APPROXIMATELY C^*-INNER PRODUCT PRESERVING MAPPINGS

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ABSTRACT. A mapping $f : \mathcal{M} \to \mathcal{N}$ between Hilbert C^*-modules approximately preserves the inner product if

$$\|\langle f(x), f(y) \rangle - \langle x, y \rangle\| \leq \varphi(x, y)$$

for an appropriate control function $\varphi(x, y)$ and all $x, y \in \mathcal{M}$. In this paper, we extend some results concerning the stability of the orthogonality equation to the framework of Hilbert C^*-modules on more general restricted domains. In particular, we investigate some asymptotic behavior and the Hyers–Ulam–Rassias stability of the orthogonality equation.

1. Introduction and preliminaries

The notion of Hilbert C^*-module can be regarded as a generalization of the concepts of Hilbert space and fibre bundle. Hilbert C^*-modules were first studied by I. Kaplansky [15] for commutative C^*-algebras and later by M. A. Rieffel [25] and W. L. Paschke [20] for more general C^*-algebras. These objects are useful tools in many areas such as AW^*-algebra theory, theory of operator algebras, operator K-theory, group representation theory, noncommutative geometry, locally compact quantum groups, and theory of operator spaces; see [17] and references therein.

Suppose that $\mathcal{A}$ is a C^*-algebra and $\mathcal{M}$ is a linear space which is an algebraic left $\mathcal{A}$-module with a compatible scalar multiplication, i.e., $\lambda(ax) = a(\lambda x) = (\lambda a)x$ for $x \in \mathcal{M}, a \in \mathcal{A}, \lambda \in \mathbb{C}$. The space $\mathcal{M}$ is called a pre-Hilbert $\mathcal{A}$-module (or an inner product $\mathcal{A}$-module) if there exists an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ with the following properties :

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(ii) $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$;

(iii) $\langle ax, y \rangle = a \langle x, y \rangle$;

(iv) $\langle x, y \rangle^* = \langle y, x \rangle$ for all $x, y, z \in \mathcal{M}, a \in \mathcal{A}, \lambda \in \mathbb{C}$.

Note that the condition (i) is understood as a statement in the C^*-algebra $\mathcal{A}$, where an element a is called positive if it can be represented as bb^* for some $b \in \mathcal{A}$. The conditions (ii) and (iv) implies the inner product to be conjugate-linear in its second variable. From the validity of a useful version of the classical Cauchy-Schwartz inequality it follows that $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ is a norm on $\mathcal{M}$ making it into a normed left $\mathcal{A}$-module. The pre-Hilbert module $\mathcal{M}$ is called a Hilbert $\mathcal{A}$-module if it is complete with respect to the above norm. Some interesting examples are the usual Hilbert spaces as Hilbert $\mathbb{C}$-modules, and any C^*-algebra $\mathcal{A}$ as a Hilbert $\mathcal{A}$-module via $\langle a, b \rangle = ab^*$ $(a, b \in \mathcal{A})$. Notice that the inner structure of a C^*-algebra is essentially more complicated than complex numbers, hence the notions such as orthogonality and theorems such as Riesz' representation in the Hilbert space theory cannot simply be generalized or transferred to the theory of Hilbert C^*-modules.

One may define an "$\mathcal{A}$-valued norm" $|\cdot|$ by $|x| = \langle x, x \rangle^{1/2}$. Clearly, $\| |x| \| = \|x\|$, for each $x \in \mathcal{M}$. It is known that $|\cdot|$ does not satisfy the triangle inequality in general; cf. [16].

Roughly speaking, a functional equation $(\mathcal{E})$ is *stable* if any mapping which approximately satisfies the equation $(\mathcal{E})$ is near to an exact solution of $(\mathcal{E})$. The equation $(\mathcal{E})$ is called *superstable* if any approximate solution of $(\mathcal{E})$ is, in fact, an exact solution.

In 1940 S. M. Ulam [26] posed a stability problem concerning the stability of group homomorphisms. In the next year, D. H. Hyers [9] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. Hyers' theorem was generalized for approximate additive mappings by T. Aoki [2] and for approximate linear mappings by Th. M. Rassias [21] by considering an unbounded Cauchy difference. The result of Th. M. Rassias has provided a lot of influence in the development of what we now call Hyers–Ulam–Rassias stability of functional equations. A number of approaches in the theory, methods and problems of the stability of functional equations have been published since the year 1940. The reader is referred to the monographs [7, 10, 14] as well as to the extensive research survey papers of [8, 12, 22] and the references cited therein. In particular, several stability results have been obtained for various equations for mappings on Hilbert C^*-modules, see [1].

A mapping $I : \mathcal{M} \to \mathcal{N}$ between Hilbert C^*-modules preserves the inner product if it is a solution of the orthogonality equation

$$\langle I(x), I(y) \rangle = \langle x, y \rangle.$$

A mapping I preserves the inner product if and only if it is both $\mathcal{A}$-linear (i.e.,

$$I(ax + \lambda y + z) = aI(x) + \lambda I(y) + I(z)$$

for all $a \in \mathcal{A}$, $x, y, z \in \mathcal{M}$, $\lambda \in \mathbb{C}$) and isometry (i.e., $\|I(x) - I(y)\| = \|x - y\|$, for all $x, y \in \mathcal{M}$). In fact, by the polarization law in Hilbert C^*-modules, if I is $\mathcal{A}$-linear and isometry, then it preserves the inner product (see [16, Theorem 3.5]). For see the converse, fix $x, y \in \mathcal{M}$ and let $u = I(x + y) - Ix - Iy$. We

have

$$\begin{aligned} \langle u, Iz \rangle &= \langle I(x+y), Iz \rangle - \langle Ix, Iz \rangle - \langle Iy, Iz \rangle \\ &= \langle x+y, z \rangle - \langle x, z \rangle - \langle y, z \rangle \\ &= 0 \end{aligned}$$

for all $z \in \mathcal{M}$. Hence

$$\langle u, u \rangle = \langle u, I(x+y) \rangle - \langle u, Ix \rangle - \langle u, Iy \rangle = 0,$$

and so $u = 0$. Therefore I is additive. Similarly, one can show that I is $\mathcal{A}$-linear. Since I preserves the inner product, it clearly is isometry.

A mapping $f : \mathcal{M} \to \mathcal{N}$ approximately preserves the inner product if it satisfies

$$\| \langle f(x), f(y) \rangle - \langle x, y \rangle \| \leq \varphi(x, y)$$

for some appropriate control function $\varphi(x, y)$ and all $x, y \in \mathcal{M}$.

Recently, the stability of the orthogonality equation (as well as of the so-called Wigner equation $|\langle f(x), f(y) \rangle| = |\langle x, y \rangle|$) has been studied in the framework of real and complex Hilbert spaces; see e.g. [4, 5, 6, 23, 24] and the Chapter (IX) of [10]. Another related work is [3] where n-inner product preserving mappings are investigated.

We generalize the main results of [4, 6] concerning the stability of orthogonality spaces to Hilbert C^*-modules, prove the stability on a general restricted domain, investigate some asymptotic aspects and prove the generalized stability of the orthogonality equation.

Throughout the paper, $\mathcal{M}$ and $\mathcal{N}$ denote a pre-Hilbert module and a Hilbert module over a C^*-algebra $\mathcal{A}$, respectively. In addition, we denote by $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}$ the set of positive integers, non-negative integers and real numbers, respectively. We refer the reader to [19] for undefined notions on C^*-algebra theory and to [16, 17] for more information on Hilbert C^*-modules.

2. Stability on restricted domains

Let D be a subset of $\mathcal{M} \times \mathcal{M}$ containing $\Delta \times \Delta$, where $\Delta = \{x \in \mathcal{M} : (x, x) \in D\}$, and suppose that there exists a positive number $c \neq 1$ such that:
  (i) for all $(x, y) \in D$ and all $m, n \in \mathbb{N}_0$, we have $(c^{-n}x, c^{-m}y) \in D$;
  (ii) for all $x, y \in \mathcal{M} \setminus \{0\}$ there are nonnegative integers m, n with

$$(c^{-n}x, c^{-m}y) \in D.$$

For instance, D can be chosen to be $\mathcal{M} \times \mathcal{M}$, $\{x \in \mathcal{M} : \|x\| \leq d\} \times \{x \in \mathcal{M} : \|x\| \leq d\}$ or $\{x \in \mathcal{M} : \|x\| \geq d\} \times \{x \in \mathcal{M} : \|x\| \geq d\}$, where d is a positive number.

Using some ideas from [4, 6], we are going to extend their main results not only to more general domains but also to a more general framework.

Theorem 2.1. *Consider a function* $\varphi : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ *satisfying*

$$\lim_{m+n \to \infty} c^{m+n} \varphi(c^{-m}x, c^{-n}y) = 0, \qquad (x, y) \in D.$$

Let $f : \mathcal{M} \to \mathcal{N}$ *be a mapping such that*

$$(2.1) \qquad \|\langle f(x), f(y) \rangle - \langle x, y \rangle\| \le \varphi(x, y), \qquad (x, y) \in D.$$

Then there exist a unique $\mathcal{A}$*-linear isometry* $I : \mathcal{M} \to \mathcal{N}$ *and a mapping* $T : \mathcal{M} \to \mathcal{N}$ *such that*

$$f(x) = I(x) + T(x),$$
$$\|f(x) - I(x)\| \le \sqrt{\varphi(x, x)},$$
$$\langle T(x), I(y) \rangle = 0$$

for all $x, y \in \Delta$.

Proof. For the sake of convenience, we introduce the functions $f_n : \mathcal{M} \to \mathcal{N}$ by $f_n(x) = c^n f(c^{-n}x)$ for any $n \in \mathbb{N}_0$. Evidently, $f_0 = f$. Recall that if a is an element of the C^*-algebra $\mathcal{A}$, then the real part $\text{Re}(a)$ of a is defined to be $\frac{a + a^*}{2}$. We have also $\|\text{Re}(a)\| \le \|a\|$.

Let $x \in \Delta$ and $m, n \in \mathbb{N}_0$. We have

$$\begin{aligned}
\|\text{Re}(\langle f_n(x), f_m(x) \rangle) - \langle x, x \rangle\| &= \|\text{Re}(\langle f_n(x), f_m(x) \rangle - \langle x, x \rangle)\| \\
&\le \|\langle f_n(x), f_m(x) \rangle - \langle x, x \rangle\| \\
&= c^{n+m} \|\langle f(c^{-n}x), f(c^{-m}x) \rangle - \langle c^{-n}x, c^{-m}x \rangle\| \\
&\le c^{n+m} \varphi(c^{-n}x, c^{-m}x),
\end{aligned}$$

whence

$$\begin{aligned}
&\|f_n(x) - f_m(x)\|^2 \\
=& \;\| \, |f_n(x) - f_m(x)| \, \|^2 \\
=& \;\| \, |f_n(x) - f_m(x)|^2 \, \| \\
=& \;\|\langle f_n(x) - f_m(x), f_n(x) - f_m(x) \rangle\| \\
\le& \;\| \, |f_n(x)|^2 + |f_m(x)|^2 - 2\text{Re}(\langle f_n(x), f_m(x) \rangle) \, \| \\
\le& \;\| \, |f_n(x)|^2 - |x|^2 \| + \| \, |f_m(x)|^2 - |x|^2 \| \\
&+ 2\|\text{Re}(\langle f_n(x), f_m(x) \rangle) - \langle x, x \rangle\| \\
(2.2) \quad \le& \;c^{2n}\varphi(c^{-n}x, c^{-n}x) + c^{2m}\varphi(c^{-m}x, c^{-m}x) + 2c^{n+m}\varphi(c^{-n}x, c^{-m}x).
\end{aligned}$$

Thus the sequence $\{f_n(x)\}$ is a Cauchy one in the complete space $\mathcal{N}$, whence it is convergent. Set

$$I_*(x) := \lim_{n \to \infty} f_n(x), \qquad x \in \Delta.$$

Let $(x, y) \in \Delta \times \Delta \subset D$. Then

$$\|\langle f_n(x), f_n(y) \rangle - \langle x, y \rangle\| \le c^{2n}\varphi(c^{-n}x, c^{-n}y)$$

for all n. Letting $n \to \infty$ we get

$$\langle I_*(x), I_*(y) \rangle = \langle x, y \rangle.$$

Putting $m = 0$ in (2.2) we get

$$\|f_n(x) - f(x)\|^2 \leq c^{2n}\varphi(c^{-n}x, c^{-n}x) + \varphi(x, x) + 2c^n\varphi(c^{-n}x, x)$$

from which we conclude that

$$(2.3) \qquad \|I_*(x) - f(x)\| \leq \sqrt{\varphi(x, x)}, \qquad x \in \Delta.$$

Let us define the mapping $I : \mathcal{M} \to \mathcal{N}$ as

$$I(x) := \begin{cases} c^{n(x)} I_*(c^{-n(x)}x), & x \in \mathcal{M} \setminus \{0\}; \\ 0, & x = 0, \end{cases}$$

where $n(x) = \min\{n \in \mathbb{N}_0 : c^{-n}x \in \Delta\}$. Note that if x is a non-zero element in $\mathcal{M}$, then $(c^{-n}x, c^{-m}x) \in D$ for some n, m. If $k = \max\{m, n\}$, then $(c^{-k}x, c^{-k}x) \in D$ and so $c^{-k}x \in \Delta$. Hence I is well-defined. If $x \in \Delta$, then $n(x) = 0$ and so $I(x) = I_*(x)$. It follows then from (2.3) that

$$(2.4) \qquad \|I(x) - f(x)\| \leq \sqrt{\varphi(x, x)}, \qquad x \in \Delta.$$

We are going to prove that I is an inner product preserving mapping and so it is an isometry. To see this, assume that $x, y \in \mathcal{M}$. If $x = 0$ or $y = 0$, then $\langle I(x), I(y) \rangle = 0 = \langle x, y \rangle$. Let $x \neq 0$ and $y \neq 0$. Then

$$\begin{aligned} \langle I(x), I(y) \rangle &= \langle c^{n(x)} I_*(c^{-n(x)}x), c^{n(y)} I_*(c^{-n(y)}y) \rangle \\ &= c^{n(x)+n(y)} \langle I_*(c^{-n(x)}x), I_*(c^{-n(y)}y) \rangle \\ &= c^{n(x)+n(y)} \langle c^{-n(x)}x, c^{-n(y)}y \rangle \\ &= \langle x, y \rangle. \end{aligned}$$

For proving the uniqueness assertion, consider inner product preserving mappings I_1, I_2 satisfying $\|I_j(x) - f(x)\| \leq \sqrt{\varphi(x, x)}$ $(j = 0, 1)$ for all $x \in \Delta$. First note that for each $x \in \Delta$ and all $n \in \mathbb{N}_0$ we have

$$\begin{aligned} \|I_1(x) - I_2(x)\| &= c^n \|I_1(c^{-n}x) - I_2(c^{-n}x)\| \\ &\leq c^n \|I_1(c^{-n}x) - f(c^{-n}x)\| + c^n \|I_2(c^{-n}x) - f(c^{-n}x)\| \\ &\leq 2\sqrt{c^{2n}\varphi(c^{-n}x, c^{-n}x)}, \end{aligned}$$

whence $I_1(x) = I_2(x)$ on Δ. Now for each $x \in \mathcal{M}$, there exists $n(x) \in \mathbb{N}_0$ such that $c^{-n(x)}x \in \Delta$. Therefore

$$I_1(x) = c^{n(x)} I_1(c^{-n(x)}x) = c^{n(x)} I_2(c^{-n(x)}x) = I_2(x).$$

Next, put $T(x) = f(x) - I(x)$.

Let $(x, y) \in D$, then $(x, c^{-n(y)}y) \in D$ and $c^{-n(y)}y \in \Delta$. Then

$$(x, c^{-n}c^{-n(y)}y) \in D$$

for all n. Therefore (2.1) yields

$$\|\langle f(x), f_n(c^{-n(y)}y) \rangle - \langle x, c^{-n(y)}y \rangle\| \leq c^n \varphi(x, c^{-n}c^{-n(y)}y).$$

Thus

$$\langle f(x), I_*(c^{-n(y)}y) \rangle = \langle x, c^{-n(y)}y \rangle,$$

whence $\langle f(x), I(y) \rangle = \langle x, y \rangle$, and

$$\langle T(x), I(y) \rangle = \langle f(x) - I(x), I(y) \rangle = \langle f(x), I(y) \rangle - \langle I(x), I(y) \rangle$$
$$= \langle f(x), I(y) \rangle - \langle x, y \rangle = 0.$$

$\square$

Remark 2.2. If f is a function such that $f(cx) = cf(x)$, then $f(0) = 0$ and $I_*(x) = \lim_{n \to \infty} c^n f(c^{-n}x) = f(x)$ for all $x \in \Delta$. It follows that $f(x) = I(x)$ for all $x \in \mathcal{M}$.

The following example, which is a slight modification of Example 1 of [5], shows that the bound $\sqrt{\varphi(x, y)}$ in (2.4) is sharp and we have no control on the bounded function T. This means that T is neither additive nor continuous in general.

Example 2.3. Let $\mathcal{M}, \mathcal{N}$ be the Hilbert space ℓ^2. Assume that $g : \mathcal{M} \to \mathbb{C}$ is an arbitrary mapping satisfying $|g(x)g(y)| \leq \sqrt{\varphi(x, y)}$ for all $x, y \in \ell^2$. Define the mapping $f : \mathcal{M} \to \mathcal{N}$ by $f(x) = (g(x), t_1, t_2, \ldots)$ where $x = (t_1, t_2, \ldots) \in \mathcal{M}$. Clearly, $\langle f(x), f(y) \rangle = g(x)\overline{g(y)} + \langle x, y \rangle$ for all $x, y \in \mathcal{M}$. Then $I((t_1, t_2, \ldots)) = (0, t_1, t_2, \ldots)$ and $T(x) = (g(x), 0, 0, \ldots)$ are the unique mappings fulfilling the required conditions in Theorem 2.1.

Corollary 2.4. *Suppose that either $p, q > 1$ or $p, q < 1$ are real numbers and $\alpha > 0$. Let $f : \mathcal{M} \to \mathcal{N}$ be a mapping such that*

$$\|\langle f(x), f(y) \rangle - \langle x, y \rangle\| \leq \alpha \|x\|^p \|y\|^q, \qquad x, y \in \mathcal{M}.$$

Then there exists a unique linear isometry $I : \mathcal{M} \to \mathcal{N}$ such that

$$\|f(x) - I(x)\| \leq \sqrt{\alpha} \|x\|^{\frac{p+q}{2}}$$

for all $x \in \mathcal{M}$.

Proof. Put $\varphi(x, y) = \alpha \|x\|^p \|y\|^q$. Consider $D = \mathcal{M} \times \mathcal{M}$ together with $c > 1$ if $p, q > 1$; and $c < 1$ if $p, q < 1$. $\square$

Remark 2.5. The above result holds true also in cases $p = 1$, $q \neq 1$ or $p \neq 1$, $q = 1$. The Corollary is not true for $p = q = 1$, in general. For a counterexample see Example 2 of [4].

In a particular case, where $\mathcal{M}$ and $\mathcal{N}$ are of the same finite dimension we can prove superstability.

Proposition 2.6. *Let $\dim \mathcal{M} = \dim \mathcal{N} < \infty$. Suppose that $f : \mathcal{M} \to \mathcal{N}$ satisfies (2.1) with φ as in Theorem 2.1. Then there exists a linear isometry $I : \mathcal{M} \to \mathcal{N}$ such that $f = I$ on Δ.*

Proof. Let $I : \mathcal{M} \to \mathcal{N}$ be the linear isometry from the assertion of Theorem 2.1 and $T = f - I$. I maps $\mathcal{M}$ onto a subspace $I(\mathcal{M})$ of $\mathcal{N}$. Since $\dim \mathcal{M} = \dim I(\mathcal{M})$, and $\mathcal{M}$ and $\mathcal{N}$ are of the same finite dimension, we get $I(\mathcal{M}) = \mathcal{N}$. For $x \in \Delta$ we have $T(x) \perp I(\mathcal{M})$, i.e., $T(x) \perp \mathcal{N}$ whence $T(x) = 0$. Thus $f = I$ on Δ. $\square$

Taking $D = \mathcal{M} \times \mathcal{M}$ we get immediately:

Corollary 2.7. *Let* $\dim \mathcal{M} = \dim \mathcal{N} < \infty$ *and suppose that* $f : \mathcal{M} \to \mathcal{N}$ *satisfies*

$$(2.5) \qquad \|\langle f(x), f(y)\rangle - \langle x, y\rangle\| \leq \varphi(x, y), \qquad x, y \in \mathcal{M}$$

where $\varphi : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ *satisfies (with some* $0 < c \neq 1$*)*

$$\lim_{m+n \to \infty} c^{m+n} \varphi(c^{-m}x, c^{-n}y) = 0, \qquad x, y \in \mathcal{M}.$$

Then f *is an inner product preserving mapping.*

3. Asymptotic behavior of orthogonality equation

Following [11, 13], a mapping $f : \mathcal{M} \to \mathcal{N}$ is called *p-asymptotically close to an isometry mapping* I if $\lim_{\|x\| \to \infty} \frac{\|f(x) - I(x)\|}{\|x\|^p} = 0$ (see also [18]).

Definition 3.1. A mapping $f : \mathcal{M} \to \mathcal{N}$ satisfies *p-asymptotically* the orthogonality equation if for each $\varepsilon > 0$ there exists $K > 0$ such that

$$(3.1) \qquad \|\langle f(x), f(y)\rangle - \langle x, y\rangle\| \leq \varepsilon \|x\|^p \, \|y\|^p$$

for all $x, y \in \mathcal{M}$ such that $\max\{\|x\|, \|y\|\} \geq K$.

We are going to show that asymptotically orthogonality preserving mappings are asymptotically close to isometries.

Theorem 3.2. *If* $0 < p < 1$ *and a mapping* $f : \mathcal{M} \to \mathcal{N}$ *satisfies p-asymptotically the orthogonality equation, then it is p-asymptotically close to a linear isometry mapping.*

Proof. By the assumption f satisfies p-asymptotically the orthogonality equation, hence there exists $K_0 > 0$ such that

$$\|\langle f(x), f(y)\rangle - \langle x, y\rangle\| \leq \|x\|^p \, \|y\|^p$$

for all $x, y \in \mathcal{M}$ with $\max\{\|x\|, \|y\|\} \geq K_0$. It follows from Theorem 2.1 (for $D = \{x : \|x\| \geq K_0\} \times \mathcal{M} \cup \mathcal{M} \times \{x : \|x\| \geq K_0\}$, $\Delta = \{x : \|x\| \geq K_0\}$, $0 < c < 1$ and $\varphi(x, y) := \|x\|^p \|y\|^p$) that there exists a linear isometry I_0 such that

$$(3.2) \qquad \|f(x) - I_0(x)\| \leq \|x\|^p$$

for all x with $\|x\| \geq K_0$.

Given $\varepsilon > 0$, the assumption gives again a number $K_\varepsilon \geq K_0$ such that

$$\|\langle f(x), f(y)\rangle - \langle x, y\rangle\| \leq \varepsilon \|x\|^p \, \|y\|^p$$

for all $x, y \in \mathcal{M}$ with $\max\{\|x\|, \|y\|\} \geq K_\varepsilon$. Applying again Theorem 2.1 we get an isometry I_ε such that

$$(3.3) \qquad \|f(x) - I_\varepsilon(x)\| \leq \sqrt{\varepsilon} \|x\|^p$$

for all x with $\|x\| \geq K_\varepsilon$.

We claim that $I_\varepsilon = I_0$. To see this, let $x \in \mathcal{M} \setminus \{0\}$ be an arbitrary element. There exists N such that for all $n > N$, $\|2^n x\| \geq K_\varepsilon \geq K_0$. By (3.2) and (3.3) we have

$$
\begin{aligned}
\|I_\varepsilon(x) - I_0(x)\| &= 2^{-n} \|I_\varepsilon(2^n x) - I_0(2^n x)\| \\
&\leq 2^{-n} \|I_\varepsilon(2^n x) - f(2^n x)\| + 2^{-n} \|f(2^n x) - I_0(2^n x)\| \\
&\leq 2^{(p-1)n} (\sqrt{\varepsilon} + 1) \|x\|^p.
\end{aligned}
$$

The right hand side tends to zero as $n \to \infty$, hence $I_\varepsilon = I_0$. Thus (3.3) implies that

$$
\frac{\|f(x) - I_0(x)\|}{\|x\|^p} < \sqrt{\varepsilon}
$$

for all x with $\|x\| \geq K_\varepsilon$. Thus f is p-asymptotically close to the isometry mapping I_0.   □

Remark 3.3. Assume that $p > 1$ and $f : \mathcal{M} \to \mathcal{N}$ is such that for each $\varepsilon > 0$ there exists $K > 0$ such that (3.1) holds for all $x, y \in \mathcal{M}$ satisfying $\min\{\|x\|, \|y\|\} \leq K$. Analogously as above, one can prove that there exists a linear isometry $I : \mathcal{M} \to \mathcal{N}$ such that

$$
\lim_{\|x\| \to 0} \frac{\|f(x) - I(x)\|}{\|x\|^p} = 0.
$$

4. Hyers–Ulam–Rassias stability

In this section, we prove the Hyers–Ulam–Rassias stability of the orthogonality equation.

Theorem 4.1. *Let $f : \mathcal{M} \to \mathcal{N}$ be an approximately inner product preserving mapping on $\mathcal{M}$ associated with a control function $\varphi : \mathcal{M} \times \mathcal{M} \to [0, \infty)$. We assume that φ and the control function ψ defined by*

$$
\begin{aligned}
\psi(x, y) :=\ & \big(\varphi(x + y, x + y) + \varphi(x, x + y) + \varphi(y, x + y) + \varphi(x + y, x) \\
& + \varphi(x, x) + \varphi(y, x) + \varphi(x + y, y) + \varphi(x, y) + \varphi(y, y) \big)^{1/2}
\end{aligned}
$$

satisfy either

$$
(4.1) \quad \widetilde{\psi}(x) := \sum_{n=0}^{\infty} 2^{-n-1} \psi(2^n x, 2^n x) < \infty \quad and \quad \lim_{n \to \infty} 1/4^n \varphi(2^n x, 2^n y) = 0
$$

or

$$
(4.2) \quad \widetilde{\psi}(x) := \sum_{n=1}^{\infty} 2^{n-1} \psi(2^{-n} x, 2^{-n} x) < \infty \quad and \quad \lim_{n \to \infty} 4^n \varphi(2^{-n} x, 2^{-n} y) = 0
$$

for all $x, y \in \mathcal{M}$. Then there exists a unique linear isometry $I : \mathcal{M} \to \mathcal{N}$ such that

$$
\|f(x) - I(x)\| \leq \widetilde{\psi}(x) \qquad x \in \mathcal{M}.
$$

Proof. Let $x, y, z \in \mathcal{M}$ and put $A = f(x + y) - f(x) - f(y)$. We have

$$
\begin{aligned}
\|\langle A, f(z) \rangle\| &\leq \|\langle f(x + y), f(z) \rangle - \langle x + y, z \rangle\| \\
&\quad + \|\langle f(x), f(z) \rangle - \langle x, z \rangle\| + \|\langle f(y), f(z) \rangle - \langle y, z \rangle\| \\
&\leq \varphi(x + y, z) + \varphi(x, z) + \varphi(y, z),
\end{aligned}
$$

whence

$$
\begin{aligned}
& \|f(x + y) - f(x) - f(y)\|^2 \\
=\ & \|\langle A, f(x + y) - f(x) - f(y) \rangle\| \\
\leq\ & \|\langle A, f(x + y) \rangle\| + \|\langle A, f(x) \rangle\| + \|\langle A, f(y) \rangle\| \\
\leq\ & \varphi(x + y, x + y) + \varphi(x, x + y) + \varphi(y, x + y) + \varphi(x + y, x) \\
& + \varphi(x, x) + \varphi(y, x) + \varphi(x + y, y) + \varphi(x, y) + \varphi(y, y).
\end{aligned}
$$

It follows that

$$
\|f(x + y) - f(x) - f(y)\| \leq \psi(x, y),
$$

whence, in particular,

$$
\|f(2x) - 2f(x)\| \leq \psi(x, x), \qquad x \in \mathcal{M}.
$$

Using the induction, one can easily verify the following inequalities:

$$
(4.3) \qquad \|2^{-n} f(2^n x) - 2^{-m} f(2^m x)\| \leq \sum_{k=m}^{n-1} 2^{-k-1} \psi(2^k x, 2^k x),
$$

$$
(4.4) \qquad \|2^n f(2^{-n} x) - 2^m f(2^{-m} x)\| \leq \sum_{k=m+1}^{n} 2^{k-1} \psi(2^{-k} x, 2^{-k} x)
$$

for all integers $n > m \geq 0$ and $x \in \mathcal{M}$. It follows that the sequence $\{c^n f(c^{-n} x)\}$ with $c = \frac{1}{2}$ or $c = 2$, respectively, is a Cauchy one, whence it is convergent. Define the mapping $I : \mathcal{M} \to \mathcal{N}$ by $I(x) := \lim_{n \to \infty} c^n f(c^{-n})$. Since f is approximately inner product preserving, we have

$$
c^{2n} \|\langle f(c^{-n} x), f(c^{-n} y) \rangle - \langle c^{-n} x, c^{-n} y \rangle\| \leq c^{2n} \varphi(c^{-n} x, c^{-n} y).
$$

Passing to the limit as n tends to infinity we get

$$
\langle I(x), I(y) \rangle = \langle x, y \rangle, \qquad x, y \in \mathcal{M}.
$$

In addition, it follows from (4.3) and (4.4) with $m = 0$ as $n \to \infty$ that

$$
\|f(x) - I(x)\| \leq \widetilde{\psi}(x).
$$

$\square$

Assuming $\|0\|^p = \infty$ for $p < 0$, we have the following result.

Corollary 4.2. *Suppose that $p \neq 2$ is a real number and $\beta > 0$. Let $f : \mathcal{M} \to \mathcal{N}$ be a mapping such that*

$$\| \langle f(x), f(y) \rangle - \langle x, y \rangle \| \leq \beta(\|x\|^p + \|y\|^p)$$

for all $x, y \in \mathcal{M}$. Then there exists a unique linear isometry $I : \mathcal{M} \to \mathcal{N}$ such that

$$\| f(x) - I(x) \| \leq \frac{\sqrt{6\beta(2^p + 2)}}{|2^{\frac{p}{2}} - 2|} \|x\|^{\frac{p}{2}}, \qquad x \in \mathcal{M}.$$

Proof. Apply Theorem 4.1 with $\varphi(x, y) = \beta(\|x\|^p + \|y\|^p)$ and consider (4.1) if $p < 2$, and (4.2) if $p > 2$. □

Remark 4.3. The case $p = 2$ remains unsolved.

References

[1] M. Amyari, *Stability of C^*-inner products*, J. Math. Anal. Appl. **322** (2006), 214–218.

[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[3] C. Baak, H. Chu, and M. S. Moslehian, *On the Cauchy-Rassias inequality and linear n-inner product preserving mappings*, Math. Inequal. Appl. **9** (2006), no. 3, 453–464.

[4] R. Badora and J. Chmieliński, *Decomposition of mappings approximately inner product preserving*, Nonlinear Anal. **62** (2005), no. 6, 1015–1023.

[5] J. Chmieliński, *On a singular case in the Hyers-Ulam-Rassias stability of the Wigner equation*, J. Math. Anal. Appl. **289** (2004), no. 2, 571–583.

[6] J. Chmieliński and S.-M. Jung, *The stability of the Wigner equation on a restricted domain*, J. Math. Anal. Appl. **254** (2001), no. 1, 309–320.

[7] S. Czerwik, *Functional equations and inequalities in several variables*, World Scientific Publishing Co., Inc., River Edge, NJ, 2002.

[8] G. L. Forti, *Hyers-Ulam stability of functional equations in several variables*, Aequationes Math. **50** (1995), no. 1-2, 143–190.

[9] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A. **27** (1941), 222–224.

[10] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of functional equations in several variables*, Progress in Nonlinear Differential Equations and their Applications, 34. Birkhauser Boston, Inc., Boston, MA, 1998.

[11] _____, *On the asymptoticity aspect of Hyers-Ulam stability of mappings*, Proc. Amer. Math. Soc. **126** (1998), no. 2, 425–430.

[12] D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math. **44** (1992), no. 2-3, 125–153.

[13] G. Isac and Th. M. Rassias, *On the Hyers-Ulam stability of ψ-additive mappings*, J. Approx. Theory **72** (1993), no. 2, 131–137.

[14] S.-M. Jung, *Hyers-Ulam-Rassias stability of functional equations in mathematical analysis*, Hadronic Press, Inc., Palm Harbor, FL, 2001.

[15] I. Kaplansky, *Modules over operator algebras*, Amer. J. Math. **75** (1953), 839–858.

[16] E. C. Lance, *Hilbert C^*-Modules*, LMS Lecture Note Series 210, Cambridge Univ. Press, 1995.

[17] V. M. Manuilov and E. V. Troitsky, *Hilbert C^*-modules*, Translations of Mathematical Monographs, 226. American Mathematical Society, Providence, RI, 2005.

[18] M. S. Moslehian, *Asymptotic behavior of the extended Jensen equation*, Studia Sci. Math. Hungar (to appear).

[19] J. G. Murphy, *C^*-algebras and operator theory*, Academic Press, Inc., Boston, MA, 1990.

[20] W. L. Paschke, *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc. **182** (1973), 443–468.

[21] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), no. 2, 297–300.

[22] _____, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), no. 1, 23–130.

[23] _____, *Stability of the Generalized Orthogonality Functional Equation*, Inner product spaces and applications, 219–240, Pitman Res. Notes Math. Ser., 376, Longman, Harlow, 1997.

[24] _____, *A new generalization of a theorem of Jung for the orthogonality equation*, Appl. Anal. **81** (2002), no. 1, 163–177.

[25] M. A. Rieffel, *Induced representations of C^*-algebras*, Advances in Math. **13** (1974), 176–257.

[26] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions John Wiley & Sons, Inc., New York 1964 .

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