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METANILPOTENT GROUPS WITH CHAIN CONDITIONS FOR NORMAL SUBGROUPS OF INFINITE ORDER OR INDEX

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Abstract. We study the structure of metanilpotent groups satisfying the maximal condition on infinite normal subgroups or the minimal condition on normal subgroups of infinite index.

1. Introduction

A group G is said to satisfy the *weak maximal condition on normal subgroups* if there are no infinite ascending chains $G_1 < G_2 < \cdots$ of normal subgroups of G such that all the indices $|G_{i+1} : G_i|$ are infinite. The *weak minimal condition on normal subgroups* is defined similarly by substituting descending for ascending chains. Kurdachenko [2] considered groups satisfying the weak maximal or weak minimal conditions on normal subgroups.

A group G is said to satisfy max-∞n (the maximal condition on infinite normal subgroups) if there are no infinite ascending chains of infinite normal subgroups of G. Similarly a group G is said to satisfy min-∞n (the minimal condition on normal subgroups of infinite index) if there are no infinite descending chains of normal subgroups with infinite index in G. Clearly max-∞n (min-∞n) implies the weak maximal condition (weak minimal condition, respectively) on normal subgroups. We note that the additive group of p-adic rationals satisfies both the weak maximal and weak minimal conditions on subgroups, but it does not satisfy max-∞ (the maximal condition on infinite subgroups) or min-∞ (the minimal condition on subgroups of infinite index). Thus we are dealing with stronger properties than the usual weak chain conditions.

Since the chain conditions max-∞n and min-∞n are weaker than the chain conditions max-n and min-n (the maximal and minimal conditions on normal subgroups, respectively), we define a group satisfies max-∞n^* if it satisfies

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max-∞n, but not max-n and a group satisfies min-∞n^* if it satisfies min-∞n, but not min-n.

De Giovanni et al. [1] characterized the structure of groups satisfying max-∞n^* or min-∞n^*. In addition, the structure of nonfinitely generated solvable groups satisfying max-∞n^* and solvable groups satisfying min-∞n^* were investigated in detail. In this paper, we consider metanilpotent groups with these chain conditions.

2. Metanilpotent groups with max-∞n^*

We describe our first result on the structure of metanilpotent groups satisfying max-∞n^*.

Theorem 2.1. *Let G be a metanilpotent group with $F = \mathrm{Fit}(G)$ and $Z = Z(F)$. Then G satisfies* max-∞n^* *if and only if the following conditions hold:*

(1) *F is infinite nilpotent and G/F has* max-n;

(2) *if L is an infinite normal subgroup of G, then $Z/Z \cap L$ is finite.*

Proof. Suppose that G satisfies max-∞n^*. Let N be a normal nilpotent subgroup of G such that G/N is nilpotent.

Case: N is finite. G/N is Prüfer-by-finite and so is G ([3, Theorem 4.2]). Let P be a normal Prüfer subgroup of G with G/P finite. Then F/P is finite and so $F = \langle a_1, a_2, \ldots, a_k, P\rangle$ for some $a_1, a_2, \ldots, a_k$. Since each a_i is contained in some normal nilpotent subgroup of G, it follows that F is a product of finitely many normal nilpotent subgroups of G. Thus F is infinite nilpotent and G/F has max-n.

Case: N is infinite. G/N is finitely generated nilpotent and so is F/N. Hence F is a product of finitely many normal nilpotent subgroups of G by the argument of the last paragraph. Thus F is an infinite nilpotent subgroup of G and G/F has max-n.

Let $G = \langle g_1, g_2, \ldots, g_r, F\rangle$ for some $g_1, g_2, \ldots, g_r$. Then any element g in G can be written as $g = aw$ where $a \in F$ and $w \in \langle g_1, g_2, \ldots, g_r\rangle$. If $x \in Z$, then $x^g = x^{aw} = x^w \in \langle g_1, g_2, \ldots, g_r, x\rangle$, which implies that $\langle x\rangle^G \leq \langle g_1, g_2, \ldots, g_r, x\rangle$. Thus $\langle x\rangle^G$ is a finitely generated module over G/F. Since G/F is finitely generated nilpotent, it is polycyclic; hence $\langle x\rangle^G$ has max-G (the maximal condition for G-invariant subgroups) ([7, 15.3.3]). It follows that $F/\langle x\rangle^G$ does not have max-G . Therefore $\langle x\rangle^G$ must be finite and so Z is torsion.

Let L be an infinite normal subgroup of G. Then $Z/Z \cap L \simeq ZL/L$ has max-G. Hence $G/Z \simeq (G/Z \cap L)/(Z/Z \cap L)$ does not have max-G and thus $Z/Z \cap L$ is finite.

Conversely, let $Z_i = Z_i(F)$. Since F is infinite, Z_i is finite and Z_{i+1} is infinite for some i. Now pass to the group G/Z_i, that is, assume that $i = 0$. Hence we can assume that Z_1 is infinite. Suppose that the theorem is false and let $G_1 < G_2 < \cdots$ be an infinite ascending chain of infinite normal subgroups.

Then $Z/Z \cap G_i$ is finite for all i. Thus $Z \cap G_i = Z \cap G_{i+1} = \cdots$ for all large i. Also $ZG_i = ZG_{i+1} = \cdots$ for all large i. Therefore

$$G_{i+1} = ZG_i \cap G_{i+1} = (Z \cap G_{i+1})G_i = G_i,$$

a contradiction. □

Proposition 2.2. *Let G be a metanilpotent group with* max-∞n^* *and let N be a normal nilpotent subgroup of G such that G/N is nilpotent. Then G is finitely generated if and only if N' is infinite.*

Proof. Suppose that G is finitely generated. Assume that N' is finite. Then G/N' is a finitely generated abelian-by-nilpotent group, so it has max-n, as must G. By this contradiction N' is infinite.

Conversely, suppose that N' is infinite. Then G/N' has max-n, so it is finitely generated. Hence let $G = XN'$ with X a finitely generated subgroup. Then $G = X\gamma_i(N)$ for all $i > 0$ ([6, Lemma 2.2]). But then since N is nilpotent, $G = X$ and so G is finitely generated. □

Example 2.3. Let $F = \langle x_i, y_j \mid i, j \in \mathbb{Z} \rangle$ be a group where

$$x_{i+1}^2 = x_i, \quad y_{i+1}^2 = y_i, \quad \text{and} \quad [x_i, y_j] \in Z(F)$$

for all integers i and j.

Note that $[x_i, x_j] = 1 = [y_i, y_j]$. Put $c_{i+j} = [x_i, y_j]$, which depends only on $i + j$. Then M is a nilpotent group of class 2. Define $\sigma \in \mathrm{Aut}(F)$ by

$$x_i^\sigma = x_{i+1} \quad \text{and} \quad y_j^\sigma = y_{j-1}.$$

Then $c_k^\sigma = c_k$. Put

$$G = \langle \sigma \rangle \ltimes F = \langle \sigma, x_0, y_0 \rangle$$

and finally let "bars" denote images modulo $\langle c_0 \rangle$; then $\overline{G} = G/\langle c_0 \rangle$ and $\overline{Z} = \langle \bar{c}_1, \bar{c}_2, \ldots \rangle$ is a Prüfer group of type 2^∞. Note that $\overline{G}/\overline{Z}$ is finitely generated metabelian, so it has max-n. Note also that $\mathrm{Fit}(\overline{G}) = \overline{F}$ is not torsion.

Let $\overline{L}$ be an infinite normal subgroup of $\overline{G}$. We claim that $\overline{Z} \leqslant \overline{L}$. If $\overline{L} \cap \overline{F} = \overline{1}$, then $[\overline{L}, \overline{F}] = \overline{1}$. Since $C_{\overline{G}}(\overline{F}) = Z(\overline{F})$, it follows that $\overline{L} \leqslant Z(\overline{F})$. This is a contradiction. Hence $\overline{L} \cap \overline{F} \neq \overline{1}$. Since $\overline{L} \cap \overline{F}$ is a non-trivial normal subgroup of a nilpotent group $\overline{F}$, we have

$$\overline{L} \cap Z(\overline{F}) = \overline{L} \cap \overline{F} \cap Z(\overline{F}) \neq 1,$$

which means that $\overline{L}$ contains an element involving $\bar{c}_i$, say. Therefore $\overline{Z} \leqslant \overline{L}$. Hence $\overline{G}$ has max-∞n^* because $\overline{Z}$ is of type 2^∞.

3. Metanilpotent groups with min-∞n^*

We now determine the structure of metanilpotent groups satisfying min-∞n^*. We note that next result is the generalization of Wilson's theorem on groups with min-n (the minimal condition on normal subgroups). This is used for investigating metanilpotent groups with min-∞n^*.

Lemma 3.1 ([5, Proposition 2.3]). *Let M be a G-operator group and let H be a subgroup of G of finite index. If M has min-G, then it has min-H.*

Lemma 3.2 ([5, Lemma 3.3]). *A polycyclic group G satisfies* min-∞n^* *if and only if it is a finite extension of a G-rationally irreducible free abelian subgroup of finite rank.*

Theorem 3.3. *Let G be a metanilpotent group with $F = \mathrm{Fit}(G)$. Then G satisfies* min-∞n^* *if and only if F is infinite normal nilpotent and:*
either

(1) *F is a G-rationally irreducible free abelian subgroup of finite rank such that G/F is finite*

or else

(2) *G/F is infinite cyclic-by-finite, F has* min-G, *and $F/[F, x]$ is finite where x is any element of infinite order in G.*

Proof. Suppose that G has min-∞n^*. Let N be a normal nilpotent subgroup of G such that G/N is nilpotent. If N is finite, then G is infinite cyclic-by-finite ([4, Lemma 3.1]). Hence G is polycyclic. So assume that N is infinite. We first argue that G/N is finitely generated nilpotent. Assume that G/N is infinite. Then N has min-G and so G/N does not satisfy min-n. Hence G/N is infinite cyclic-by-finite ([4, Lemma 3.1]). Therefore G/N is finitely generated nilpotent. It follows that $N \leqslant F$ and F/N is finitely generated; hence F is infinite nilpotent.

Case: *G/F is finite.* We will show that G is polycyclic in this case. Let $Z_i = Z_i(F)$ and let i be maximum subject to F/Z_i being infinite. Then Z_i has min-G, so F/Z_i does not. Since G/F is finite, Z_i has min-F by Lemma 3.1. Hence Z_i has min and so G/Z_i does not have min-n. Now pass to the group G/Z_i, that is, assume that $i = 0$. Put $Z = Z(F)$. Then G/Z is finite. Let $a \in Z$ have infinite order and put $B = \langle a \rangle^G$. Then G/B is finite since otherwise B has min-G. Therefore G is polycyclic. Thus it will suffice to show that Z is not torsion.

Next after factoring out Z_i we can suppose that Z is torsion: we will argue that this cannot occur. Let $a \in Z$: then $\langle a \rangle^G$ is finite since G/Z is finite. Hence Z contains minimal normal subgroups of G. Also, if N is a minimal G-invariant subgroup of Z, then let $1 \neq a \in N$, and note that $N = \langle a \rangle^G$ is finite. Let S be the G-socle of Z. Then S is a direct product of finitely many minimal G-invariant subgroups of Z. Thus S is finite. Now Z does not have min-G, so there exists an infinite descending chain $N_1 > N_2 > \cdots$ of G-invariant subgroups of Z. Put $I = \bigcap_i N_i$. Then Z/I is infinite, so I has min-G, but Z/I does not. Pass to the group G/I, i.e., assume that $I = 1$. Since S is finite, $S \cap N_i = S \cap N_{i+1} = \cdots$ for some i; hence $S \cap N_i = 1$ since $S \cap N_j \leqslant N_j$ for all j. But this implies that $N_i = 1$, a contradiction. Therefore Z is not torsion and our discussion of this case is complete.

Case: *G/F is infinite.* Assume that $z \in Z$ has infinite order. Now G/F does not have min-n since F has min-G. Thus G/F is infinite cyclic-by-finite. Hence $\langle z\rangle^G$ is a finitely generated $\mathbb{Z}(G/F)$-module, so it is $\mathbb{Z}(G/F)$-noetherian. Also $\langle z\rangle^G$ is $\mathbb{Z}(G/F)$-artinian. Thus $\langle z\rangle^G$ is finite ([7, 15.4.4]), which is a contradiction. Hence Z is torsion. Pass to the group $\overline{G} = G/Z(F)$. Then same argument shows that $Z(\overline{F})$ is torsion; hence so is $Z_2(F)$. Repetition of this argument shows that every term of the upper central series of F is torsion. Therefore F is torsion.

Now if $\overline{F} = F/[F, x]$, then $\overline{F}/\overline{F}'$ is finite. Since $\overline{F}$ is nilpotent, $\overline{F}$ is finite for all $k > 0$ where x is any element of infinite order in G.

Conversely, if (1) holds, then G is polycyclic and the result follows from Lemma 3.2. Hence we assume that (2) holds. Suppose that the theorem is false and let $G_1 > G_2 > \cdots$ be an infinite descending chain of normal subgroups of G with infinite index. Assume that G_iF/F is infinite for some i. Then G_iF/F contains an element xF of infinite order where $x \in G_i$ and G/G_iA is finite. Since $[F, x] \leqslant F \cap G_i$, it follows that $F/F \cap G_i \simeq FG_i/G_i$ is finite and so is G/G_i, a contradiction. Thus each G_iF/F is finite. Hence there is an i such that $G_iF = G_{i+1}F$ and $G_i \cap F = G_{i+1} \cap F$, which implies that

$$G_{i+1} = G_{i+1} \cap G_iF = G_i(G_{i+1} \cap F) = G_i.$$

This is a contradiction.

Therefore G has min-∞n. Finally if G has min-n, then it is locally finite ([6, Theorem 5.25]). Hence G/F is finitely generated locally finite and so is finite, a contradiction. □

References

[1] F. De Giovanni, D. H. Paek, D. J. S. Robinson, and A. Russo, *The maximal and minimal conditions for normal subgroups of infinite order or index*, Comm. Algebra **33** (2005), no. 1, 183–199.

[2] L. A. Kurdachenko, *Groups that satisfy weak minimality and maximality conditions for normal subgroups*, Siberian Math. J. **20** (1980), 755–761.

[3] D. H. Paek, *Chain conditions for subgroups of infinite order or index*, J. Algebra **249** (2002), no. 2, 291–305.

[4] ———, *Locally nilpotent groups with the minimal condition on normal subgroups of infinite index*, Bull. Korean Math. Soc. **41** (2004), no. 4, 779–783.

[5] ———, *Abelian-by-nilpotent groups with chain conditions for normal subgroups of infinite order or index*, Bull. Korean Math. Soc. **44** (2007), No. 4, 763–769.

[6] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, Springer-Verlag, New York-Berlin, 1972.

[7] ———, *A Course in the Theory of Groups*, Springer-Verlag, New York, 1996.

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