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ON SOME NEW NONLINEAR RETARDED INTEGRAL INEQUALITIES WITH ITERATED INTEGRALS AND THEIR APPLICATIONS

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Abstract. Some new nonlinear retarded integral inequalities of Gronwall-like type are established, which mainly generalized some results given by Cho, Dragomir and Kim (J. Korean Math. Soc. **43** (2006), No. 3, pp. 563–578) and can be used in the analysis of various problems in the theory of certain classes of differential equations and integral equations. Applications examples are also indicated.

1. Introduction

Integral inequalities are indispensable for us in the quantitative study of various differential equations and integral equations. Besides the famous Gronwall-Bellman inequality and its first nonlinear generalization by Bihari (see [1], [2], [13], [20], [21]), there are several other very useful Gronwall-like inequality. Haraux [12, Corollary 16, p.139] derived a Gronwall-like inequality and used it to prove the existence of solutions of wave equations with logarithmic nonlinearities. On the other hand, Engler [11] utilized the following slight variant of inequality due to Haraux [12] in the study of global regular solutions for the dynamic antiplane shear problem in nonlinear viscoelasticity.

Lemma 1.1. *Let $c > 1$ and $a \in C(I = [0,T], R_+), R_+ = [0,\infty)$, and assume that the function $w \to [1,\infty)$ satisfies*

$$w(t) \le c\left(1 + \int_0^t a(s)w(s)\log w(s)ds\right),\ t \in I.$$

Then

$$w(t) \le c\exp\left(\int_0^t a(s)ds\right),\ t \in I.$$

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Recently, Cho, Dragomir and Kim [6] proved the following Haraux-Engler-Bykov type inequality and other interesting related results:

Lemma 1.2. *Let $u(t), b(t), k(t,s)$ and $h(t,s,\tau)$ be nonnegative continuous for $\alpha \le \sigma \le \tau \le s \le t \le \beta$ with $u(t) \ge 1$. Suppose that*

$$
\begin{aligned}
u(t) \le a &+ \int_\alpha^t b(s)u(s)\log u(s)ds + \int_\alpha^t \left(\int_\alpha^s k(s,\tau)u(\tau)\log u(\tau)d\tau\right) ds \\
&+ \int_\alpha^t \left(\int_\alpha^s \left(\int_\alpha^\tau h(s,\tau,\sigma)u(\sigma)\log u(\sigma)d\sigma\right) d\tau\right) ds
\end{aligned}
\tag{1.1}
$$

for $t \in I = [\alpha, \beta]$, where $a \ge 1$ is a constant. Then, for any $t \in I$,

$$u(t) \le a^{\exp(A(t))},$$

where

$$
\begin{aligned}
A(t) = &\int_\alpha^t b(s)ds + \int_\alpha^t \left(\int_\alpha^s k(s,\tau)d\tau\right) ds \\
&+ \int_\alpha^t \left(\int_\alpha^s \left(\int_\alpha^\tau h(s,\tau,\sigma)d\sigma\right) d\tau\right) ds.
\end{aligned}
$$

The aim of the present paper is to establish some new nonlinear retarded inequalities, which generalize the inequalities discussed in the above lemmas and other results appeared in [6] and [14, 18]. Application examples are also included.

2. Some new nonlinear retarded inequalities

In what follows, R denotes the set of all real numbers, $R_+ = [0, +\infty), R_1 = [1, +\infty), I = [\alpha, \beta]$ are the give subsets of R. $C^i(M, S)$ denotes the class of all i-times continuously differentiable functions defined on set M with range in the set $S (i = 1, 2, \dots)$ and $C^0(M, S) = C(M, S)$.

Lemma 2.1. *Let $u(t), b(t), k(t,s)$ and $h(t,s,\tau)$ be nonnegative continuous functions for $\alpha \le \sigma \le \tau \le s \le t \le \beta$. Let $\phi(t) \in C^1(I, I)$ with $\phi(t) \le t$ and $g(u)$ be a nonnegative and nondecreasing continuous functions for $u \in R_+$ with $g(u) > 0$ for $u > 0$. Suppose that*

$$
\begin{aligned}
u(t) \le a &+ \int_{\phi(\alpha)}^{\phi(t)} b(s)g(u(s))ds + \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^s k(s,\tau)g(u(\tau))d\tau\right) ds \\
&+ \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^s \left(\int_{\phi(\alpha)}^\tau h(s,\tau,\sigma)g(u(\sigma))d\sigma\right) d\tau\right) ds
\end{aligned}
\tag{2.1}
$$

for $t \in I$, where $a \geq 0$ is a constant. Then

$$(2.2)\quad \begin{aligned} u(t) \leq G^{-1}\Bigg[G(a) &+ \int_{\phi(\alpha)}^{\phi(t)} b(s)ds + \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} k(s,\tau)d\tau\right) ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} \left(\int_{\phi(\alpha)}^{\tau} h(s,\tau,\sigma)d\sigma\right) d\tau\right) ds\Bigg] \end{aligned}$$

for $t \in [\alpha, T_1]$, where

$$(2.3)\quad G(r) = \int_{r_0}^{r} \frac{ds}{g(s)}, \ r \geq r_0 > 0$$

G^{-1} denotes the inverse function of G, and $T_1 \in I$ is chosen so that the quantity in the square brackets of (2.2) is in the range of G.

Proof. Let $a > 0$ and define a function $v_1(t)$ by the right-hand side of (2.1). Then $v_1(\alpha) = a, v_1(t)$ is positive and nondecreasing, $u(t) \leq v_1(t)$ for $t \in I$ and

$$\begin{aligned} v_1'(t) &= b(\phi(t))g(u(\phi(t)))\phi'(t) + \left(\int_{\phi(\alpha)}^{\phi(t)} k(\phi(t),\tau)g(u(\tau))d\tau\right)\phi'(t) \\ &\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{\tau} h(\phi(t),\tau,\sigma)g(u(\sigma))d\sigma\right) d\tau\right)\phi'(t) \\ &\leq b(\phi(t))g(v_1(\phi(t)))\phi'(t) + \left(\int_{\phi(\alpha)}^{\phi(t)} k(\phi(t),\tau)g(v_1(\tau))d\tau\right)\phi'(t) \\ &\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{\tau} h(\phi(t),\tau,\sigma)g(v_1(\sigma))d\sigma\right) d\tau\right)\phi'(t) \\ &\leq g(v_1(t))\Bigg[b(\phi(t))\phi'(t) + \left(\int_{\phi(\alpha)}^{\phi(t)} k(\phi(t),\tau)d\tau\right)\phi'(t) \\ &\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{\tau} h(\phi(t),\tau,\sigma)d\sigma\right) d\tau\right)\phi'(t)\Bigg], \end{aligned}$$

i.e.,

$$\begin{aligned} \frac{v_1'(t)}{g(v_1(t))} &\leq b(\phi(t))\phi'(t) + \left(\int_{\phi(\alpha)}^{\phi(t)} k(\phi(t),\tau)d\tau\right)\phi'(t) \\ &\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{\tau} h(\phi(t),\tau,\sigma)d\sigma\right) d\tau\right)\phi'(t). \end{aligned}$$

Setting $t = s$ and then integrating from α to $t, t \in I$, changing the variable and using the definition of function G we get

$$G(v_1(t)) \leq G(a) + \int_{\phi(\alpha)}^{\phi(t)} b(s)ds + \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} k(s,\tau)d\tau \right) ds + \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} \left(\int_{\phi(\alpha)}^{\tau} h(s,\tau,\sigma)d\sigma \right) d\tau \right) ds$$

or

$$v_1(t) \leq G^{-1} \left[G(a) + \int_{\phi(\alpha)}^{\phi(t)} b(s)ds + \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} k(s,\tau)d\tau \right) ds + \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} \left(\int_{\phi(\alpha)}^{\tau} h(s,\tau,\sigma)d\sigma \right) d\tau \right) ds \right]$$

for $t \in [\alpha, T_1]$. Now by $u(t) \leq v_1(t)$ and the last inequality, we get the desired inequality (2.2) for $a > 0$. By continuity, (2.2) also holds for any $a \geq 0$. □

Theorem 2.2. *Let a, u, b, k, h, g and ϕ be as in Lemma 2.1. Let $\varphi \in C^1(R_+, R_+)$ with $\varphi'(u) > 0$ and $\varphi'(u)$ is increasing for $u > 0$. Suppose that*

$$\begin{aligned} \varphi(u(t)) \leq a &+ \int_{\phi(\alpha)}^{\phi(t)} b(s)\varphi'(u(s))g(u(s))ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} k(s,\tau)\varphi'(u(\tau))g(u(\tau))d\tau \right) ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} \left(\int_{\phi(\alpha)}^{\tau} h(s,\tau,\sigma)\varphi'(u(\sigma))g(u(\sigma))d\sigma \right) d\tau \right) ds \end{aligned} \tag{2.4}$$

for $t \in I$. Then

$$\begin{aligned} u(t) \leq G^{-1} \Bigg[& G(\varphi^{-1}(a)) + \int_{\phi(\alpha)}^{\phi(t)} b(s)ds + \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} k(s,\tau)d\tau \right) ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} \left(\int_{\phi(\alpha)}^{\tau} h(s,\tau,\sigma)d\sigma \right) d\tau \right) ds \Bigg] \end{aligned} \tag{2.5}$$

for $t \in [\alpha, T_2]$, where $T_2 \in I$ is chosen so that the quantity in the square brackets of (2.5) is in the range of G, G and G^{-1} are as in Lemma 2.1.

Proof. Let $a > 0$ and define a function $v_2(t)$ by the right-hand side of (2.4). Then $v_2(\alpha) = a, v_2(t)$ is positive and nondecreasing, $u(t) \leq \varphi^{-1}(v_2(t))$ for $t \in I$

and

$$\begin{aligned}
v_2'(t) &= b(\phi(t))\varphi'(u(\phi(t)))g(u(\phi(t)))\phi'(t) \\
&\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} k(\phi(t),\tau)\varphi'(u(\tau))g(u(\tau))d\tau\right)\phi'(t) \\
&\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{\tau} h(\phi(t),\tau,\sigma)\varphi'(u(\sigma))g(u(\sigma))d\sigma\right)d\tau\right)\phi'(t) \\
&\le b(\phi(t))\varphi'[\varphi^{-1}(v_2(\phi(t))]g[\varphi^{-1}(v_2(\phi(t)))]\phi'(t) \\
&\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} k(\phi(t),\tau)\varphi'[\varphi^{-1}(v_2(\tau))]g[\varphi^{-1}(v_2(\tau))]d\tau\right)\phi'(t) \\
&\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{\tau} h(\phi(t),\tau,\sigma)\varphi'[\varphi^{-1}(v_2(\sigma))]g[\varphi^{-1}(v_2(\sigma))]d\sigma\right)d\tau\right)\phi'(t) \\
&\le \varphi'[\varphi^{-1}(v_2(t))]\left[b(\phi(t))g[\varphi^{-1}(v_2(\phi(t)))]\phi'(t)\right. \\
&\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} k(\phi(t),\tau)g[\varphi^{-1}(v_2(\tau))]d\tau\right)\phi'(t) \\
&\quad \left. + \left(\int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{\tau} h(\phi(t),\tau,\sigma)g[\varphi^{-1}(v_2(\sigma))]d\sigma\right)d\tau\right)\phi'(t)\right],
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\frac{v_2'(t)}{\varphi'[\varphi^{-1}(v_2(t))]} \\
&\le b(\phi(t))g[\varphi^{-1}(v_2(\phi(t)))]\phi'(t) \\
&\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} k(\phi(t),\tau)g[\varphi^{-1}(v_2(\tau))]d\tau\right)\phi'(t) \\
&\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{\tau} h(\phi(t),\tau,\sigma)g[\varphi^{-1}(v_2(\sigma))]d\sigma\right)d\tau\right)\phi'(t).
\end{aligned} \tag{2.6}$$

Observe that for any continuously differentiable and invertible function $f(\xi)$, by a change of variable $\eta = f^{-1}(\xi)$, we have

$$\int \frac{d\xi}{f'[f^{-1}(\xi)]} = \int \frac{f'(\eta)}{f'(\eta)}d\eta = \eta + c = f^{-1}(\xi) + c. \tag{2.7}$$

Setting $t = s$ in (2.6), integrating with respect to s from α to t and using (2.7) to the left-hand side and changing variable in the right-hand side we obtain

$$\varphi^{-1}(v_2(t)) \le \varphi^{-1}(a) + \int_{\phi(\alpha)}^{\phi(t)} b(s)g[\varphi^{-1}(v_2(s))]ds$$

$$+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s}k(s,\tau)g[\varphi^{-1}(v_2(\tau))]d\tau\right)ds$$
$$+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s}\left(\int_{\phi(\alpha)}^{\tau}h(s,\tau,\sigma)g[\varphi^{-1}(v_2(\sigma))]d\sigma\right)d\tau\right)ds.$$

Now, using Lemma 2.1 to the last inequality we have

$$\varphi^{-1}(v_2(t))\leq G^{-1}\left[G(\varphi^{-1}(a))+\int_{\phi(\alpha)}^{\phi(t)}b(s)ds+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s}k(s,\tau)d\tau\right)ds\right.$$
$$\left.+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s}\left(\int_{\phi(\alpha)}^{\tau}h(s,\tau,\sigma)d\sigma\right)d\tau\right)ds\right]$$

for $t\in[\alpha,T_2]$. By $u(t)\leq\varphi^{-1}(v_2(t))$, we get the desired inequality (2.5). The case $a\geq 0$ follows as mentioned in the proof of Lemma 2.1. □

Letting $\varphi(u)=u^p(p\geq 1$ is a constant) in Theorem 2.2, we obtain the following Ou-Iang type retarded integral inequality with iterated integrals. About Ou-Iang type inequalities and their generalizations and applications, one can see [21].

Corollary 2.3. *Let a,u,b,k,h,g and ϕ be as in Lemma 2.1, $p\geq 1$ be a constant. Suppose that*

$$\begin{aligned}u^p(t)\leq a&+\int_{\phi(\alpha)}^{\phi(t)}b(s)u^{p-1}(s)g(u(s))ds\\&+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s}k(s,\tau)u^{p-1}(\tau)g(u(\tau))d\tau\right)ds\\&+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s}\left(\int_{\phi(\alpha)}^{\tau}h(s,\tau,\sigma)u^{p-1}(\sigma)g(u(\sigma))d\sigma\right)d\tau\right)ds\end{aligned}\tag{2.8}$$

for $t\in I$. Then

$$\begin{aligned}u(t)\leq G^{-1}&\left[G(a^{1/p})+\int_{\phi(\alpha)}^{\phi(t)}\frac{b(s)}{p}ds+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s}\frac{k(s,\tau)}{p}d\tau\right)ds\right.\\&\left.+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s}\left(\int_{\phi(\alpha)}^{\tau}\frac{h(s,\tau,\sigma)}{p}d\sigma\right)d\tau\right)ds\right]\end{aligned}\tag{2.9}$$

for $t\in[\alpha,T_3]$, where $T_3\in I$ is chosen so that the quantity in the square brackets of (2.9) is in the range of G, G and G^{-1} are as in Lemma 2.1.

Corollary 2.4. *Let b, k, h, g and ϕ be as in Lemma 2.1, $a > 1$ and $p > 0$ be constants and $u \in C(I, R_1)$. Suppose that*

$$\begin{aligned} u^p(t) \le a &+ \int_{\phi(\alpha)}^{\phi(t)} b(s)u^p(s)g(\log u(s))ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} k(s,\tau)u^p(\tau)g(\log u(\tau))d\tau\right) ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} \left(\int_{\phi(\alpha)}^{\tau} h(s,\tau,\sigma)u^p(\sigma)g(\log u(\sigma))d\sigma\right) d\tau\right) ds \end{aligned} \tag{2.10}$$

for $t \in I$. Then

(2.11)

$$\begin{aligned} u(t) \le \exp\Bigg(G^{-1}\Bigg[&G(\frac{1}{p}\log a) + \int_{\phi(\alpha)}^{\phi(t)} \frac{b(s)}{p}ds + \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} \frac{k(s,\tau)}{p}d\tau\right) ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} \left(\int_{\phi(\alpha)}^{\tau} \frac{h(s,\tau,\sigma)}{p}d\sigma\right) d\tau\right) ds\Bigg]\Bigg) \end{aligned}$$

for $t \in [\alpha, T_4]$, where $T_4 \in I$ is chosen so that the quantity in the square brackets of (2.9) is in the range of G, G and G^{-1} are as in Lemma 2.1.

Proof. Taking $v(t) = \log u(t)$, then inequality (2.10) reduces to

$$\begin{aligned} \exp(pv(t)) \le a &+ \int_{\phi(\alpha)}^{\phi(t)} b(s)\exp(pv(s))g(v(s))ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} k(s,\tau)\exp(pv(\tau))g(v(\tau))d\tau\right) ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} \left(\int_{\phi(\alpha)}^{\tau} h(s,\tau,\sigma)\exp(pv(\sigma))g(v(\sigma))d\sigma\right) d\tau\right) ds \end{aligned}$$

which is a special case of inequality (2.4) when $\varphi = \exp(pv)$. By Theorem 2.2, we get the desired inequality (2.11) directly. □

Remark 2.1. (i) When $p = 1, g(u) = u$ and $\phi(t) = t$, from Corollary 2.4 we can derive Theorem 2.1 [6].

(ii) If constant a is placed by a nondecreasing continuous function $a(t)$ on I, all conclusions in Lemma 2.1 to Corollary 2.4 are still valid.

Theorem 2.5. *Let u, b, k, h and ϕ be as in Corollary 2.4, g be as in Lemma 2.1 with the subadditivity and multiplicity and $\pi \in C(I, R_1)$. Suppose that*

$$u^p(t) \le \pi(t)\Bigg\{a + \int_{\phi(\alpha)}^{\phi(t)} b(s)u^q(s)g(\log u(s))ds$$

$$
\begin{aligned}
&+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s} k(s,\tau)u^q(\tau)g(\log u(\tau))d\tau\right)ds \\
&+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s}\left(\int_{\phi(\alpha)}^{\tau} h(s,\tau,\sigma)u^q(\sigma)g(\log u(\sigma))d\sigma\right)d\tau\right)ds\Bigg\}
\end{aligned}
\tag{2.12}
$$

for $t \in I$, where $p \geq q > 0$ and $a \geq 1$ are a constant. Then, if $p = q$

$$
\begin{aligned}
u(t) \leq \pi^{1/p}(t)\exp\Bigg\{\frac{1}{p}G^{-1}\Bigg[&G(B_1(t)) + \int_{\phi(\alpha)}^{\phi(t)} b(s)ds \\
&+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s} k(s,\tau)d\tau\right)ds \\
&+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s}\left(\int_{\phi(\alpha)}^{\tau} h(s,\tau,\sigma)d\sigma\right)d\tau\right)ds\Bigg]\Bigg\}
\end{aligned}
\tag{2.13}
$$

for $t \in [\alpha, T_5]$, where

$$
\begin{aligned}
B_1(t) = {}& \log a + \int_{\phi(\alpha)}^{\phi(t)} b(s)g(\log \pi^{1/p}(s))ds \\
&+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s} k(s,\tau)g(\log \pi^{1/p}(\tau))d\tau\right)ds \\
&+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s}\left(\int_{\phi(\alpha)}^{\tau} h(s,\tau,\sigma)g(\log \pi^{1/p}(\sigma))d\sigma\right)d\tau\right)ds
\end{aligned}
\tag{2.14}
$$

for $t \in [\alpha, T_5]$; if $p > q$,

$$
\begin{aligned}
u(t) \leq \pi^{1/p}(t)\exp\Bigg\{\frac{1}{p-q}G^{-1}\Bigg[&G(B_1(t)) + \int_{\phi(\alpha)}^{\phi(t)} b(s)ds \\
&+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s} k(s,\tau)d\tau\right)ds \\
&+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s}\left(\int_{\phi(\alpha)}^{\tau} h(s,\tau,\sigma)d\sigma\right)d\tau\right)ds\Bigg]\Bigg\}
\end{aligned}
\tag{2.15}
$$

for $t \in [\alpha, T_6]$, where $T_5, T_6 \in I$ are chosen so that the quantity in the square brackets of (2.13) and (2.15) are in the range of G respectively, G and G^{-1} are as in Lemma 2.1.

Proof. Define a function $v_3(t)$ by

$$v_3(t) = a + \int_{\phi(\alpha)}^{\phi(t)} b(s)g(\log u(s))ds + \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} k(s,\tau)g(\log u(\tau))d\tau \right) ds + \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{s} \left(\int_{\phi(\alpha)}^{\tau} h(s,\tau,\sigma)g(\log u(\sigma))d\sigma \right) d\tau \right) ds.$$

Then $v_3(t)$ is nondecreasing for $t \in I$, $v_3(\alpha) = a$ and

$$u(t) \leq \pi^{1/p}(t)v_3^{1/p}(t) \tag{2.16}$$

for $t \in I$. Differentiating $v_3(t)$, we get

$$\begin{aligned}
v_3'(t) &= b(\phi(t))u^q(\phi(t))g(\log u(\phi(t)))\phi'(t) \\
&\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} k(\phi(t),\tau)u^q(\tau)g(\log u(\tau))d\tau \right) \phi'(t) \\
&\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{\tau} h(\phi(t),\tau,\sigma)u^q(\sigma)g(\log u(\sigma))d\sigma \right) d\tau \right) \phi'(t) \\
&\leq b(\phi(t))\pi^{1/p}(\phi(t))v_3^{q/p}(\phi(t))g(\log[\pi^{1/p}(\phi(t))v_3^{q/p}(\phi(t))])\phi'(t) \\
&\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} k(\phi(t),\tau)\pi^{1/p}(\tau)v_3^{q/p}(\tau)g(\log[\pi^{1/p}(\tau)v_3^{q/p}(\tau)])d\tau \right) \phi'(t) \\
&\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{\tau} h(\phi(t),\tau,\sigma)\pi^{1/p}(\sigma)v_3^{q/p}(\sigma)g(\log[\pi^{1/p}(\sigma)v_3^{q/p}(\sigma)])d\sigma \right) d\tau \right) \phi'(t) \\
&\leq v_3^{q/p}(t) \Bigg[b(\phi(t))\pi^{1/p}(\phi(t))g(\log[\pi^{1/p}(\phi(t))v_3^{q/p}(\phi(t))])\phi'(t) \\
&\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} k(\phi(t),\tau)\pi^{1/p}(\tau)g(\log[\pi^{1/p}(\tau)v_3^{q/p}(\tau)])d\tau \right) \phi'(t) \\
&\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{\tau} h(\phi(t),\tau,\sigma)\pi^{1/p}(\sigma)g(\log[\pi^{1/p}(\sigma)v_3^{q/p}(\sigma)])d\sigma \right) d\tau \right) \phi'(t) \Bigg].
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{v_3'(t)}{v_3^{q/p}(t)} &\leq b(\phi(t))\pi^{1/p}(\phi(t))g(\log[\pi^{1/p}(\phi(t))v_3^{q/p}(\phi(t))])\phi'(t) \\
&\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} k(\phi(t),\tau)\pi^{1/p}(\tau)g(\log[\pi^{1/p}(\tau)v_3^{q/p}(\tau)])d\tau \right) \phi'(t) \\
&\quad + \left(\int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{\tau} h(\phi(t),\tau,\sigma)\pi(\sigma)g(\log[\pi^{1/p}(\sigma)v_3^{q/p}(\sigma)])d\sigma \right) d\tau \right) \phi'(t).
\end{aligned} \tag{2.17}$$

If $p=q$, setting $t=s$ in (2.17) and then integrating from α to $t, t\in I$, changing the variables and using the subadditivity of function g we get

$$\begin{aligned}\log v_3(t) \le B_1(t) &+ \int_{\phi(\alpha)}^{\phi(t)} b(s)g(\log v_3(s))ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s} k(s,\tau)g(\log v_3(\tau))d\tau\right)ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s}\left(\int_{\phi(\alpha)}^{\tau} h(s,\tau,\sigma)g(\log v_3(\sigma))d\sigma\right)d\tau\right)ds,\end{aligned}$$

where $B_1(t)$ is defined by (2.14) and obviously $B_1(t)$ is nondecreasing continuous on I. By Lemma 2.1 and Remark 2.1 we get

(2.18)
$$\begin{aligned}v_3(t) \le \exp\Bigg\{G^{-1}\Bigg[&G(B_1(t)) + \int_{\phi(\alpha)}^{\phi(t)} b(s)ds + \int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s} k(s,\tau)d\tau\right)ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s}\left(\int_{\phi(\alpha)}^{\tau} h(s,\tau,\sigma)d\sigma\right)d\tau\right)ds\Bigg]\Bigg\}\end{aligned}$$

for $t\in[\alpha, T_5]$.

If $p>q$, setting $t=s$ in (2.17) and then integrating from α to $t, t\in I$, changing the variables and using the multiplicity of function g we get

$$\begin{aligned}v_3^{\frac{p-q}{p}}(t) \le B_1(t) &+ \int_{\phi(\alpha)}^{\phi(t)} b(s)g(\frac{q}{p-q})g(\log v_3^{\frac{p-q}{p}}(s))ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s} k(s,\tau)g(\frac{q}{p-q})g(\log v_3^{\frac{p-q}{p}}(\tau))d\tau\right)ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s}\left(\int_{\phi(\alpha)}^{\tau} h(s,\tau,\sigma)g(\frac{q}{p-q})g(\log v_3^{\frac{p-q}{p}}(\sigma))d\sigma\right)d\tau\right)ds,\end{aligned}$$

where $B_1(t)$ is defined by (2.14). By Lemma 2.1 and Remark 2.1 we get

$$\begin{aligned}v_3^{\frac{p-q}{p}}(t) \le \exp\Bigg\{G^{-1}\Bigg[&G(B_1(t)) + \int_{\phi(\alpha)}^{\phi(t)} b(s)ds + \int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s} k(s,\tau)d\tau\right)ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s}\left(\int_{\phi(\alpha)}^{\tau} h(s,\tau,\sigma)d\sigma\right)d\tau\right)ds\Bigg]\Bigg\}\end{aligned}$$

or

$$v_3(t) \le \exp\Bigg\{\frac{p}{p-q}G^{-1}\Bigg[G(B_1(t)) + \int_{\phi(\alpha)}^{\phi(t)} b(s)ds + \int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s} k(s,\tau)d\tau\right)ds$$

$$+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{s}\left(\int_{\phi(\alpha)}^{\tau}h(s,\tau,\sigma)d\sigma\right)d\tau\right)ds\Bigg]\Bigg\}$$

for $t \in [\alpha, T_6]$. Now, by using (2.18) and the last inequality in (2.16), we get the desired inequalities (2.13) and (2.15), respectively. This completes the proof. □

3. Some more generalized inequalities

In this section, we give some more generalizations of the results obtained in section 2. Let $\alpha < \beta$ and set $J_i = \{(t_1, t_2, \ldots, t_i) \in R^i : \alpha \le t_i \le \cdots \le t_1 \le \beta\}$ for $i = 1, \ldots, n$.

Theorem 3.1. *Let $a, u(t), \phi(t)$ and g be as in Lemma 2.1, φ be as in Theorem 2.2. Suppose that*

(3.1)

$$\begin{aligned}&\varphi(u(t))\\ &\le a+\int_{\phi(\alpha)}^{\phi(t)}k_1(t,t_1)\varphi'(u(t_1))g(u(t_1))dt_1+\cdots\\ &+\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{t_1}\cdots\left(\int_{\phi(\alpha)}^{t_{n-1}}k_n(t,t_1,\ldots,t_n)\varphi'(u(t_n))g(u(t_n))dt_n\right)\cdots\right)dt_1\end{aligned}$$

for $t \in I$, where $k_i(t, t_1, \ldots, t_i)$ are nonnegative, continuous functions in J_{i+1} and the partial derivative $\frac{\partial k_i}{\partial t}(t, t_1, \ldots, t_i)$ exist and are nonnegative, continuous in J_{i+1} for $i = 1, 2, \ldots, n$. Then

$$u(t) \le G^{-1}\left\{G(\varphi^{-1}(a)) + \int_{\phi(\alpha)}^{\phi(t)} R[1](\tau,\tau)d\tau + \int_{\alpha}^{t} Q[1](\tau)d\tau\right\} \tag{3.2}$$

for $t \in [\alpha, T_7]$, where T_7 is chosen so that the quality in the braces of (3.2) in the range of G,

(3.3)

$$\begin{aligned}R[x](t,\hat{t}) &= k_1(t,\hat{t})x(t)+\int_{\phi(\alpha)}^{\phi(t)}k_2(t,\hat{t},t_2)x(t_2)dt_2\\ &\quad+\sum_{i=3}^{n}\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{t_2}\cdots\left(\int_{\phi(\alpha)}^{t_{i-1}}k_i(t,\hat{t},t_2,\ldots,t_i)x(t_i)dt_i\right)\cdots\right)dt_2,\\ Q[x](t) &= \int_{\phi(\alpha)}^{\phi(t)}\frac{\partial k_1}{\partial t}(t,t_1)x(t)dt_1\\ &\quad+\sum_{i=2}^{n}\int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{t_2}\cdots\left(\int_{\phi(\alpha)}^{t_{i-1}}\frac{\partial k_i}{\partial t}(t,t_1,\ldots,t_i)x(t)dt_i\right)\cdots\right)dt_1\end{aligned}$$

for any $x(t) \in C(I, I)$ and $t, \hat{t} \in I, G$ and G^{-1} are defined as in Lemma 2.1.

Proof. Define a function $v(t)$ by the right side of (3.1). Then $v(t)$ is nondecreasing continuous, $u(t) \leq \varphi^{-1}(v(t))$ for $t \in I$ and $v(\alpha) = a$. Taking derivative to $v(t)$, we have

$$\begin{aligned} v'(t) &= R[\varphi'(u)g(u)](t, \phi(t))\phi'(t) + Q[\varphi'(u)g(u)](t) \\ &\leq \Big\{R[g(\varphi^{-1}(v)](t, \phi(t))\phi'(t) + Q[g(\varphi^{-1}(v)](t)\Big\}\varphi'(\varphi^{-1}(v(t))), \end{aligned}$$

or

$$\frac{v'(t)}{\varphi'(\varphi^{-1}(v(t)))} \leq R[g(\varphi^{-1}(v)](t, \phi(t))\phi'(t) + Q[g(\varphi^{-1}(v)](t).$$

As the procedures in the proof of Theorem 2.2, setting $t = s$ in the last inequality and then integrating from α to $t, t \in I$, changing the variables to the right-hand side first integral we get

$$\varphi^{-1}(v(t)) \leq \varphi^{-1}(a) + \int_{\phi(\alpha)}^{\phi(t)} R[g(\varphi^{-1}(v)](\tau, \tau)d\tau + \int_{\alpha}^{t} Q[g(\varphi^{-1}(v)](\tau)d\tau. \tag{3.4}$$

Now, we denote the right-hand side of (3.4) by $\bar{v}(t)$. Then $\bar{v}(\alpha) = \varphi^{-1}(a)$, $\varphi^{-1}(v(t)) \leq \bar{v}(t)$, the function $\bar{v}(t)$ is nondecreasing in $t \in [\alpha, \beta]$ and

$$\bar{v}'(t) \leq \Big\{R[1](t, \phi(t))\phi'(t) + Q[1](t)\Big\}g(\bar{v}(t)).$$

Integrating the last inequality from α to $t, t \in I$ and changing the variables to the first integral we get

$$G(\bar{v}(t)) \leq G(\bar{v}(\alpha)) + \int_{\phi(\alpha)}^{\phi(t)} R[1](\tau, \tau)d\tau + \int_{\alpha}^{t} Q[1](\tau)d\tau$$

or

$$\bar{v}(t) \leq G^{-1}\left\{G(\varphi^{-1}(a)) + \int_{\phi(\alpha)}^{\phi(t)} R[1](\tau, \tau)d\tau + \int_{\alpha}^{t} Q[1](\tau)d\tau\right\} \tag{3.5}$$

for $t \in [\alpha, T]$, where T is chosen so that the quality in the braces of (3.5) in the range of G. Now the desired inequality (3.2) follows by the inequalities $u(t) \leq \varphi^{-1}(v(t)) \leq \bar{v}(t)$ and (3.5). This completes the proof. □

Let $\varphi(u) = u^p$ in Theorem 3.1, we get the following Ou-Iang type retarded integral inequality with iterated integrals immediately.

Corollary 3.2. *Let a, u, g, ϕ and $k_i (i = 1, 2, \ldots, n)$ be as in Theorem 3.1, $p \geq 1$ is a constant. Suppose that*

(3.6)

$$\begin{aligned} &u^p(t) \\ &\leq a + \int_{\phi(\alpha)}^{\phi(t)} k_1(t, t_1)u^{p-1}(t_1)g(u(t_1))dt_1 + \cdots \\ &+ \int_{\phi(\alpha)}^{\phi(t)} \left(\int_{\phi(\alpha)}^{t_1} \cdots \left(\int_{\phi(\alpha)}^{t_{n-1}} k_n(t, t_1, \ldots, t_n)u^{p-1}(t_n)g(u(t_n))dt_n\right) \cdots\right) dt_1 \end{aligned}$$

for $t \in I$. *Then*

$$u(t) \le G^{-1}\left\{G(a^{1/p}) + \frac{1}{p}\int_{\phi(\alpha)}^{\phi(t)} R[1](\tau,\tau)d\tau + \frac{1}{p}\int_{\alpha}^{t} Q[1](\tau)d\tau\right\} \tag{3.7}$$

for $t \in [\alpha, T_8]$, *where* T_8 *is chosen so that the quality in the braces of* (3.7) *in the range of* G,R, Q, G *and* G^{-1} *are defined as in Theorem 3.1.*

As similar argument as in Corollary 2.4, we can get the following interesting result from Theorem 3.1:

Corollary 3.3. *Let* g, ϕ *and* $k_i (i = 1, 2, \ldots, n)$ *be as in Theorem 3.1,* $a > 1, p \ge 0$ *be constants and* $u \in C(I, R_1)$. *Suppose that*

$$\begin{aligned} &u^p(t) \\ &\le a + \int_{\phi(\alpha)}^{\phi(t)} k_1(t,t_1)u^p(t_1)g(\log u(t_1))dt_1 + \cdots \\ &+ \int_{\phi(\alpha)}^{\phi(t)}\left(\int_{\phi(\alpha)}^{t_1} \cdots \left(\int_{\phi(\alpha)}^{t_{n-1}} k_n(t,t_1,\ldots,t_n)u^p(t_n)g(\log u(t_n))dt_n\right)\cdots\right)dt_1 \end{aligned} \tag{3.8}$$

for $t \in I$. *Then*

$$u(t) \le \exp\left(G^{-1}\left\{G(a^{1/p}) + \frac{1}{p}\int_{\phi(\alpha)}^{\phi(t)} R[1](\tau,\tau)d\tau + \frac{1}{p}\int_{\alpha}^{t} Q[1](\tau)d\tau\right\}\right) \tag{3.9}$$

for $t \in [\alpha, T_9]$, *where* T_9 *is chosen so that the quality in the braces of* (3.9) *in the range of* G, R, Q, G *and* G^{-1} *are defined as in Theorem 3.1.*

Remark 3.1. Let $p = 1, \phi(t) \equiv t$ and $g(u) = u$ in Corollary 3.3, we can get a similar result as Theorem 3.1 in [6].

Theorem 3.4. *Let* u *and* ϕ *be as in Lemma 2.1,* $a \in C^{n-1}(R_+, R_+), p_1, \ldots, p_n$, $f_1, \ldots, f_n$ *be nonnegative, continuous functions in* I. *Let* $g(u)$ *be a continuously differentiable function defined on* R_+, $g(u) > 0$ *on* $(0, \infty)$, $g'(u) \ge 0$ *for* $u \in R_+$ *and* $\varphi \in C^2(R_+, R_+)$ *with* $\varphi'(u) > 0$ *for* $u > 0$. *Suppose that*

$$\begin{aligned} &\varphi(u(t)) \\ &\le a(t) + \int_{\phi(\alpha)}^{\phi(t)} p_1(s)f_1(s)\varphi'(u(s))g(u(s))ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\int_{\phi(\alpha)}^{t_1} p_2(s)f_2(s)\varphi'(u(s))g(u(s))dsdt_1 + \cdots \\ &+ \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\left(\int_{\phi(\alpha)}^{t_1} p_2(t_2)\cdots\left(\int_{\phi(\alpha)}^{t_{n-1}} p_n(s)f_n(s)\varphi'(u(s))g(u(s))ds\cdots\right)dt_2\right)dt_1 \end{aligned} \tag{3.10}$$

for $t \in I$. Then

$$u(t) \le G^{-1}\left\{G[\varphi^{-1}(a(t))] + F(t)\right\} \tag{3.11}$$

for $t \in [\alpha, T_{10}]$, where

$$\begin{aligned} F(t) = &\int_{\phi(\alpha)}^{\phi(t)} p_1(s)f_1(s)ds + \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\int_{\phi(\alpha)}^{t_1} p_2(s)f_2(s)dsdt_1 + \cdots \\ &+ \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\left(\int_{\phi(\alpha)}^{t_1} p_2(t_2)\cdots\left(\int_{\phi(\alpha)}^{t_{n-1}} p_n(s)f_n(s)ds\cdots\right)dt_2\right)dt_1, \end{aligned} \tag{3.12}$$

T_{10} is chosen so that the quality in the braces of (3.11) in the range of G, G and G^{-1} are same as in Lemma 2.1.

Proof. Define a function $v(t)$ by the right side of (3.10). Then $v(t)$ is nondecreasing continuous, $u(t) \le \varphi^{-1}(v(t))$ for $t \in I$ and $v(\alpha) = a(t)$. Then differentiating $v(t)$ and rewriting we have

$$\frac{v'(t) - a'(t)}{\phi'(t)p_1(\phi(t))} - f_1(\phi(t))\varphi'(u)g(u) = v_1(t), \tag{3.13}$$

where

$$\begin{aligned} &v_1(t) \\ &= \int_{\phi(\alpha)}^{\phi(t)} p_2(s)f_2(s)\varphi'(u(s))g(u(s))ds + \cdots \\ &+ \int_{\phi(\alpha)}^{\phi(t)} p_2(t_2)\left(\int_{\phi(\alpha)}^{t_2} p_3(t_3)\cdots\left(\int_{\phi(\alpha)}^{t_{n-1}} p_n(s)f_n(s)\varphi'(u(s))g(u(s))ds\cdots\right)dt_3\right)dt_2. \end{aligned}$$

From the definition of $v_1(t)$ we get

$$\frac{v_1'(t)}{\phi'(t)p_2(\phi(t))} - f_2(\phi(t))\varphi'(u)g(u) = v_2(t), \tag{3.14}$$

where

$$\begin{aligned} &v_2(t) \\ &= \int_{\phi(\alpha)}^{\phi(t)} p_3(s)f_3(s)\varphi'(u(s))g(u(s))ds + \cdots \\ &+ \int_{\phi(\alpha)}^{\phi(t)} p_3(t_3)\left(\int_{\phi(\alpha)}^{t_3} p_4(t_4)\cdots\left(\int_{\phi(\alpha)}^{t_{n-1}} p_n(s)f_n(s)\varphi'(u(s))g(u(s))ds\cdots\right)dt_4\right)dt_3. \end{aligned}$$

Continuing in this way, we obtain

$$\frac{v_{n-2}'(t)}{\phi'(t)p_{n-1}(\phi(t))} - f_{n-1}(\phi(t))\varphi'(u)g(u) = v_{n-1}(t), \tag{3.15}$$

where

$$v_{n-1}(t) = \int_{\phi(\alpha)}^{\phi(t)} p_n(s)f_n(s)\varphi'(u(s))g(u(s))ds,$$

from the definition of $v_{n-1}(t)$ and using $u(t) \leq \varphi^{-1}(v(t))$ we can get

$$\frac{v'_{n-1}}{\varphi'(\varphi^{-1}(v(t)))} \leq \phi'(t)p_n(\phi(t))f_n(\phi(t))g(\varphi^{-1}(v(\phi(t)))).$$

Integrating the last inequality and changing the variable to the right-hand side integral we have

$$\int_\alpha^t \frac{v'_{n-1}}{\varphi'(\varphi^{-1}(v(s)))}ds \leq \int_{\phi(\alpha)}^{\phi(t)} p_n(s)f_n(s)g(\varphi^{-1}(v(s)))ds. \tag{3.16}$$

By integrating by parts to the left-hand side of (3.16) we obtain

$$\begin{aligned}
&\int_\alpha^t \frac{v'_{n-1}(s)}{\varphi'(\varphi^{-1}(v(s)))}ds \\
&= \frac{v_{n-1}}{\varphi'(\varphi^{-1}(v(t)))} \\
&\quad + \int_\alpha^t \frac{v_{n-1}(s)}{\varphi'^2(\varphi^{-1}(v(s)))} \times \frac{\varphi''(\varphi^{-1}(v(s)))}{\varphi'((v(s)))} \times v'(s)ds \\
&\geq \frac{v_{n-1}}{\varphi'(\varphi^{-1}(v(t)))}.
\end{aligned} \tag{3.17}$$

From (3.16) and (3.17) we have

$$\frac{v_{n-1}}{\varphi'(\varphi^{-1}(v(t)))} \leq \int_{\phi(\alpha)}^{\phi(t)} p_n(s)f_n(s)g(\varphi^{-1}(v(s)))ds. \tag{3.18}$$

Now by (3.15) and (3.18) we observe that

$$\begin{aligned}
&\int_\alpha^t \frac{v'_{n-2}(s)}{\varphi'(\varphi^{-1}(v(s)))}ds \\
&\leq \frac{v'_{n-1}(t)}{\varphi'(\varphi^{-1}(v(t)))}p_{n-1}(\phi(t))\phi'(t) \\
&\quad + p_{n-1}(\phi(t))f_{n-1}(\phi(t))g(\varphi^{-1}(v(\phi(t))))\phi'(t).
\end{aligned}$$

Using the same procedures from (3.16) to (3.17) to the last inequality we have

$$\begin{aligned}
\frac{v_{n-2}}{\varphi'(\varphi^{-1}(v(t)))} \leq &\int_{\phi(\alpha)}^{\phi(t)} p_{n-1}(s)f_{n-1}(s)g(\varphi^{-1}(v(s)))ds \\
&+ \int_{\phi(\alpha)}^{\phi(t)} p_{n-1}(t_1)\int_{\phi(\alpha)}^{t_1} p_n(s)f_n(s)g(\varphi^{-1}(v(s)))dsdt_1.
\end{aligned}$$

Proceeding in this way we can get
(3.19)

$$\begin{aligned}&\frac{v_1}{\varphi'(\varphi^{-1}(v(t)))}\\ &\leq \int_{\phi(\alpha)}^{\phi(t)} p_2(s)f_2(s)g(\varphi^{-1}(v(s)))ds + \cdots\\ &+\int_{\phi(\alpha)}^{\phi(t)} p_2(t_2)\left(\int_{\phi(\alpha)}^{t_2} p_3(t_3)\cdots\left(\int_{\phi(\alpha)}^{t_{n-1}} p_n(s)f_n(s)g(\varphi^{-1}(v(s)))ds\cdots\right)dt_3\right)dt_2.\end{aligned}$$

From (3.19), (3.13) and using $u(t) \leq \varphi^{-1}(v(t))$, we observe that

$$\begin{aligned}&\frac{v'}{\varphi'(\varphi^{-1}(v(t)))} - \frac{a'}{\varphi'(\varphi^{-1}(a(t)))}\\ &\leq \frac{v'}{\varphi'(\varphi^{-1}(v(t)))} - \frac{a'}{\varphi'(\varphi^{-1}(v(t)))}\\ &= p_1(\phi(t))f_1(\phi(t))\varphi'(u)g(u)\phi'(t) + \frac{v_1}{\varphi'(\varphi^{-1}(v(t)))}p_1(\phi(t))\phi'(t).\end{aligned}\tag{3.20}$$

Integrating (3.20) from α to t, and then using (2.7) to the left-hand side integrals and using (3.19) to the right-hand side integrals and changing variables we can get
(3.21)

$$\begin{aligned}&\varphi^{-1}(v(t)) - \varphi^{-1}(a(t))\\ &\leq \int_{\phi(\alpha)}^{\phi(t)} p_1(s)f_1(s)g(\varphi^{-1}(v(s)))ds\\ &+\int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\int_{\phi(\alpha)}^{t_1} p_2(s)f_2(s)g(\varphi^{-1}(v(s)))dsdt_1 + \cdots\\ &+\int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\left(\int_{\phi(\alpha)}^{t_1} p_2(t_2)\cdots\left(\int_{\phi(\alpha)}^{t_{n-1}} p_n(s)f_n(s)g(\varphi^{-1}(v(s)))ds\cdots\right)dt_2\right)dt_1.\end{aligned}$$

Denoting $z(t) = \varphi^{-1}(v(t))$ and $l(t) = \varphi^{-1}(a(t))$, from (3.21) we have
(3.22)

$$\begin{aligned}z(t) \leq l(t) &+ \int_{\phi(\alpha)}^{\phi(t)} p_1(s)f_1(s)g(z(s))ds\\ &+\int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\int_{\phi(\alpha)}^{t_1} p_2(s)f_2(s)g(z(s))dsdt_1 + \cdots\\ &+\int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\left(\int_{\phi(\alpha)}^{t_1} p_2(t_2)\cdots\left(\int_{\phi(\alpha)}^{t_{n-1}} p_n(s)f_n(s)g(z(s))ds\cdots\right)dt_2\right)dt_1.\end{aligned}$$

Now, let $V(t)$ be the right-hand side of (3.22), then $z(t) \le V(t)$, using same steps as used from (3.13) to (3.20) we can get

(3.23)
$$\frac{V_1}{g(V(t))} \le \int_{\phi(\alpha)}^{\phi(t)} p_2(s)f_2(s)ds + \cdots + \int_{\phi(\alpha)}^{\phi(t)} p_2(t_2)\left(\int_{\phi(\alpha)}^{t_2} p_3(t_3)\cdots\left(\int_{\phi(\alpha)}^{t_{n-1}} p_n(s)f_n(s)ds\cdots\right)dt_3\right)dt_2.$$

and

$$\begin{aligned}&\frac{V'}{g(V(t))} - \frac{l'}{g(l(t))}\\ &\le \frac{V'}{g(V(t))} - \frac{l'}{g(V(t))}\\ &= p_1(\phi(t))f_1(\phi(t))\phi'(t) + \frac{V_1}{g(V(t))}p_1(\phi(t))\phi'(t),\end{aligned} \tag{3.24}$$

where

$$V_1(t) = \frac{V'(t) - l'(t)}{\phi'(t)p_1(\phi(t))} - f_1(\phi(t))g(z). \tag{3.25}$$

Integrating (3.24) from α to t, using the definition of G and $z(t) \le V(t)$ we obtain

$$\begin{aligned}&G(V(t)) - G(l(t))\\ &\le \int_{\phi(\alpha)}^{\phi(t)} p_1(s)f_1(s)ds + \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\int_{\phi(\alpha)}^{t_1} p_2(s)f_2(s)dsdt_1 + \cdots\\ &\quad + \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\left(\int_{\phi(\alpha)}^{t_1} p_2(t_2)\cdots\left(\int_{\phi(\alpha)}^{t_{n-1}} p_n(s)f_n(s)ds\cdots\right)dt_2\right)dt_1\end{aligned}$$

or

(3.26)
$$\begin{aligned}&V(t)\\ &\le G^{-1}\left\{G(l(t)) + \int_{\phi(\alpha)}^{\phi(t)} p_1(s)f_1(s)ds + \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\int_{\phi(\alpha)}^{t_1} p_2(s)f_2(s)dsdt_1 + \cdots\right.\\ &\quad \left.+ \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1)\left(\int_{\phi(\alpha)}^{t_1} p_2(t_2)\cdots\left(\int_{\phi(\alpha)}^{t_{n-1}} p_n(s)f_n(s)ds\cdots\right)dt_2\right)dt_1\right\}\end{aligned}$$

for $t \in [\alpha, T_{10}]$, T_{10} is chosen so that the quality in the braces of (3.26) in the range of G. Now from $u(t) \le \varphi^{-1}(v(t)) = z(t) \le V(t)$ and (3.26) we can get the desired inequality (3.11). This completed the proof. □

Remark 3.2. (i) When $\varphi(u) = u, \phi(t) = t$ in Theorem 3.4, we can get Theorem 1 which was given by Medveď [18]; (ii) When $\varphi(u) = u$, a slightly different version of Theorem 3.4 was also given in [18]; (iii) When $f_1 = f_2 = \cdots = f_{n-2} = 0, \phi(t) = t$, form Theorem 3.4 we can get Theorem 2.1 given in [14].

By a similar argument as in the proof of Corollary 2.4, we have an interesting result as follows:

Corollary 3.5. *Let ϕ, p_i, and $f_i (i = 1, 2, \ldots, n)$ be as in Theorem 3.4, $u \in C(R_+, R_1)$, $a \in C^1(R_+, R_1)$ and $p > 0$ be a constant. Suppose that*

(3.27)

$$\begin{aligned} u^p(t) &\le a(t) + \int_{\phi(\alpha)}^{\phi(t)} p_1(s) f_1(s) u^p(s) g(\log u(s)) ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1) \int_{\phi(\alpha)}^{t_1} p_2(s) f_2(s) u^p(s) g(\log u(s)) ds dt_1 + \cdots \\ &+ \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1) \left(\int_{\phi(\alpha)}^{t_1} p_2(t_2) \cdots \left(\int_{\phi(\alpha)}^{t_{n-1}} p_n(s) f_n(s) u^p(s) g(\log u(s)) ds \cdots \right) dt_2 \right) dt_1 \end{aligned}$$

for $t \in I$, then

$$u(t) \le \exp\left(G^{-1} \left\{ G[\frac{1}{p} \log a(t))] + F_p(t) \right\} \right), \tag{3.28}$$

for $t \in [\alpha, T_{10}]$, where T_{10} is chosen so that the quality in the braces of (3.28) *in the range of G, G and G^{-1} are same as in Lemma 2.1, $F_p(t) = \frac{1}{p} F(t), F(t)$ is defined as in* (3.12).

Let $p = 1, g(u) = u$ in Corollary 3.5 we have

Corollary 3.6. *Let u, a, ϕ, p_i, and $f_i (i = 1, 2, \ldots, n)$ be as in Corollary 3.5. Suppose that*

(3.29)

$$\begin{aligned} u(t) &\le a(t) + \int_{\phi(\alpha)}^{\phi(t)} p_1(s) f_1(s) u(s) \log u(s) ds \\ &+ \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1) \int_{\phi(\alpha)}^{t_1} p_2(s) f_2(s) u(s)) \log u(s) ds dt_1 + \cdots \\ &+ \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1) \left(\int_{\phi(\alpha)}^{t_1} p_2(t_2) \cdots \left(\int_{\phi(\alpha)}^{t_{n-1}} p_n(s) f_n(s) u(s) \log u(s) ds \cdots \right) dt_2 \right) dt_1 \end{aligned}$$

for $t \in I$. Then

$$u(t) \le (a(t))^{\exp(F(t))} \tag{3.30}$$

for $t \in I$, where $F(t)$ is defined in (3.12).

Remark 3.3. When $a(t) = a$ (a is a constant) $f_i = 1 (i = 1, 2, \ldots, n)$ and $\phi(t) = t$ in Corollary 3.6, [6] established a explicit bound of the solutions of (3.29), which is not only different to the bound given in (3.30) but also is valid only on a subset of I.

4. Applications

(A) Consider the following retarded integro-differential equation

$$(h(t)x')' = F\left(t, x(t-\tau(t)), \int_\alpha^t G(\sigma, x(\sigma - \tau(\sigma)))d\sigma\right) \tag{4.1}$$

for $t \in I$, where h is positive continuous on I, $\tau \in C^1(I, I)$ is nonincreasing with $t - \tau(t) \geq 0, \tau(\alpha) = 0, t - \tau(t) \in C^1(I, I), \tau'(t) < 1, F \in C(I \times R \times R, R), G \in C(I \times R, R)$.

Theorem 4.1. *Assume that the conditions*

$$|F(t, u, v)| \leq b_1(t)|u|g(\log |u|) + b_2(t)|v|, \tag{4.2}$$

$$|G(s, v)| \leq b_3(s)|v|g(\log |v|), \tag{4.3}$$

are satisfied, where $b_i(t) \in C(I, R_+)(i = 1, 2, 3)$ *and function* g *is defined in Lemma 2.1. If* $x(t)$ *is any solution of* (4.1), *then*

$$\begin{aligned} |x(t)| \leq \exp\Bigg(G^{-1}\Bigg[G(\log a) &+ \int_\alpha^{t-\tau(t)} MH(t, \eta + \tau(s))b_1(\eta + \tau(s))d\eta \\ &+ \int_\alpha^{t-\tau(t)} \int_\alpha^\eta M^2 H(t, \eta + \tau(s))b_2(\eta + \tau(s))b_3(\zeta + \tau(\xi))d\zeta d\eta \Bigg]\Bigg) - 1 \end{aligned} \tag{4.4}$$

for $t, s, \xi \in [\alpha, T_1^*]$, *where* $M = \max_{t \in I} \dfrac{1}{1 - \tau'(t)}, T_1^* \in I$ *is chosen so that the quantity in the square brackets of* (4.4) *is in the range of* G, G *and* G^{-1} *are as in Lemma 2.1.*

Proof. Let $x(t)$ be any solution of (4.1) which is passing through the initial point $(x(\alpha), x'(\alpha))$. Integrating the both side of (4.1) two times and using the well-known Dirichlet formula we obtain

$$\begin{aligned} x(t) &= x(\alpha) + h(\alpha)x'(\alpha)H(t, \alpha) \\ &+ \int_\alpha^t H(t, s)F\left[s, x(s - \tau(s)), \int_\alpha^s G(\sigma, x(\sigma - \tau(\sigma)))d\sigma\right] ds, \end{aligned} \tag{4.5}$$

where $H(t,s)=\int_s^t \frac{1}{h(u)}du, t\geq s\geq \alpha$. Applying (4.2) and (4.3) to (4.5) and making the change of variables we obtain

(4.6)

$$\begin{aligned}
&|x(t)|\\
&\leq l(t)+\int_\alpha^t H(t,s)b_1(s)|x(s-\tau(s))|g(\log|x(s-\tau(s))|ds\\
&\quad+\int_\alpha^t\int_\alpha^s H(t,s)b_2(s)b_3(\xi)|x(\xi-\tau(\xi))|g(\log|x(\xi-\tau(\xi))|d\xi ds\\
&\leq l(t)+\int_\alpha^{t-\tau(t)} MH(t,\eta+\tau(s))b_1(\eta+\tau(s))|x(\eta)|g(\log|x(\eta)|d\eta\\
&\quad+\int_\alpha^{t-\tau(t)}\int_\alpha^\eta M^2H(t,\eta+\tau(s))b_2(\eta+\tau(s))b_3(\zeta+\tau(\xi))|x(\zeta)|g(\log|x(\zeta)|)d\zeta d\eta
\end{aligned}$$

for $t\in I$, where $l(t)=1+|x(\alpha)|+h(\alpha)H(t,\alpha)|x'(\alpha)|$. Obviously, function $H(t,s)$ is nondecreasing in variable t for every s fixed. Fix $T\in[\alpha,T_1^*]$, then for $\alpha\leq t\leq T$, from (4.6) we have

$$\begin{aligned}
&|x(t)|+1\\
&\leq l(t)+\int_\alpha^{t-\tau(t)} MH(T,\eta+\tau(s))b_1(\eta+\tau(s))(|x(\eta)|+1)g(\log(|x(\eta)|+1))d\eta\\
&\quad+\int_\alpha^{t-\tau(t)}\int_\alpha^\eta M^2H(T,\eta+\tau(s))b_2(\eta+\tau(s))b_3(\zeta+\tau(\xi))(|x(\zeta)|+1)g(\log(|x(\zeta)|+1))d\zeta d\eta.
\end{aligned}$$

If $l(t)\leq a$ (a is a constant), an application of Corollary 2.4 to the last inequality yields

$$\begin{aligned}
|x(t)|\leq\ &\exp\Bigg(G^{-1}\Bigg[G(\log a)+\int_\alpha^{t-\tau(t)} MH(T,\eta+\tau(s))b_1(\eta+\tau(s))d\eta\\
&+\int_\alpha^{t-\tau(t)}\int_\alpha^\eta M^2H(T,\eta+\tau(s))b_2(\eta+\tau(s))b_3(\zeta+\tau(\xi))d\zeta d\eta\Bigg]\Bigg)-1
\end{aligned}$$

for $t\in[\alpha,T]$. Setting $t=T$ in the last inequality we get

(4.7)

$$\begin{aligned}
|x(T)|\leq\ &\exp\Bigg(G^{-1}\Bigg[G(\log a)+\int_\alpha^{T-\tau(T)} MH(T,\eta+\tau(s))b_1(\eta+\tau(s))d\eta\\
&+\int_\alpha^{T-\tau(T)}\int_\alpha^\eta M^2H(T,\eta+\tau(s))b_2(\eta+\tau(s))b_3(\zeta+\tau(\xi))d\zeta d\eta\Bigg]\Bigg)-1.
\end{aligned}$$

Since T is arbitrary, from (4.7) with T replaced by t we have

$$|x(t)| \leq \exp\left(G^{-1}\left[G(\log a) + \int_\alpha^{t-\tau(t)} MH(t,\eta+\tau(s))b_1(\eta+\tau(s))d\eta \right.\right.$$
$$\left.\left. + \int_\alpha^{t-\tau(t)}\int_\alpha^{\eta} M^2H(t,\eta+\tau(s))b_2(\eta+\tau(s))b_3(\zeta+\tau(\xi))d\zeta d\eta\right]\right) - 1,$$

which is the desired inequality (4.4). □

(B) Consider the retarded differential equation

$$D_n x(t) = F(t, x(t-\tau(t))),\ t \in I, \tag{4.8}$$

$$D_i x(\alpha) = C_{i-1},\ i = 1, 2, \ldots, n, \tag{4.9}$$

where

$$D_0 x(t) = x(t),\ D_i x(t) = \frac{1}{p_i(t)}\frac{d}{dt}(D_{i-1}x(t)),\ 1 \leq i \leq n,$$

with $p_n(t) = 1, p_i(t) > 0 (i = 1, 2, \ldots, n-1)$ are some continuous functions defined on I; $C_{i-1}, 1 \leq i \leq n$ are constants; function τ is defined as in Theorem 4.1. For convenience, for $t \in I$ and some continuous functions $q_j(t) > 0, j = 1, 2, \ldots, n-1$ and $q \geq 0$ defined on I, we define
(4.10)

$$N[t, q_1, q_2, \ldots, q_{n-1}] = \int_\alpha^t q_1(t_1)\int_\alpha^{t_1} q_2(t_2)\cdots\int_\alpha^{t_{n-1}} q(t_n)dt_n dt_{n-1}\cdots dt_1,$$

where $t_0 = t$.

Theorem 4.2. *Assume that the condition*

$$|F(t,u)| \leq f(t)|u|g(\log|u|) \tag{4.11}$$

is satisfied, where $f(t) \in C(I, R_+), F \in C(I \times R, R)$ *and function* g *is defined in Lemma 2.1. If* $x(t)$ *is any solution of* (4.1), *then*

$$|x(t)| \leq \exp\left(G^{-1}\{G[\log(1+|b(t)|)] + \widetilde{N}(t)\}\right) - 1, \tag{4.12}$$

for $t, t_1, \ldots, t_{n-1} \in [\alpha, T_2^*]$, *where*

$$\widetilde{N}(t) = M^{n-1}\int_\alpha^{t-\tau(t)} q_1(s_1+\tau(t_1))\int_\alpha^{s_1} q_2(s_2+\tau(t_2))\cdots$$
$$\times\int_\alpha^{s_{n-2}} q_{n-1}(s_{n-1}+\tau(t_{n-1}))\int_\alpha^{s_{n-1}} f(s_n+\tau(t_n))ds_n ds_{n-1}\cdots ds_1,$$

T_2^* *is chosen so that the quality in the braces of* (4.12) *in the range of* G, G *and* G^{-1} *are same as in Lemma 2.1.*

Proof. Let $x(t)$ be any solution of (4.8) with initial value (4.9). Then it is easy to observe that $x(t)$ satisfies

$$x(t) = b(t) + N[t, p_1, p_2, \ldots, p_{n-1}, F(t, x(t-\tau(t)))],\ t \in I, \tag{4.13}$$

where $b(t) = C_0 + \sum_{i=1}^{n-1} C_i N[t, p_1, \ldots, p_i]$. Using the condition (4.11) to (4.13) and changing the variables to $N[t, p_1, p_2, \ldots, p_{n-1}, F(t, x(t-\tau(t)))]$ we can get

$$\begin{aligned}
&|x(t)| + 1 \\
&\le 1 + |b(t)| + N[t, p_1, p_2, \ldots, |F(t, x(t-\tau(t)))|] \\
&\le 1 + |b(t)| + N[t, p_1, p_2, \ldots, f(t)(|x(t-\tau(t)| + 1) g(\log|x(t-\tau(t))| + 1))] \\
&\le 1 + |b(t)| + N[t-\tau(t), \tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{n-1}, \tilde{f} \cdot (|x| + 1) g(\log(|x| + 1))],
\end{aligned}$$

where

$$\begin{aligned}
&N[t-\tau(t), \tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{n-1}, \tilde{f} \cdot (|x| + 1) g(\log(|x| + 1))] \\
&= M^{n-1} \int_\alpha^{t-\tau(t)} q_1(s_1 + \tau(t_1)) \int_\alpha^{s_1} q_2(s_2 + \tau(t_2)) \cdots \int_\alpha^{s_{n-2}} q_{n-1}(s_{n-1} + \tau(t_{n-1})) \\
&\quad \times \int_\alpha^{s_{n-1}} f(s_n + \tau(t_n))(|x(s_n)| + 1) g(\log(|x(s_n)| + 1)) ds_n ds_{n-1} \cdots ds_1.
\end{aligned}$$

An application of Corollary 3.5 to the last inequality yields the desired inequality (4.12). □

Obviously, under our assumptions to equations (4.1) and (4.8)-(4.9), a sufficient condition for boundedness of all solutions defined on suitable intervals is that the integrals in (4.4) and (4.12) are boundedness.

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