum_{n=1}^M F(n, x_n) \\ &= \frac{1}{2} \sum_{n=1}^M \sum_{s=-k}^k d_s x_{n+s} x_n - \sum_{n=1}^M F(n, x_n), \end{aligned}$$

where

$$\begin{aligned} d_0 &= \sum_{i=1}^k (-1)^{i+1} r_i C_{2i}^i, \\ d_s &= \sum_{i=s}^k (-1)^{i-s+1} r_i C_{2i}^{i-s}, \end{aligned}$$

and  $d_{-s} = d_s, s = 1, 2, \dots, k$ .

If we define  $x_0 := x_M$ , then

$$\begin{aligned} \frac{\partial J(x)}{\partial x_n} &= -\left[ \sum_{s=-k}^k d_s x_{n+s} + f(n, x_n) \right] \\ &= -\left[ \sum_{i=1}^k r_i \Delta^{2i} x_{n-i} + f(n, x_n) \right], \end{aligned}$$

where  $n \in \mathbf{Z}(1, M)$ . Therefore,  $x \in E_M$  is a critical point of  $J$ , that is,  $J'(x) = 0$  if and only if

$$\sum_{i=1}^k r_i \Delta^{2i} x_{n-i} + f(n, x_n) = 0, \quad n \in \mathbf{Z}(1, M).$$

On the other hand,  $\{x_n\} \in E_M$  is  $M$ -periodic in  $n$ , and  $f(t, z)$  is  $M$ -periodic in  $t$ , hence,  $x \in E_M$  is a critical point of  $J$  if and only if  $\sum_{i=1}^k r_i \Delta^{2i} x_{n-i} + f(n, x_n) = 0$  for any  $n \in \mathbf{Z}$ , and  $x = \{x_n\}$  is a  $M$ -periodic solution of (1.2).

Thus we reduce the problem of finding  $M$ -periodic solutions of (1.2) to that of seeking critical points of the functional  $J$  in  $E_M$ .

For convenience, we write  $x \in E_M$  as  $x = (x_1, x_2, \dots, x_M)^T$ .

When  $M \geq 2k + 1$ ,  $J(x)$  is rewritten as

$$J(x) = \frac{1}{2}x^T D x - \sum_{n=1}^M F(n, x_n),$$

where  $x = (x_1, x_2, \dots, x_M)^T$ .

Let the eigenvalues of  $D$  be  $\lambda'_0, \lambda'_1, \dots, \lambda'_{M-1}$ , and  $D$  be a circulant matrix [3] denoted by

$$D \stackrel{def}{=} \text{Circ}\{d_0, d_1, d_2, \dots, d_k, 0, \dots, 0, d_k, d_{k-1}, \dots, d_2, d_1\}.$$

By [3], the eigenvalues of  $D$  are

$$\begin{aligned} \lambda'_j &= d_0 + \sum_{s=1}^k d_s \{\exp i \frac{2j\pi}{M}\}^s + \sum_{s=1}^k d_s \{\exp i \frac{2j\pi}{M}\}^{M-s} \\ (2.9) \quad &= d_0 + 2 \sum_{s=1}^k d_s \cos(\frac{2js\pi}{M}) \\ &= \sum_{s=1}^k (-1)^{s+1} r_s [2(1 - \cos \frac{2j\pi}{M})]^s, \end{aligned}$$

where  $j = 0, 1, \dots, M - 1$ .

Let

$$\begin{aligned} \lambda'_{\max} &= \max\{\lambda'_j | \lambda'_j \neq 0, j = 0, 1, \dots, M - 1\}, \\ \lambda'_{\min} &= \min\{\lambda'_j | \lambda'_j \neq 0, j = 0, 1, \dots, M - 1\}. \end{aligned}$$

According to (2.9), for any positive integer  $M$  with  $M \geq 2k + 1$ , we have

**Case 4.** If  $r_{2t+1} \geq 4r_{2t+2}$  when  $M$  is even, or  $r_{2t+1} \geq 2(1 + \cos \frac{\pi}{M})r_{2t+2}$  when  $M$  is odd,  $t = 0, 1, \dots, [\frac{k}{2}] - 1$ , and  $r_k > 0$ , then the matrix  $D$  is semi-positive definite.

### 3. Main lemmas

In this section, we give several lemmas which will play important roles in the proofs of our main results.

**Lemma 3.1** ([16]). *For any given  $u_j, v_j \geq 0 (j = 1, 2, \dots, k)$ ,  $q > 1$ ,  $s > 1$ , and  $\frac{1}{q} + \frac{1}{s} = 1$ , the following inequality hold:*

$$\sum_{j=1}^k u_j v_j \leq \left( \sum_{j=1}^k u_j^s \right)^{\frac{1}{s}} \left( \sum_{j=1}^k v_j^q \right)^{\frac{1}{q}}.$$

According to Lemma 3.1, for any  $s > 1$ , we can give another norm as follows:

$$\|X\|_s = \left( \sum_{j=1}^M |X_j|^s \right)^{\frac{1}{s}}, \quad \forall X \in E_M,$$



apparently,  $\|X\|_2 = \|X\|_{E_M}$ .

Because  $E_M$  is equivalent to the finite dimensional Hilbert space  $\mathbf{R}^{mM}$ ,  $(E_M, \|X\|_2)$  and  $(E_M, \|X\|_s)$  is equivalent, that is, there exist constants  $c_2 \geq c_1 > 0$  such that

$$(3.1) \quad c_1 \|X\|_s \leq \|X\|_2 \leq c_2 \|X\|_s, \quad \forall X \in E_M.$$

**Definition 3.1** (Palais-Smale condition [14]). Let  $X$  be a real Banach space,  $I \in C^1(X, \mathbf{R})$ , that is,  $I$  is a continuously Fréchet differentiable functional defined on  $X$ .  $I$  is said to satisfy the Palais-Smale condition if any sequence  $\{u_n\} \subset X$  for which  $\{I(u_n)\}$  is bounded and  $I'(u_n) \rightarrow 0 (n \rightarrow \infty)$  possesses a convergent subsequence in  $X$ .

Let  $B_r$  denote the open ball in  $X$  about 0 of radius  $r$  and  $\partial B_r$  denote its boundary.

**Lemma 3.2** (Linking Theorem [14]). Let  $X$  be a real Banach space,  $X = X_1 \oplus X_2$ , where  $X_1$  is a finite dimensional subspace of  $X$ . Assume that  $I \in C^1(X, \mathbf{R})$  satisfies the P.S. condition, and

- (I<sub>1</sub>) there exist constants  $\sigma > 0$  and  $\rho > 0$  such that  $I|_{\partial B_\rho \cap X_2} \geq \sigma$ ;
- (I<sub>2</sub>) there is an  $\theta \in \partial B_1 \cap X_2$  and a constant  $R_2 > \rho$  such that  $I|_{\partial Q} \leq 0$ , and

$$Q \stackrel{def}{=} (B_{R_2} \cap X_1) \oplus \{r\theta | 0 < r < R_2\}.$$

Then  $I$  possesses a critical value  $c \geq \sigma$ , where

$$c = \inf_{h \in \Gamma} \max_{u \in Q} I(h(u)),$$

and  $\Gamma = \{h \in C(\bar{Q}, x) : h|_{\partial Q} = id\}$ .

**Lemma 3.3** (Ambrosetti-Rabinowitz Mountain Theorem [18]). Let  $I \in C^1(X, \mathbf{R}^1)$  satisfy the P.S. condition, even, and

- (S<sub>1</sub>) there exist constants  $\sigma > 0$ ,  $\rho > 0$ , and there exists a finite subspace  $E$  such that

$$I|_{E \cap B_\rho} \geq \alpha;$$

- (S<sub>2</sub>) there exist a series subspace  $E_j$ ,  $\dim(E_j) = j$  and  $R_j > 0$  such that

$$I(x) \leq 0, \forall x \in E_j / B_{R_j}, j = 1, 2, \dots$$

Then  $I$  possesses infinite critical points corresponding to positive critical values.

**Lemma 3.4** (Linking Theorem [18]). Let  $X$  be a real Banach space,  $\beta > \alpha$ , and  $I \in C^1(X, \mathbf{R})$  satisfy

- a)  $\sup_{x \in \partial Q} I(x) \leq \alpha$ ;
- b)  $\inf_{x \in S} I(x) \geq \beta$ , and  $I^{-1}[\alpha, +\infty)$  satisfies the P.S. condition;
- c)  $\sup_{x \in Q} I(x) < +\infty$ .

Then  $I$  possesses a critical value  $c \geq \beta$ , where

$$c = \inf_{h \in \Gamma} \max_{u \in Q} I(h(u)),$$

and  $\Gamma = \{h \in C(\bar{Q}, x) : h|_{\partial Q} = id\}$ .

Consider the following functional

$$I(X) = \frac{1}{2}(LX)^T A(LX) - \sum_{n=1}^M F(n, X_n).$$

We have the following lemmas.

**Lemma 3.5.** *Suppose that  $F(t, Z)$  satisfies  $(f_1)$  and  $(f_3)$ . Then the functional  $I(X)$  is bounded from above in  $E_M$ .*

*Proof.* According to  $(f_3)$ , if we let

$$\gamma_1 = \max\{|F(n, Z) - \beta|Z|^2 + \gamma| : n \in \mathbf{Z}, |Z| \leq \rho\}, \gamma' = \gamma + \gamma_1.$$

Then

$$(3.2) \quad F(n, Z) \geq \beta|Z|^2 - \gamma', n \in \mathbf{Z}, Z \in \mathbf{R}^m.$$

For any  $X \in E_M$ , by (3.2),

$$\begin{aligned} I(X) &= \frac{1}{2}(LX)^T A(LX) - \sum_{n=1}^M F(n, X_n) \\ (3.3) \quad &\leq \frac{1}{2}\lambda_{\max}|LX|^2 - \sum_{n=1}^M F(n, X_n) \\ &\leq \frac{1}{2}\lambda_{\max}\|X\|^2 - \sum_{n=1}^M (\beta|X|^2 - \gamma') \\ &= \left(\frac{1}{2}\lambda_{\max} - \beta\right)\|X\|_{E_M}^2 + M\gamma'. \end{aligned}$$

Since by  $(f_3)$ , we see that  $\beta > \frac{1}{2}\lambda_{\max}$ , thus  $I(X) \leq M\gamma'$ . The proof of Lemma 3.5 is complete.  $\square$

**Lemma 3.6.** *Suppose that  $F(t, Z)$  satisfies  $(f_1)$  and  $(f_3)$ . Then the functional  $I(X)$  satisfies the P.S. condition.*

*Proof.* Let  $\{I(X^{(k)})\}$  be a bounded sequence from below, that is, there exists a positive constant  $c$  such that

$$-c \leq I(X^{(k)}), \forall k \in \mathbf{N}.$$

By the proof of Lemma 3.5, it is easy to see that

$$-c \leq I(X^{(k)}) \leq \left(\frac{1}{2}\lambda_{\max} - \beta\right)\|X^{(k)}\|_{E_M}^2 + M\gamma',$$

which implies

$$\|X^{(k)}\|_{E_M}^2 \leq \left(\beta - \frac{1}{2}\lambda_{\max}\right)^{-1}(M\gamma' + c).$$

That is,  $\{X^{(k)}\}$  is a bounded sequence in the finite dimensional space  $E_M$ . Consequently, it has a convergent subsequence. Thus, we obtain Lemma 3.6.  $\square$

**Lemma 3.7.** *Assume  $F(t, Z)$  satisfies  $(f_1)$  and  $(f_5)$ . Then the functional  $I(X)$  is bounded from above in  $E_M$ .*

*Proof.* By  $(f_5)$ , we have

$(f_5^*)$  there exist some constants  $\beta_1 > 0$  and  $\beta_2 > 0$ , such that

$$(3.4) \quad F(t, Z) \geq \beta_1 |Z|^{\alpha_2} - \beta_2, \forall Z \in \mathbf{R}^m.$$

Therefore, for any  $X \in E_M$ ,

$$\begin{aligned} I(X) &= \frac{1}{2}(LX)^T A(LX) - \sum_{n=1}^M F(n, X_n) \\ &\leq \frac{1}{2}\lambda_{\max}|LX|^2 - \beta_1 \sum_{n=1}^M |X_n|^{\alpha_2} + \beta_2 M \\ &= \frac{1}{2}\lambda_{\max}\|X\|_2^2 - \beta_1\|X\|^{\alpha_2} + \beta_2 M. \end{aligned}$$

According to (3.1), there exists a positive constant  $\frac{1}{c_2}$ , such that  $\|X\|_{\alpha_2} \geq \frac{1}{c_2}\|X\|_2$ , thus,

$$I(X) \leq \frac{1}{2}\lambda_{\max}\|X\|_2^2 - \beta_1\left(\frac{1}{c_2}\right)^{\alpha_2}\|X\|_2^{\alpha_2} + \beta_2 M.$$

For  $\alpha_2 > 2$ , there exists a positive constant  $M_1 > 0$  such that

$$I(X) \leq M_1, \forall X \in E_M.$$

$\square$

**Lemma 3.8.** *Assume  $F(t, Z)$  satisfies  $(f_1)$  and  $(f_5)$ . Then the functional  $I(X)$  satisfies the P.S. condition.*

*Proof.* Let  $\{I(X^{(k)})\}$  be a bounded sequence from below, that is, there exists a positive constant  $M_2$  such that

$$-M_2 \leq I(X^{(k)}), \forall k \in \mathbf{N}.$$

From Lemma 3.7, it follows that

$$-M_2 \leq I(X^{(k)}) \leq \frac{1}{2}\lambda_{\max}\|X^{(k)}\|_2^2 - \beta_1\left(\frac{1}{c_2}\right)^{\alpha_2}\|X^{(k)}\|_2^{\alpha_2} + \beta_2 M.$$

Therefore,

$$\beta_1\left(\frac{1}{c_2}\right)^{\alpha_2}\|X^{(k)}\|_2^{\alpha_2} - \frac{1}{2}\lambda_{\max}\|X^{(k)}\|_2^2 \leq M_2 + \beta_2 M, \forall k \in \mathbf{N}.$$

Because of  $\alpha_2 > 2$ , there exists a constant  $M_3 > 0$  such that  $\|X^{(k)}\|_2 \leq M_3$  for any  $k \in \mathbf{N}$ . That is,  $\{X^{(k)}\}$  is a bounded sequence in the finite dimensional space  $E_M$ . Consequently, it has a convergent subsequence. The proof of Lemma 3.8 is complete.  $\square$

For equation (1.2), by  $(g_2)$ , we have

$$\lim_{z \rightarrow 0} \frac{f(t, z)}{z} = 0,$$

which means  $f(t, z)$  is superlinear at 0, according to the following integrating inequality

$$zf(t, z) \geq \beta' \int_0^z f(t, s) ds > 0,$$

we see that there are constants  $\beta'_1 > 0$  and  $\beta'_2 > 0$  such that

$$\int_0^z f(t, s) ds \geq \beta'_1 |z|^{\beta'_1} - \beta'_2, \forall z \in \mathbf{R}.$$

We have

$(g_3^*)$  there are constants  $\beta'_1 > 0$  and  $\beta'_2 > 0$ , such that

$$(3.5) \quad \int_0^z f(t, s) ds \geq \beta'_1 |z|^{\beta'_1} - \beta'_2, \forall z \in \mathbf{R}.$$

By  $(g_3)$  and  $(g_3^*)$ , we see that

$$zf(t, z) \geq \beta' \int_0^z f(t, s) ds \geq \beta' \beta'_1 |z|^{\beta'_1} - \beta' \beta'_2, \forall |z| \geq R.$$

It follows that

$$\lim_{z \rightarrow +\infty} \frac{f(t, z)}{z} = +\infty,$$

which means that  $f(t, z)$  is superlinear at infinity. So, equation (1.2) is called superlinear at 0 and at infinity.

**Lemma 3.9.** *Suppose  $f(t, z)$  satisfies  $(g_1)$  and  $(g_3)$ . Then*

$$J(x) = \frac{1}{2} x^T D x - \sum_{n=1}^M F(n, x_n)$$

is bounded above in  $E_M$ .

*Proof.* By (3.5), for any  $x \in E_M$ ,

$$\begin{aligned} J(x) &= \frac{1}{2} x^T D x - \sum_{n=1}^M F(n, x_n) \\ &\leq \frac{1}{2} \lambda'_{\max} \|x\|_2^2 - \beta'_1 \sum_{n=1}^M |x_n|^{\beta'_1} + \beta'_2 M \\ &= \frac{1}{2} \lambda'_{\max} \|x\|_2^2 - \beta'_1 \|x\|^{\beta'_1} + \beta'_2 M. \end{aligned}$$

Also according to (3.1), there exists a constant  $\frac{1}{c_2}$ , such that

$$\|x\|_{\beta} \geq \frac{1}{c_2} \|x\|_2,$$

we have

$$J(x) \leq \frac{1}{2}\lambda'_{\max}\|x\|_2^2 - \beta'_1\left(\frac{1}{c_2}\right)^{\beta'}\|x\|_2^{\beta'} + \beta'_2M.$$

Because of  $\beta' > 2$ , there exists a positive constant  $M_4$  such that

$$J(x) \leq M_4, \forall x \in E_M.$$

□

**Lemma 3.10.** *Suppose  $f(t, z)$  satisfies  $(g_1)$  and  $(g_3)$ . Then the functional  $J(x)$  satisfies the P.S. condition.*

*Proof.* Let  $\{J(x^{(k)})\}$  be a bounded sequence from below, then there exists a positive constant  $M_5$  such that

$$-M_5 \leq J(x^{(k)}), \forall k \in \mathbf{N}.$$

By the proof of Lemma 3.9, it is easy to see that

$$-M_5 \leq J(x^{(k)}) \leq \frac{1}{2}\lambda'_{\max}\|x^{(k)}\|_2^2 - \beta'_1\left(\frac{1}{c_2}\right)^{\beta'}\|x^{(k)}\|_2^{\beta'} + \beta'_2M.$$

Thus,

$$\beta'_1\left(\frac{1}{c_2}\right)^{\beta'}\|x^{(k)}\|_2^{\beta'} - \frac{1}{2}\lambda'_{\max}\|x^{(k)}\|_2^2 \leq M_5 + \beta'_2M, \forall k \in \mathbf{N}.$$

Because of  $\beta' > 2$ , there exists a constant  $M_6 > 0$  such that  $\|x^{(k)}\|_2 \leq M_6$  for  $k \in \mathbf{N}$ . That is,  $\{x^{(k)}\}$  is a bounded sequence in the finite dimensional sequence  $E_M$ . Consequently, it has a convergent subsequence. □

#### 4. Proofs of main results

In what follows, we will prove that Theorems 1.1-1.5 hold respectively.

*Proof of Theorem 1.1.* From  $a_0 + \sum_{s=1}^k |a_s| \leq 0$  and the fact that there exists  $i \in \mathbf{N}(0, M - 1)$  such that  $\sum_{s=0}^k a_s \cos \frac{2s\pi}{M}i = 0$ , it follows that  $\lambda_j \geq 0 (j = 0, 1, 2, \dots, M - 1)$ . Apparently,

$$0 < \lambda_{\max} \leq -2a_0 + 2 \sum_{s=1}^k |a_s|,$$

$$\lambda_{\min} \geq -2a_0 - 2 \sum_{s=1}^k |a_s| > 0.$$

In view of  $(f_2)$  and the fact  $f(t, Z) \in C^1(\mathbf{R} \times \mathbf{R}^m, \mathbf{R}^m)$  for any  $t \in \mathbf{R}$ , we have  $f(t, 0) = 0$  and  $F(t, 0) \equiv 0$ . Therefore  $\{X_n\} \equiv 0$ , where  $X_n \equiv 0 (n \in \mathbf{Z})$  is a trivial periodic solution of (1.1) with period  $M$ .

By Lemma 3.5,  $I$  is bounded from above on  $E_M$ . We denote  $c_0 = \sup_{X \in E_M} I(X)$ , then there exists a sequence  $\{X^{(k)}\}_{k=1}^\infty$ , where  $X^{(k)} \in E_M$ , such that  $c_0 = \lim_{k \rightarrow \infty} I(X^{(k)})$ . On the other hand, by (3.3), we have

$$I(X) \leq \left(\frac{1}{2}\lambda_{\max} - \beta\right)\|X\|_{E_M}^2 + M\gamma', \forall X \in E_M.$$

Therefore,  $I(X) \rightarrow -\infty$  as  $\|X\|_{E_M}^2 \rightarrow \infty$ , which implies that  $\{X^{(k)}\}$  is bounded, it has a convergent subsequence  $\{X^{(k_i)}\}$ . Let  $\bar{X} = \lim_{i \rightarrow \infty} X^{(k_i)}$ . By the continuity of  $I$ ,  $I(\bar{X}) = c_0$ . Clearly,  $\bar{X} \in E_M$  is a critical point of  $I \in E_M$ .

Let 0 be the  $l$  multiple eigenvalue of the matrix  $A$ ,  $\varphi_0, \dots, \varphi_{l-1}$  be linearly independent eigenvectors of  $A$  associated to 0. Denote  $W = \text{span}\{\varphi_0, \dots, \varphi_{l-1}\} \in E_M$ , then  $W$  is an invariable subspace of  $E_M$ ,  $E_M = Y \oplus W$ .

We next claim that  $c_0 > 0$ .

In fact, by  $(f_2)$ , there exists a constant  $\alpha$ ,  $0 < \alpha < \frac{1}{2}\lambda_{\min}$ , such that

$$(3.6) \quad F(n, Z) \leq \alpha|Z|^2.$$

According to (3.6), for any  $X = (X_1, X_2, \dots, X_M)^T \in Y$ ,  $\|X\|_2 \leq \delta$ , we obtain

$$\begin{aligned} I(X) &= \frac{1}{2}(LX)^T A(LX) - \sum_{n=1}^M F(n, X_n) \\ &\geq \frac{1}{2}\lambda_{\min}\|X\|^2 - \alpha \sum_{n=1}^M |X_n|^2 \\ &= \left(\frac{1}{2}\lambda_{\min} - \alpha\right)\|X\|_{E_M}^2. \end{aligned}$$

Let  $\sigma = \left(\frac{1}{2}\lambda_{\min} - \alpha\right)\delta^2$ . Then  $I(X) \geq \sigma > 0, \forall X \in Y \cap \partial B_\delta$ . Thus we have shown that  $c_0 = \sup_{X \in E_M} I(X) \geq \sigma > 0$ . At the same time, we have also proved that  $I|_{\partial B_\delta \cap Y} \geq \sigma$  for  $\sigma > 0$  and  $\delta > 0$ . This implies that  $I$  satisfies assumption  $(I_1)$  of Lemma 3.2.

Note that  $ALX = 0$  for any  $X \in W$ , we have

$$I(X) = \frac{1}{2}\langle ALX, LX \rangle - \sum_{n=1}^M F(n, X_n) = - \sum_{n=1}^M F(n, X_n) \leq 0.$$

It follows that  $\bar{X} \in Y$  and the critical point  $\bar{X}$  of  $I$  corresponding to the critical value  $c_0$  is a nontrivial periodic solution of (1.1) with period  $M$ .

In order to obtain another nontrivial  $M$ -periodic solution of (1.1) different from  $\bar{X}$ , we will use Lemma 3.2. In view of Lemma 3.6, It is obvious that  $I$  satisfies the Palais-Smales condition. Furthermore we have also verified that  $I$  satisfies the condition  $(I_1)$  of Lemma 3.2. In the following, we will show the condition  $(I_2)$  is also satisfied.

In fact, let  $\theta = (\theta_1, \theta_2, \dots, \theta_M)^T \in \partial B_1 \cap Y$  and  $X = r\theta + Z$ . Then, for any  $Z \in W$  and  $r \in \mathbf{R}$ , we have

$$\begin{aligned} I(X) &= \frac{1}{2} \langle AL(r\theta + Z), L(r\theta + Z) \rangle - \sum_{n=1}^M F(n, X_n) \\ &= \frac{1}{2} \langle AL(r\theta), L(r\theta) \rangle - \sum_{n=1}^M F(n, X_n) \\ &\leq \frac{1}{2} \lambda_{\max} |L(r\theta)|^2 - \beta \sum_{n=1}^M (|r\theta_n + Z_n|^2 - \gamma') \\ &= \frac{1}{2} \lambda_{\max} r^2 - \beta \sum_{n=1}^M (r^2 |\theta_n|^2 + |Z_n|^2 - \gamma') \\ &= (\frac{1}{2} \lambda_{\max} - \beta) r^2 - \beta \|Z\|_{E_M}^2 + M\beta\gamma' \\ &\leq -\beta \|Z\|_{E_M}^2 + M\beta\gamma'. \end{aligned}$$

It follows that there exists some constant  $R_2 > \delta$  such that

$$I(X) \leq 0 \quad \text{for } X \in \partial Q, \quad Q \stackrel{\text{def}}{=} (\bar{B}_{R_2} \cap W) \oplus \{r\theta \mid 0 < r < R_2\}.$$

By Lemma 3.2,  $I$  possesses a critical value  $c \geq \sigma \geq 0$ , where

$$c = \inf_{h \in \Gamma} \max_{u \in Q} I(h(u)), \quad \Gamma = \{h \in C(\bar{Q}, E_M) : h|_{\partial Q} = id\}.$$

Let  $\tilde{X} \in E_M$  be a critical point corresponding to the critical value  $c$  of  $I$ , that is  $I(\tilde{X}) = c$ . If  $\tilde{X} \neq \bar{X}$ , then Theorem 1.1 holds. Otherwise,  $\tilde{X} = \bar{X}$ , then  $c_0 = I(\tilde{X}) = I(\bar{X}) = c$ , which is the same as

$$\sup_{X \in E_M} I(X) = \inf_{h \in \Gamma} \sup_{u \in Q} I(h(u)).$$

Choosing  $h = id$ , then we have  $\sup_{X \in Q} I(X) = c_0$ . Because the choice of  $\theta \in \partial B_1 \cap Y \subset Q$  is arbitrary, we can take  $-\theta \in \partial B_1 \cap Y$ . Similarly, there exists a positive number  $R_3 > \delta$  such that

$$I(X) \leq 0 \quad \text{for } X \in \partial Q_1,$$

where  $Q_1 \stackrel{\text{def}}{=} (\bar{B}_{R_3} \cap W) \oplus \{-r\theta \mid 0 < r < R_3\}$ . Again by Lemma 3.2,  $I$  possesses a critical value  $c' \geq \sigma > 0$ , and

$$c' = \inf_{h \in \Gamma_1} \max_{u \in Q_1} I(h(u)),$$

where  $\Gamma_1 = \{h \in C(\bar{Q}_1, E_M) : h|_{\partial Q_1} = id\}$ .

If  $c' \neq c_0$ , then the proof of Theorem 1.1 is complete. Otherwise,  $c' = c_0 = \sup_{X \in Q_1} I(X)$ . Because of the fact that  $I|_{\partial Q} \leq 0$  and  $I|_{\partial Q_1} \leq 0$ ,  $I$  attains its maximum at some points in the interior of sets  $Q$  and  $Q_1$ . But  $Q \cap Q_1 \subset W$ , and  $I(X) \leq 0, \forall X \in W$ . Thus, there is a critical point  $\hat{X} \in E_M, \hat{X} \neq \bar{X}$ , and  $I(\hat{X}) = c' = c_0$ .

The proof of Theorem 1.1 is now complete. □

Proof of Theorem 1.2 is similar to that of Theorem 1.1. So we will omit it.

*Proof of Theorem 1.3.* Note that  $-a_0 - \sum_{s=1}^k |a_s| > 0$ , this is,  $a_0 < -\sum_{s=1}^k |a_s|$ , we have  $\lambda_j > 0, j = 0, 1, 2, \dots, M - 1$ .

According to Lemmas 3.7 and 3.8, the functional  $I(X)$  is continuous and satisfies the P.S. condition in  $E_M$ . Again by  $(f_6)$ ,  $I(X)$  is even.

Let  $0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{M-1}$ , and the corresponding linearly independent eigenvectors be  $\phi_0, \phi_1, \phi_2, \dots, \phi_{M-1}$  respectively. Assume  $E_j = \text{span}\{\phi_0, \dots, \phi_j\}$ , then  $E_j \subseteq E_i$  for  $j < i, E_M = \cup_{j=0}^{M-1} E_j$ .

At first, we will prove that the functional  $I(X)$  satisfies the condition  $(S_1)$  of Lemma 3.3.

By  $(f_4)$ , for any  $X = (X_1, X_2, \dots, X_M)^T \in E_M, \|X\|_2 \leq \rho$ ,

$$\begin{aligned} I(X) &= \frac{1}{2}(LX)^T A(LX) - \sum_{n=1}^M F(n, X_n) \\ &\geq \frac{1}{2}\lambda_{\min}\|X\|^2 - \alpha_1 \sum_{n=1}^M |X_n|^2 \\ &= \left(\frac{1}{2}\lambda_{\min} - \alpha_1\right)\|X\|_{E_M}^2. \end{aligned}$$

Let  $\sigma = (\frac{1}{2}\lambda_{\min} - \alpha_1)\rho^2$ . Then  $I(X) \geq \sigma$  for  $X \in E_M \cap \partial B_\rho$ .

Next, we will prove that the functional  $I(X)$  satisfies the conditions  $(S_2)$  of Lemma 3.3.

By  $(f_5^*)$ , we obtain

$$F(t, Z) \geq \beta_1 |Z|^{\alpha_2} - \beta_2, \forall Z \in \mathbf{R}^m.$$

For any  $X \in E_j$ , we have

$$\begin{aligned} I(X) &= \frac{1}{2}(LX)^T A(LX) - \sum_{n=1}^M F(n, X_n) \\ &\leq \frac{1}{2}\lambda_{\max}\|X\|_2^2 - \beta_1 \sum_{n=1}^M |X_n|^{\alpha_2} + \beta_2 M \\ &\leq \frac{1}{2}\lambda_{\max}\|X\|_2^2 - \beta_1 \left(\frac{1}{c_2}\right)^{\alpha_2} \|X\|_2^{\alpha_2} + \beta_2 M. \end{aligned}$$

It follows that there exists a positive constant  $R_j > \rho$  such that

$$I(X) \leq 0, X \in E_j/B_{R_j}.$$

According to Lemma 3.3, we know that the functional  $I(X)$  has infinite critical points, that is, system (1.1) has infinite  $M$ -periodic solutions. The proof of Theorem 1.3 is complete.  $\square$

*Proof of Theorem 1.4.* From the fact that there exist  $i, j \in \mathbf{N}(0, M - 1)$  such that

$$\sum_{s=0}^k a_s \cos \frac{2s\pi}{M} i \leq 0 \leq \sum_{s=0}^k a_s \cos \frac{2s\pi}{M} j,$$

it follows that the matrix  $A$  either has positive eigenvalues, zero eigenvalues, or negative eigenvalues,  $\lambda_{\min} < 0 < \lambda_{\max}$ .



Let  $H_1$  be the vector space consisting of those linearly independent eigenvectors corresponding to the negative eigenvalues and zero eigenvalues of  $A$ , and  $H_2$  be the vector space consisting of those linearly independent eigenvectors corresponding to the positive eigenvalues of  $A$ , then  $E_M = H_1 \oplus H_2$ .

Assume that  $\epsilon_+$  is the minimum positive eigenvalue of  $A$ ,  $\epsilon_-$  is the maximum negative eigenvalue of  $A$ , then  $\epsilon_- < 0 < \epsilon_+$ .

According to the conditions in Theorem 1.4, it is not difficult to prove that the functional  $I(X)$  is bounded from above and satisfies the P.S. condition in  $H_1$  and  $H_2$  respectively, and hence  $I(X)$  is bounded from above and satisfies the P.S. condition in  $E_M$ .

By  $(f_2)$ , there exists a constant  $\alpha_4 > 0$  such that  $\frac{\epsilon_+}{2} - \alpha_4 > 0$  with  $F(n, X_n) \leq \alpha_4 |X_n|$ . For any  $X \in H_2$ , we have

$$\begin{aligned} I(X) &= \frac{1}{2}(LX)^T A(LX) - \sum_{n=1}^M F(n, X_n) \\ &\geq \frac{\epsilon_+}{2} \|LX\|^2 - \alpha_4 \sum_{n=1}^M |X_n|^2 \\ &= (\frac{\epsilon_+}{2} - \alpha_4) \|X\|_{E_M}^2. \end{aligned}$$

Therefore,  $I(X) \rightarrow +\infty$  as  $\|X\| \rightarrow +\infty$ , thus there must exist real number  $\eta$  such that

$$I(X)|_{H_2} \geq \eta > 0.$$

On the other hand, for any  $X \in H_1$ , we have

$$\begin{aligned} I(X) &= \frac{1}{2}(LX)^T A(LX) - \sum_{n=1}^M F(n, X_n) \\ &\leq \frac{\epsilon_-}{2} \|X\|^2 - \sum_{n=1}^M F(n, X_n). \end{aligned}$$

Again by  $(f_3)$ , it follows that

$$F(n, X_n) \geq \beta |X|^2 - \gamma', n \in \mathbf{Z}, X_n \in E_M.$$

Thus,

$$-\sum_{n=1}^M F(n, X_n) \rightarrow -\infty \quad \text{as} \quad \|X\| \rightarrow +\infty,$$

and hence  $I(X) \rightarrow -\infty$  if  $\|X\| \rightarrow +\infty$ . It follows that there exists a positive constant  $R$  such that

$$I(X) < \eta - 1 \quad \text{if} \quad \|X\| \geq R.$$

Now we denote

$$S = H_2, Q = B_R \cap H_1,$$

then  $I$  satisfies the conditions of Lemma 3.4, it has at least one nontrivial  $M$ -periodic solution. The proof of Theorem 1.4 is complete.  $\square$

*Proof of Theorem 1.5.* According to the conditions of Theorem 1.5, the matrix  $D$  is semi-positive definite. Let 0 be  $l$  multiple eigenvalue of  $D$ ,  $\varphi_0, \dots, \varphi_{l-1}$  be linearly independent eigenvectors of  $D$  associated to 0.

Define  $W' = \text{span}\{\varphi_0, \dots, \varphi_{l-1}\}^T \in E_M$ , let  $E_M = Y' \oplus W'$ .

By Lemma 3.9, Lemma 3.10 and proof of Theorem 1.1, Theorem 1.5 is apparent.  $\square$

*Remark 4.1.* When  $k = 1$  and  $r_1 = 1$ , the equation (1.2) has the same form as [7]. Our results extend and improve some earlier publications.

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