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BINDING NUMBER CONDITIONS FOR (a, b, k)-CRITICAL GRAPHS

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ABSTRACT. Let G be a graph, and let a, b, k be integers with $0 \le a \le b, k \ge 0$. Then graph G is called an (a, b, k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a, b]$-factor. In this paper, the relationship between binding number bind(G) and (a, b, k)-critical graph is discussed, and a binding number condition for a graph to be (a, b, k)-critical is given.

1. Introduction

In this paper, we consider finite and undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, the degree of x in G is denoted by $d_G(x)$, the minimum vertex degree of $V(G)$ is denoted by $\delta(G)$. For any $S \subseteq V(G)$, we denote by $N_G(S)$ the neighborhood set of S in G, by $G[S]$ the subgraph of G induced by S, by $G - S$ the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. Let S and T be disjoint subsets of $V(G)$. We denote by $e_G(S, T)$ the number of edges joining S and T. We write $i(G)$ for the number of isolated vertices in G. The binding number of G is defined by Woodall [6] as

$$\text{bind}(G) = \min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

Let a and b be integers with $0 \le a \le b$. An $[a, b]$-factor of graph G is defined as a spanning subgraph F of G such that $a \le d_F(x) \le b$ for each $x \in V(F)$. And if $a = b = k$, then an $[a, b]$-factor is called an k-factor. A graph G is called an (a, b, k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a, b]$-factor. If G is an (a, b, k)-critical graph, then we also say that G is (a, b, k)-critical. If $a = b = n$, then an (a, b, k)-critical graph is simply called an (n, k)-critical graph. If $n = 1$, then a (n, k)-critical graph is simply

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called a k-critical graph. The other terminologies and notations not given in this paper can be found in [1].

Yu [7] gave the characterization of k-critical graphs. Cai et al [2, 5] studied the relationship between toughness $t(G)$ and (n,k)-critical graphs ($n = 2,3$). Enomoto et al [3] showed the relationship between toughness $t(G)$ and (n,k)-critical graphs. Liu et al [4] gave a necessary and sufficient condition for a graph to be (a,b,k)-critical. Zhou [8] gave two sufficient conditions for graphs to be (a,b,k)-critical. In this paper, we give a binding number condition for a graph to be (a,b,k)-critical.

Liu et al [4] obtained a necessary and sufficient condition for a graph to be (a,b,k)-critical, where $1 \le a < b$ and $k \ge 0$ are integers.

Lemma 1.1 ([4]). *Let G be a graph with $|V(G)| \ge a+k+1$. Then G is (a,b,k)-critical if and only if for any $S \subseteq V(G)$ and $|S| \ge k$*

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \le b|S| - bk,$$

where $p_j(G-S) = |\{x : d_{G-S}(x) = j\}|$.

2. The proof of main results

Theorem 1. *Let G be a graph, and let $2 < b$ and $k \ge 0$ be integers, $\delta(G) \ge k+2$. If $\mathrm{bind}(G) \ge \frac{(k+3)b+4}{3b}$, then graph G is $(2,b,k)$-critical.*

Proof. Suppose that $bind(G) \ge \frac{(k+3)b+4}{3b}$ and G is not $(2,b,k)$-critical. By Lemma 1.1, there exists $S_0 \subseteq V(G)$ and $|S_0| \ge k$ such that

$$2p_0(G-S_0) + p_1(G-S_0) > b|S_0| - bk. \tag{1}$$

Let

$$T_0 = \{x : x \in V(G) \setminus S_0, d_{G-S_0}(x) = 0\},$$

and

$$T_1 = \{x : x \in V(G) \setminus S_0, d_{G-S_0}(x) = 1\}.$$

We have $|T_0| = p_0(G-S_0) = i(G-S_0)$ and $|T_1| = p_1(G-S_0)$. The proof splits into three cases.

Case 1. $T_1 = \varnothing$.

By (1), we have

$$bk \le b|S_0| < 2p_0(G-S_0) + bk.$$

So

$$p_0(G-S_0) > 0.$$

Subcase 1.1. $1 \le p_0(G-S_0) \le 2$.

Since $\delta(G) \ge k+2$, we get

$$|S_0| \ge k+2. \tag{2}$$

It follows from (1) and (2) that

$$\begin{aligned}4 \geq 2p_0(G-S_0) &= 2p_0(G-S_0)+p_1(G-S_0)\\ &> b|S_0| - bk \geq b(k+2) - bk = 2b > 4.\end{aligned}$$

which is contradicted.

Subcase 1.2. $p_0(G-S_0) \geq 3$.

We write X for the set of isolated vertices of $G-S_0$, it is easily seen that $3 \leq |X| = p_0(G-S_0)$ and $N_G(X) \leq |S_0|$. Thus

$$\frac{(k+3)b+4}{3b} \leq \mathrm{bind}(G) \leq \frac{|N_G(X)|}{|X|} \leq \frac{|S_0|}{p_0(G-S_0)}.$$

So

$$\begin{aligned}b|S_0| &\geq \frac{(k+3)b+4}{3}p_0(G-S_0) = 2p_0(G-S_0) + \frac{(k+3)b-2}{3}p_0(G-S_0)\\ &\geq 2p_0(G-S_0) + (k+3)b - 2\\ &> 2p_0(G-S_0) + bk = 2p_0(G-S_0) + p_1(G-S_0) + bk.\end{aligned}$$

This contradicts (1).

Case 2. $T_1 \neq \varnothing$ and $T_1 = N_{G-S_0}(T_1)$.

By the definition of T_1, we have $|T_1| = 2r$ (r be positive-integer).

Subcase 2.1. $|T_1| = 2$.

Since $\delta(G) \geq k+2$, we get

$$|S_0| \geq k+1. \tag{3}$$

Subcase 2.1.1. $p_0(G-S_0) = 0$.

By (1) and (3),

$$\begin{aligned}2 = |T_1| = p_1(G-S_0) &= 2p_0(G-S_0) + p_1(G-S_0)\\ &> b|S_0| - bk \geq b(k+1) - bk = b > 2\end{aligned}$$

a contradiction.

Subcase 2.1.2. $p_0(G-S_0) \geq 1$.

Let $X = T_0 \cup T_1$, then $|N_G(X)| \leq |S_0| + p_1(G-S_0)$. Thus

$$\frac{(k+3)b+4}{3b} \leq \mathrm{bind}(G) \leq \frac{|N_G(X)|}{|X|} \leq \frac{|S_0| + p_1(G-S_0)}{p_0(G-S_0)+p_1(G-S_0)}.$$

So

$$\begin{aligned}&b|S_0|\\ &\geq \frac{(k+3)b+4}{3}p_0(G-S_0) + \frac{bk+4}{3}p_1(G-S_0)\\ &= 2p_0(G-S_0) + p_1(G-S_0) + \frac{(k+3)b-2}{3}p_0(G-S_0) + \frac{bk+1}{3}p_1(G-S_0)\\ &\geq 2p_0(G-S_0) + p_1(G-S_0) + \frac{(k+3)b-2}{3} + \frac{2(bk+1)}{3}\\ &> 2p_0(G-S_0) + p_1(G-S_0) + bk.\end{aligned}$$

Which contradicts (1).

Subcase 2.2. $|T_1| \geq 4$.

For any $u \in T_1$, let $X = T_0 \cup (T_1 - u)$, then $|X| = p_0(G-S_0)+p_1(G-S_0)-1$, $|N_G(X)| \leq |S_0| + p_1(G - S_0) - 1$, we have

$$\frac{(k+3)b+4}{3b} \leq \mathrm{bind}(G) \leq \frac{|N_G(X)|}{|X|} \leq \frac{|S_0| + p_1(G - S_0) - 1}{p_0(G - S_0) + p_1(G - S_0) - 1}.$$

So

$$\begin{aligned} 3b|S_0| &\geq (bk+4)p_1(G - S_0) + (bk + 3b + 4)p_0(G - S_0) - bk - 4 \\ &= 6p_0(G - S_0) + 3p_1(G - S_0) + (bk + 3b - 2)p_0(G - S_0) \\ &\quad + (bk+1)p_1(G - S_0) - bk - 4 \\ &\geq 6p_0(G - S_0) + 3p_1(G - S_0) + 4(bk+1) - bk - 4 \\ &= 6p_0(G - S_0) + 3p_1(G - S_0) + 3bk \end{aligned}$$

i.e.,

$$b|S_0| \geq 2p_0(G - S_0) + p_1(G - S_0) + bk.$$

This contradicts (1).

Case 3. $T_1 \neq \varnothing$ and $T_1 \neq N_{G-S_0}(T_1)$.

There exists $u \in N_{G-S_0}(T_1) \setminus T_1$, such that $d_{G-S_0}(u) \geq 2$. Let r be the edge number in $[T_1]_{G-S_0}$, we write $X = S_0 \cup (N_{G-S_0}(T_1) - T_1 - u)$. By the definition of T_1, we can easily obtain

$$|X| \leq |S_0| + p_1(G - S_0) - 2r - 1$$

and

$$\begin{aligned} i(G - X) &\geq i(G - S_0) + (p_1(G - S_0) - 2r) + r \\ &= p_0(G - S_0) + p_1(G - S_0) - r \\ &\geq p_0(G - S_0) + \frac{1}{2}p_1(G - S_0) > 0. \end{aligned}$$

We write Y for the set of isolated vertices of $G - X$, then $|Y| = i(G - X)$ and $|N_G(Y)| \leq |X| \leq |S_0| + p_1(G - S_0) - 2r - 1$. Thus

$$\begin{aligned} \frac{(k+3)b+4}{3b} &\leq \mathrm{bind}(G) \leq \frac{|N_G(Y)|}{|Y|} \leq \frac{|S_0| + p_1(G - S_0) - 2r - 1}{p_0(G - S_0) + p_1(G - S_0) - r} \\ &\leq \frac{|S_0| + p_1(G - S_0) - 1}{p_0(G - S_0) + p_1(G - S_0)}. \end{aligned}$$

So

$$\begin{aligned} 3b|S_0| &\geq 6p_0(G - S_0) + 3p_1(G - S_0) + (bk + 3b - 2)p_0(G - S_0) \\ &\quad + (bk+1)p_1(G - S_0) + 3b. \end{aligned} \tag{4}$$

Since $T_1 \neq \varnothing$, then $|S_0| \geq k + 1$. By (1), we get

$$2p_0(G - S_0) + p_1(G - S_0) \geq b|S_0| - bk + 1 \geq b + 1 \geq 4. \tag{5}$$

Obviously, $p_1 \geq 1$ since $p_1 = |T_1|$. Then, by the integrity of p_0, p_1 and (5) we obtain

$$\begin{array}{llll} p_0 \geq 2 & \text{if} & p_1 = 1; & \text{or} \\ p_0 \geq 1 & \text{if} & p_1 = 2; & \text{or} \\ p_0 \geq 0 & \text{if} & p_1 \geq 3. & \end{array}$$

Thus, we have

$$(bk + 3b - 2)p_0(G - S_0) + (bk + 1)p_1(G - S_0) + 3b \geq 3bk + 3b + 3. \tag{6}$$

It follows from (4) and (6) that

$$\begin{aligned} 3b|S_0| &\geq 6p_0(G - S_0) + 3p_1(G - S_0) + 3bk + 3b + 3 \\ &> 6p_0(G - S_0) + 3p_1(G - S_0) + 3bk, \end{aligned}$$

that is,

$$b|S_0| > 2p_0(G - S_0) + p_1(G - S_0) + bk.$$

Which contradicts (1).

From all the cases above, we deduced the contradiction, so the hypothesis can not hold. Hence, G is $(2, b, k)$-critical. □

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