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STABILITY OF THE MULTI-JENSEN EQUATION

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Abstract. Given an $m \in \mathbb{N}$ and two vector spaces V and W, a function $f : V^m \to W$ is called multi-Jensen if it satisfies Jensen's equation in each variable separately. In this paper we unify these m Jensen equations to obtain a single functional equation for f and prove its stability in the sense of Hyers-Ulam, using the so-called direct method.

1. Introduction

Throughout this paper we assume that V and W are vector spaces over $\mathbb{Q}$ and that m is a natural number.

Definition 1. A function $f : V^m \to W$ is called *multi-Jensen* (or *m-Jensen*) if it satisfies Jensen's equation in each of its m arguments,

$$\begin{aligned} & f(x_1, \ldots, x_{i-1}, \frac{1}{2}(x_i + y_i), x_{i+1}, \ldots, x_m) \\ (1) \quad & = \frac{1}{2} f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m) + \frac{1}{2} f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_m), \\ & i = 1, \ldots, m, \end{aligned}$$

for all $x_1, \ldots, x_i, y_i, \ldots, x_m \in V$.

All solutions of (1) have been characterized in detail in [4]. The present paper deals with a proof of Hyers-Ulam stability of (1) without referring to any knowledge of solutions. In [1] the stability problem was treated for the bi-Jensen equation in the sense of Hyers-Ulam-Rassias. As a result the authors obtain boundedness of $f(x_1, x_2) - f(0, x_2) - F(x_1, x_2)$ and $f(x_1, x_2) - f(x_1, 0) - F'(x_1, x_2)$, F, F' being bi-Jensen functions, by functions with three arguments. To get a general idea of stability concepts for functional equations, see, for instance, [3] and [2].

For our purpose it will be convenient to introduce the following notation. For $n \in \mathbb{N}_0$ the boldfaced symbol $\mathbf{n}$ shall denote the set of all natural numbers from 1 to n, i.e., $\mathbf{n} = \{1, \ldots, n\}$ in case $n \geq 1$ and $\mathbf{n} = \emptyset$ in case $n = 0$. For

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a subset $S = \{j_1, j_2, \ldots, j_i\} \subseteq \mathbf{m}$ with $1 \le j_1 < j_2 < \cdots < j_i \le m$ and for $x = (x_1, \ldots, x_m) \in V^m$,

$$x_S := (0, \ldots, 0, x_{j_1}, 0, \ldots, 0, x_{j_2}, 0, \ldots, 0, x_{j_i}, 0, \ldots, 0) \in V^m$$

denotes the vector which coincides with x in exactly those components, which are indexed by the elements of S and whose other components are set equal zero. It follows immediately that $x_\emptyset = 0$, $x_{\mathbf{m}} = x$ and that $(x_S)_T = (x_T)_S = x_{S \cap T}$ for any $S, T \subseteq \mathbf{m}$. The cardinality of a set S will be denoted by $|S|$.

As a first step system (1) of m functional equations is unified into one functional equation characterizing multi-Jensen functions.

Lemma 1.1. *A function $f : V^m \to W$ is multi-Jensen if and only if it satisfies*

$$f(\frac{1}{2}(x+y)) = \frac{1}{2^m} \sum_{S \subseteq \mathbf{m}} f(x_S + y_{\mathbf{m} \setminus S}) \tag{2}$$

for all $x, y \in V^m$.

Proof. Necessity of (2) is shown by proving

$$f(\frac{1}{2}(x+y)) = \frac{1}{2^k} \sum_{S \subseteq \mathbf{k}} f(x_S + y_{\mathbf{k} \setminus S} + \frac{1}{2}(x_{\mathbf{m} \setminus \mathbf{k}} + y_{\mathbf{m} \setminus \mathbf{k}})), \quad k = 0, 1, \ldots, m, \tag{3}$$

by induction on k, the number of equations of (1) taken into account. For $k = 0$, (3) is obviously true. Assuming that (3) holds true for any $k \in \mathbf{m-1}$, one obtains by (1)

$$\begin{aligned}
& f(\frac{1}{2}(x+y)) \\
= \ & \frac{1}{2^k} \sum_{S \subseteq \mathbf{k}} f(x_S + y_{\mathbf{k} \setminus S} + \frac{1}{2}(x_{\{k+1\}} + y_{\{k+1\}}) + \frac{1}{2}(x_{\mathbf{m} \setminus \mathbf{k+1}} + y_{\mathbf{m} \setminus \mathbf{k+1}})) \\
= \ & \frac{1}{2^k} \sum_{S \subseteq \mathbf{k}} (\frac{1}{2} f(x_S + y_{\mathbf{k} \setminus S} + x_{\{k+1\}} + y_\emptyset + \frac{1}{2}(x_{\mathbf{m} \setminus \mathbf{k+1}} + y_{\mathbf{m} \setminus \mathbf{k+1}})) \\
& \qquad + \frac{1}{2} f(x_S + y_{\mathbf{k} \setminus S} + x_\emptyset + y_{\{k+1\}} + \frac{1}{2}(x_{\mathbf{m} \setminus \mathbf{k+1}} + y_{\mathbf{m} \setminus \mathbf{k+1}}))) \\
= \ & \frac{1}{2^{k+1}} \sum_{S \subseteq \mathbf{k+1}} f(x_S + y_{\mathbf{k+1} \setminus S} + \frac{1}{2}(x_{\mathbf{m} \setminus \mathbf{k+1}} + y_{\mathbf{m} \setminus \mathbf{k+1}})).
\end{aligned}$$

Sufficiency of (2) is shown by taking an $i \in \mathbf{m}$, defining $\tilde{x} := (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_m)$ and splitting up the argument on the left hand side of the i-th equation of (1) by $(x_1, \ldots, x_{i-1}, \frac{1}{2}(x_i + y_i), x_{i+1}, \ldots, x_m) = \frac{1}{2}x + \frac{1}{2}\tilde{x}$. Then for any $S \subseteq \mathbf{m}$ it follows $x_S + \tilde{x}_{\mathbf{m} \setminus S} = x$ in case $i \in S$ and $x_S + \tilde{x}_{\mathbf{m} \setminus S} = \tilde{x}$ in case $i \notin S$. Hence

$$\begin{aligned}
f(\frac{1}{2}(x+\tilde{x})) &= \frac{1}{2^m} \sum_{S \subseteq \mathbf{m}} f(x_S + \tilde{x}_{\mathbf{m} \setminus S}) = \frac{1}{2^m}(\sum_{S \subseteq \mathbf{m}, i \in S} f(x) + \sum_{S \subseteq \mathbf{m}, i \notin S} f(\tilde{x})) \\
&= \frac{1}{2} f(x) + \frac{1}{2} f(\tilde{x}),
\end{aligned}$$

showing that f satisfies the i-th equation of (1). $\square$

Equation (2) is utilized now to define functions which are approximately multi-Jensen in the following sense.

Definition 2. Let $(W, \|\cdot\|)$ be a Banach-space and let $\varepsilon \geq 0$. A function $f : V^m \to W$ is called *ε-multi-Jensen* if there exists a function $\varphi : V^m \times V^m \to W$ satisfying $\|\varphi(x, y)\| \leq \varepsilon$ for all $x, y \in V^m$, such that

$$f(\frac{1}{2}(x+y)) = \frac{1}{2^m} \sum_{S \subseteq \mathbf{m}} f(x_S + y_{\mathbf{m}\setminus S}) + \varphi(x, y)$$

for all $x, y \in V^m$.

The remainder of the paper deals with the proof of the following theorem.

Theorem 1.2. *Let $f : V^m \to W$ be an ε-multi-Jensen function. Then there exists a multi-Jensen function $g : V^m \to W$, such that*

$$\|f(x) - g(x)\| \leq 2m\varepsilon \tag{4}$$

for all $x \in V^m$. Moreover, g is uniquely determined up to a suitable constant.

2. Proof of the theorem

Our proof is based on the so-called direct method. For a given function $f : V^m \to W$ let a sequence of functions $g_n : V^m \to W$ be defined by

$$g_n(x) := \sum_{S \subseteq \mathbf{m}} 2^{-|S|n} f(2^n x_S), \quad x \in V^m, \quad n = 1, 2, \ldots \tag{5}$$

Lemma 2.1. *Let $f : V^m \to W$ be ε-multi-Jensen and let g_n be defined by (5). Then g_n is $2^{-n}(2^m - 1)\varepsilon$-multi-Jensen for each $n \in \mathbb{N}$.*

Proof. Let be $x, y \in V^m$. Estimation of the multi-Jensen kernel of g_n yields for each $n \in \mathbb{N}$

$$\begin{aligned}
&\|g_n(\frac{1}{2}(x+y)) - \frac{1}{2^m} \sum_{T \subseteq \mathbf{m}} g_n(x_T + y_{\mathbf{m}\setminus T})\| \\
&= \|\sum_{S \subseteq \mathbf{m}} 2^{-|S|n} f(\frac{1}{2}(2^n x_S + 2^n y_S)) - \frac{1}{2^m} \sum_{T \subseteq \mathbf{m}} \sum_{S \subseteq \mathbf{m}} 2^{-|S|n} f(2^n (x_T)_S + 2^n (y_{\mathbf{m}\setminus T})_S)\| \\
&= \|\sum_{S \subseteq \mathbf{m}} 2^{-|S|n} \frac{1}{2^m} \sum_{T \subseteq \mathbf{m}} f(2^n (x_S)_T + 2^n (y_S)_{\mathbf{m}\setminus T}) + \sum_{S \subseteq \mathbf{m}} 2^{-|S|n} \varphi(2^n x_S, 2^n y_S) \\
&\quad - \frac{1}{2^m} \sum_{T \subseteq \mathbf{m}} \sum_{S \subseteq \mathbf{m}} 2^{-|S|n} f(2^n (x_T)_S + 2^n (y_{\mathbf{m}\setminus T})_S)\| \\
&= \|\sum_{\emptyset \neq S \subseteq \mathbf{m}} 2^{-|S|n} \varphi(2^n x_S, 2^n y_S)\| \leq \sum_{i=1}^{m} \binom{m}{i} 2^{-in} \varepsilon \leq 2^{-n}(2^m - 1)\varepsilon,
\end{aligned}$$

where we have used $\varphi(0, 0) = 0$. $\square$

Lemma 2.2. *Let M be a finite set, R a commutative ring with unity, $\mathcal{R}$ an R-module and let $F : \mathcal{P}(M) \to \mathcal{R}$ be any mapping from the power set of M to $\mathcal{R}$. Given coefficients $\alpha_{S,T} \in R$ for $S, T \subseteq M$, then*

$$\sum_{S\subseteq M}\sum_{T\subseteq M} \alpha_{S,T} F(S\cap T) = \sum_{U\subseteq M} \beta_U F(U),$$

where

$$\beta_U = \sum_{S'\subseteq M\setminus U}\ \sum_{T'\subseteq M\setminus(U\cup S')} \alpha_{U\cup S',U\cup T'}.$$

Proof. For $U \subseteq M$ let $D_U := \{(S,T) \in \mathcal{P}(M) \times \mathcal{P}(M) \mid S \cap T = U\}$ and for $U, S' \subseteq M$ with $U \cap S' = \emptyset$ let $E_{S',U} := \{(U \cup S', U \cup T') \in \mathcal{P}(M) \times \mathcal{P}(M) \mid T' \subseteq M \setminus (U \cup S')\}$. Then for fixed $U \subseteq M$ and for all $S'_1, S'_2 \subset M$ with $S'_1 \cap U = S'_2 \cap U = \emptyset$ it follows $E_{S'_1,U} \cap E_{S'_2,U} = \emptyset$ and $\bigcup_{S'\subseteq M\setminus U} E_{S',U} = D_U$. Therefore

$$\begin{aligned}
\beta_U &= \sum_{(S,T)\in D_U} \alpha_{S,T} = \sum_{(U\cup S',U\cup T')\in \bigcup_{S'\subseteq M\setminus U} E_{S',U}} \alpha_{U\cup S',U\cup T'} \\
&= \sum_{S'\subseteq M\setminus U}\ \sum_{T'\subseteq M\setminus(U\cup S')} \alpha_{U\cup S',U\cup T'}.
\end{aligned}$$

□

We will use this lemma several times in a form stated as in the corollary.

Corollary 2.3. *Let $M, R, \mathcal{R}, F$ be as in Lemma 2.2 and let $a, b, c, d \in R$. Then*
$$\sum_{S\subseteq M}\sum_{T\subseteq M} a^{|T|} b^{|S|} c^{|M|-|T|} d^{|M|-|S|} F(S\cap T) = \sum_{U\subseteq M} (ab)^{|U|} \left((b+d)c + ad\right)^{|M|-|U|} F(U).$$

Proof. For any $U \subseteq M$, abbreviating $m := |M|, u := |U|$, we have

$$\begin{aligned}
\beta_U &= \sum_{S'\subseteq M\setminus U}\ \sum_{T'\subseteq M\setminus(U\cup S')} a^{u+|T'|} b^{u+|S'|} c^{m-u-|T'|} d^{m-u-|S'|} \\
&= \sum_{j=0}^{m-u}\sum_{k=0}^{m-u-j} \binom{m-u}{j}\binom{m-u-j}{k} a^{u+k} b^{u+j} c^{m-u-k} d^{m-u-j} \\
&= \sum_{j=0}^{m-u}\sum_{k=0}^{m-u-j} \binom{m-u}{k}\binom{m-u-k}{j} a^{u+k} b^{u+j} c^{m-u-k} d^{m-u-j} \\
&= (ab)^u \sum_{k=0}^{m-u} \binom{m-u}{k} a^k d^k \Big(\sum_{j=0}^{m-u-k} \binom{m-u-k}{j} b^j d^{m-u-k-j} \Big) c^{m-u-k} \\
&= (ab)^u \left((b+d)c + ad\right)^{m-u}.
\end{aligned}$$

□

Looking at (5), we need a representation of $f(2^n x_S)$ in terms of $f(x_T)$, that is, with scalar factor 1 in the argument of f.

Lemma 2.4. *Let $f : V^m \to W$ be any function and denote the multi-Jensen kernel of f by*

$$\varphi(x,y) := f(\frac{1}{2}(x+y)) - \frac{1}{2^m}\sum_{S\subseteq \mathbf{m}} f(x_S + y_{\mathbf{m}\setminus S}), \quad x,y\in V^m.$$

Then

$$\begin{aligned} f(2^n x) = & \sum_{S\subseteq\mathbf{m}} 2^{|S|n}(1-2^n)^{m-|S|} f(x_S) \\ & - \sum_{i=1}^{n}\sum_{S\subseteq\mathbf{m}} 2^{|S|(n+1-i)}(1-2^{n+1-i})^{m-|S|}\varphi(2^i x_S,0) \end{aligned} \tag{6}$$

for all $n\in\mathbb{N}$ and for all $x\in V^m$.

Proof. For any $x\in V^m$ the proof is by induction on n. In order to establish the induction basis, let $S\subseteq \mathbf{m}$ and consider

$$f(x_S) = f(\frac{1}{2}(2x_S+0)) = \frac{1}{2^m}\sum_{T\subseteq\mathbf{m}} f(2x_{S\cap T}) + \varphi(2x_S,0).$$

Multiplying with $2^{|S|}(-1)^{m-|S|}$, taking the sum over all $S\subseteq\mathbf{m}$ and then applying Corollary 2.3 with $a=c=1, b=2, d=-1$ and with $F(U)=f(2x_U)$, one obtains

$$\begin{aligned} & \sum_{S\subseteq\mathbf{m}} 2^{|S|}(-1)^{m-|S|}f(x_S) \\ = & \frac{1}{2^m}\sum_{S\subseteq\mathbf{m}}\sum_{T\subseteq\mathbf{m}} 2^{|S|}(-1)^{m-|S|}f(2x_{S\cap T}) \\ & + \sum_{S\subseteq\mathbf{m}} 2^{|S|}(-1)^{m-|S|}\varphi(2x_S,0) \\ = & \frac{1}{2^m}\sum_{U\subseteq\mathbf{m}} 2^{|U|}0^{m-|U|}f(2x_U) + \sum_{S\subseteq\mathbf{m}} 2^{|S|}(-1)^{m-|S|}\varphi(2x_S,0). \end{aligned}$$

The only active term in the sum over all $U\subseteq\mathbf{m}$ is that for $U=\mathbf{m}$, so the value of this sum reduces to $2^m f(2x)$, hence it follows

$$f(2x) = \sum_{S\subseteq\mathbf{m}} 2^{|S|}(-1)^{m-|S|}f(x_S) - \sum_{S\subseteq\mathbf{m}} 2^{|S|}(-1)^{m-|S|}\varphi(2x_S,0), \tag{7}$$

which is (6) for $n=1$. In order to establish the induction step, we use at first (7) and then as induction hypothesis (6), to obtain

$$\begin{aligned} & f(2^{n+1}x) \\ = & f\big(2(2^n x)\big) = \sum_{T\subseteq\mathbf{m}} 2^{|T|}(-1)^{m-|T|}\big(f(2^n x_T) - \varphi(2^{n+1}x_T,0)\big) \\ = & \sum_{S\subseteq\mathbf{m}}\sum_{T\subseteq\mathbf{m}} (-1)^{m-|T|}2^{|T|+|S|n}(1-2^n)^{m-|S|}f(x_{S\cap T}) \end{aligned}$$

$$-\sum_{i=1}^{n}\sum_{S\subseteq\mathbf{m}}\sum_{T\subseteq\mathbf{m}}(-1)^{m-|T|}2^{|T|+|S|(n+1-i)}(1-2^{n+1-i})^{m-|S|}\varphi(2^i x_{S\cap T},0)$$
$$-\sum_{T\subseteq\mathbf{m}}2^{|T|}(-1)^{m-|T|}\varphi(2^{n+1}x_T,0).$$

The double sum over $S, T \subseteq \mathbf{m}$ containing terms with f is transformed to a single sum by Corollary 2.3 with $a = 2, b = 2^n, c = -1, d = 1 - 2^n$, in the same way as that containing terms with φ with $a = 2, b = 2^{n+1-i}, c = -1, d = 1 - 2^{n+1-i}$, resulting in

$$\begin{aligned}
f(2^{n+1}x) &= \sum_{U\subseteq\mathbf{m}}2^{|U|(n+1)}(1-2^{n+1})^{m-|U|}f(x_U)\\
&\quad-\sum_{i=1}^{n}\sum_{U\subseteq\mathbf{m}}2^{|U|(n+2-i)}(1-2^{n+2-i})^{m-|U|}\varphi(2^i x_U,0)\\
&\quad-\sum_{U\subseteq\mathbf{m}}2^{|U|((n+2-(n+1))}(1-2^{n+2-(n+1)})^{m-|U|}\varphi(2^{n+1}x_U,0)\\
&= \sum_{S\subseteq\mathbf{m}}2^{|S|(n+1)}(1-2^{n+1})^{m-|S|}f(x_S)\\
&\quad-\sum_{i=1}^{n+1}\sum_{S\subseteq\mathbf{m}}2^{|S|(n+2-i)}(1-2^{n+2-i})^{m-|S|}\varphi(2^i x_S,0).
\end{aligned}$$

□

Now we are in a position to estimate f against g_n.

Lemma 2.5. *Let $f : V^m \to W$ be ε-multi-Jensen and let g_n be given by (5). Then*

$$\|g_n(x)-f(x)\| \le 2^{-n}\sum_{U\subsetneq\mathbf{m}}2^{-(m-1-|U|)n}\|f(x_U)\|+2m(1+2^{-n})^{m-1}(1-2^{-n})\varepsilon$$

for all $x \in V^m$ and for all $n \in \mathbb{N}$.

Proof. By (6) we have

$$\begin{aligned}
g_n(x) &= \sum_{S\subseteq\mathbf{m}}2^{-|S|n}f(2^n x_S)=\sum_{S\subseteq\mathbf{m}}\sum_{T\subseteq\mathbf{m}}2^{(|T|-|S|)n}(1-2^n)^{m-|T|}f(x_{S\cap T})\\
&\quad-\sum_{i=1}^{n}\sum_{S\subseteq\mathbf{m}}\sum_{T\subseteq\mathbf{m}}2^{(|T|-|S|)n}2^{|T|(1-i)}(1-2^{n+1-i})^{m-|T|}\varphi(2^i x_{S\cap T},0).
\end{aligned}$$

Application of Corollary 2.3 with $a = 2^n, b = 2^{-n}, c = 1 - 2^n, d = 1$ to the double sum containing f, and with $a = 2^{n+1-i}, b = 2^{-n}, c = 1 - 2^{n+1-i}, d = 1$

to the double sums containing φ renders
(8)

$$g_n(x) = \sum_{U\subseteq\mathbf{m}} 2^{-(m-|U|)n} f(x_U) - \sum_{i=1}^{n}\sum_{U\subseteq\mathbf{m}} 2^{(1-i)|U|}(1+2^{-n}-2^{1-i})^{m-|U|}\varphi(2^i x_U, 0).$$

The term for $U = \mathbf{m}$ in the sum containing f is $f(x)$. Subtracting it, taking the norm and observing $\varphi(0,0) = 0$ renders

$$\begin{aligned}
&\|g_n(x) - f(x)\| \\
\leq\ & \sum_{U\subsetneq\mathbf{m}} 2^{-(m-|U|)n}\big\|f(x_U)\big\| + \sum_{i=1}^{n}\sum_{\emptyset\neq U\subseteq\mathbf{m}} 2^{(1-i)|U|}(1+2^{-n}-2^{1-i})^{m-|U|}\varepsilon \\
=\ & 2^{-n}\sum_{U\subsetneq\mathbf{m}} 2^{-(m-1-|U|)n}\|f(x_U)\| + \sum_{i=1}^{n}\sum_{j=1}^{m}\binom{m}{j}(2^{1-i})^j(1+2^{-n}-2^{1-i})^{m-j}\varepsilon.
\end{aligned}$$

The double sum in the second line may be rewritten (without ε) as

$$\sum_{i=1}^{n}((1+2^{-n})^m - (1+2^{-n}-2^{1-i})^m).$$

With $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(t) := t^m$, $t_i := 1 + 2^{-n} - 2^{1-i}$ and $h_i := 2^{1-i}$, $i = 1, \dots, n$, we obtain by the mean value theorem

$$\begin{aligned}
&|\sum_{i=1}^{n}(\phi(t_i+h_i) - \phi(t_i))| = |\sum_{i=1}^{n} h_i\phi'(\tau_i)| = |\sum_{i=1}^{n} 2^{1-i} m\tau_i^{m-1}| \\
\leq\ & 2m(1+2^{-n})^{m-1}\sum_{i=1}^{n} 2^{-i} = 2m(1+2^{-n})^{m-1}(1-2^{-n}),
\end{aligned}$$

where τ_i was estimated by $1 + 2^{-n}$ for $i = 1, \dots, n$. □

Lemma 2.6. *Let $f : V^m \to W$ be ε-multi-Jensen and let g_n be given by* (5). *Then $(g_n(x))_{n\in\mathbb{N}}$ is a Cauchy-sequence for every $x \in V^m$.*

Proof. Taking $x \in V^m$ and $n, k \in \mathbb{N}$, we have by (8)

$$\begin{aligned}
& g_{n+k}(x) - g_n(x) \\
=\ & \sum_{U\subseteq\mathbf{m}} (2^{-(m-|U|)(n+k)} f(x_U) - 2^{-(m-|U|)n} f(x_U)) \\
& -\sum_{i=1}^{n+k}\sum_{U\subseteq\mathbf{m}} 2^{(1-i)|U|}(1+2^{-(n+k)}-2^{1-i})^{m-|U|}\varphi(2^i x_U, 0) \\
& +\sum_{i=1}^{n}\sum_{U\subseteq\mathbf{m}} 2^{(1-i)|U|}(1+2^{-n}-2^{1-i})^{m-|U|}\varphi(2^i x_U, 0) \\
=:\ & S_f + S_\varphi.
\end{aligned}$$

In the sum containing f, the terms for $U = \mathbf{m}$ cancel each other, such that

$$\begin{aligned}\|S_f\| &= \|\sum_{U\subsetneq \mathbf{m}} 2^{-(m-|U|)n}(2^{-(m-|U|)k}-1)f(x_U)\| \\ &\le 2^{-n}\sum_{U\subsetneq \mathbf{m}} 2^{-(m-1-|U|)n}\|f(x_U)\|.\end{aligned}$$

Observing $\varphi(0,0)=0$, the sums containing φ are at first rewritten as

$$\begin{aligned}S_\varphi \;=\; -\sum_{\emptyset\neq U\subseteq \mathbf{m}} \Bigg(&\sum_{i=1}^{n} 2^{(1-i)|U|}((1+2^{-(n+k)}-2^{1-i})^{m-|U|} \\ &\qquad -(1+2^{-n}-2^{1-i})^{m-|U|})\varphi(2^i x_U,0) \\ &+\sum_{i=n+1}^{n+k} 2^{(1-i)|U|}(1+2^{-(n+k)}-2^{1-i})^{m-|U|}\varphi(2^i x_U,0)\Bigg).\end{aligned}$$

For $U\subseteq \mathbf{m}$ we have

$$\begin{aligned}&\left|(1+2^{-(n+k)}-2^{1-i})^{m-|U|}-(1+2^{-n}-2^{1-i})^{m-|U|}\right| \\ =\;&\left|\sum_{l=0}^{m-|U|} 2^{-nl}\binom{m-|U|}{l}(2^{-kl}-1)(1-2^{1-i})^{m-|U|-l}\right| \\ \le\;& 2^{-n}\sum_{l=1}^{m-|U|}\binom{m-|U|}{l} = 2^{-n}(2^{m-|U|}-1),\end{aligned}$$

and therefore

$$\begin{aligned}&\|\sum_{i=1}^{n} 2^{(1-i)|U|}((1+2^{-(n+k)}-2^{1-i})^{m-|U|}-(1+2^{-n}-2^{1-i})^{m-|U|})\varphi(2^i x_U,0)\| \\ \le\;& 2^{-n}(2^{m-|U|}-1)2^{|U|}\sum_{i=1}^{n}2^{-|U|i}\varepsilon = 2^{-n}(2^{m-|U|}-1)\frac{1-2^{-|U|n}}{1-2^{-|U|}}\varepsilon.\end{aligned}$$

Furthermore

$$\begin{aligned}&\|\sum_{i=n+1}^{n+k} 2^{(1-i)|U|}(1+2^{-(n+k)}-2^{1-i})^{m-|U|}\varphi(2^i x_U,0)\| \\ \le\;& \sum_{i=1}^{k}2^{(1-n-i)|U|}2^{m-|U|}\varepsilon = 2^{-|U|n}2^{m-|U|}\frac{1-2^{-|U|k}}{1-2^{-|U|}}\varepsilon,\end{aligned}$$

such that

$$\begin{aligned}\|S_\varphi\| \;\le\;& \sum_{\emptyset\neq U\subseteq \mathbf{m}}\left(2^{-n}(2^{m-|U|}-1)\frac{1-2^{-|U|n}}{1-2^{-|U|}}+2^{-|U|n}2^{m-|U|}\frac{1-2^{-|U|k}}{1-2^{-|U|}}\right)\varepsilon \\ \le\;& 2^{-n}2(2^m-1)^2\varepsilon.\end{aligned}$$

The estimations for S_f and S_φ together imply

$$\|g_{n+k}(x) - g_n(x)\| \leq 2^{-n}\Big(\sum_{U \subsetneq \mathbf{m}} 2^{-(m-1-|U|)n} \|f(x_U)\| + 2(2^m - 1)^2 \varepsilon \Big).$$

□

Finally we have all ingredients for the proof of Theorem 1.2.

Proof. By Lemma 2.6 and completeness of W there exists a function $g : V^m \to W$, $g(x) := \lim_{n\to\infty} g_n(x)$. Taking the limit for n to infinity in Lemma 2.1, it follows that g is multi-Jensen and doing the same in Lemma 2.5, one obtains (4) for all $x \in V^m$.

In order to show uniqueness of g up to a constant, assume that there is another multi-Jensen function $\tilde{g} : V^m \to W$, satisfying $\|f(x) - \tilde{g}(x)\| \leq 2m\varepsilon$ for all $x \in V^m$. Defining a sequence $\tilde{g}_n : V^m \to W$, by

$$\tilde{g}_n(x) := \sum_{S \subseteq \mathbf{m}} 2^{-|S|n} \tilde{g}(2^n x_S), \quad n = 1, 2, \ldots,$$

it follows on replacing f by $\tilde{g}$ from Lemma 2.6 that $(\tilde{g}_n(x))_{n\in\mathbb{N}}$ is convergent for all $x \in V^m$ and by Lemma 2.5 (with $\varepsilon = 0$) that $\lim_{n\to\infty} \tilde{g}_n(x) = \tilde{g}(x)$ for all $x \in V^m$. Furthermore, taking in

$$\begin{aligned}\|g_n(x) - \tilde{g}_n(x)\| &\leq \sum_{S \subseteq \mathbf{m}} 2^{-|S|n} \|f(2^n x_S) - \tilde{g}(2^n x_S)\| \\ &\leq \|f(0) - \tilde{g}(0)\| + \sum_{\emptyset \neq S \subseteq \mathbf{m}} 2^{-|S|n} 2m\varepsilon,\end{aligned}$$

the limit for n to infinity, one obtains $\tilde{g} = g$ in case $f(0) = \tilde{g}(0)$. If $f(0) \neq \tilde{g}(0)$, let $\hat{g} : V^m \to W$ be defined by $\hat{g}(x) := \tilde{g}(x) - \tilde{g}(0) + f(0)$. Then $\hat{g}$ is multi-Jensen, because it differs from a multi-Jensen function only by a constant. Defining a sequence $\hat{g}_n : V^m \to W$ by

$$\hat{g}_n(x) := \sum_{S \subseteq \mathbf{m}} 2^{-|S|n} \hat{g}(2^n x_S), \quad n = 1, 2, \ldots,$$

one has because of $\hat{g}(0) = f(0)$

$$\begin{aligned}\|g_n(x) - \hat{g}_n(x)\| &\leq \sum_{\emptyset \neq S \subseteq \mathbf{m}} 2^{-|S|n} \|f(2^n x_S) - \tilde{g}(2^n x_S) + \tilde{g}(0) - f(0)\| \\ &\leq (2m\varepsilon + \|\tilde{g}(0) - f(0)\|) 2^{-n} (2^m - 1),\end{aligned}$$

and therefore, taking the limit for n to infinity, $\hat{g}(x) = g(x)$ for all $x \in V^m$. But this in turn yields $\tilde{g}(x) = g(x) + \tilde{g}(0) - f(0)$. □

References

[1] J.-H. Bae and W.-G. Park, *On the solution of a bi-Jensen functional equation and its stability*, Bull. Korean Math. Soc. **43** (2006), no. 3, 499–507.

[2] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Co., Inc., River Edge, NJ, 2002.

[3] G. L. Forti, *Hyers-Ulam stability of functional equations in several variables*, Aequationes Math. **50** (1995), no. 1-2, 143–190.

[4] W. Prager and J. Schwaiger, *Multi-affine and multi-Jensen functions and their connection with generalized polynomials*, Aequationes Math. **69** (2005), no. 1-2, 41–57.

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